IOSR Journal of Mathematics (IOSR-JM)
e-ISSN: 2278-5728, p-ISSN: 2319-765X. Volume 14, Issue 3 Ver. III (May - June 2018), PP 19-20
www.iosrjournals.org

Numerical Index and Extremely Non-Complex Banach Spaces

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Abstract: There exist infinite-dimensional extremely non-complex Banach spaces, i.e. spaces X such that the norm equality $\|Id + T^2\| = 1 + \|T^2\|$ holds for every bounded linear operator T on X .show that every extremely non-complex Banach space has positive numerical index

Keywords.Banachspace ;numerical index; extremely non-complex; diameter of slices.

The concept of numerical index of a Banach space it is a parameter relating the norm and the numerical range of operators on the space. The numerical range was first introduced for matrices, and it was extended in the sixties to bounded linear operators on an arbitrary Banach space [1] , and we defined it for an operator$T \in L(x)$ as :$V(T) = \{x^*(Tx) : x \in SX, x^* \in S_{X^*}, x^*(x) = 1\}$

whereX^* stands for the dual space of X and S_X its unit sphere. This definition of numerical range was introduced by [2].

The name of extremely non-complex comes from the fact that a real Banach space X is said to have a complex structure if there exists $T \in L(X)$ such that $T^2 = -Id$, so extremely noncomplex spaces lack of complex structures in a very strong way. Let us also comment that nohyperplane (actually no finite-codimensional subspace) of an extremely non-complex Banach space admits a complex structure. [4]

Suppose the seminorm$v$ on $L(X)$ satisfying $v(T) \leq ||T||$ for every $T \in L(X)$ and the equation

$$n(X) = \inf \{v(T): T \in S_L(X)\}$$

Represent the numerical index of the space X.

Note that $0 \leq n(X) \leq 1$ and $n(X) > 0$ if and only if v and $||.||$ are equivalent norms on $L(X)$.

When$n(X) \geq 1/e,$ the numerical radius is always an equivalent norm (complex case), the real case is more different, since every real Hilbert space of dimension greater than one has numerical index zero[2].

Corollary 1

Let $(X, ||.||)$ be a finite-dimensional real Banach space. Then, the following are equivalent:

(i) The numerical index of X is zero.

(ii) There are nonzero complex vector spaces $X_1, X_2 ..., X_n$, a real vector space X_0, and positive integer numbers$c_1, c_2 ..., c_n$ such that :

$$X = X_0 \bigoplus X_1 \bigoplus X_2 \bigoplus \cdots \bigoplus X_n$$

and

$$|| X_0 + e^{ic_1\rho} x_1 + \cdots + e^{ic_n\rho} x_n || = || x_0 + x_1 + \cdots + x_n ||$$

for all $\rho \in R,\ x_i \in X_i\ ,\ i = 1, \ldots, n+1.$

The above corollary is derivation from [2, theorem 1].

For every real Banachspace X, let us denote by $Z(X)$ the subspace of $L(X)$ consisting of those operators T on $X$with $v(T) = 0$.

Corollary 2. ([2, Corollary 2]) Let X be a real Banach space of dimension$n \in N$. Then we have:

$\dim(Z(X)) \leq \frac{n^2-n}{2}$. Moreover, X is a Hilbert space if and only if $\dim(Z(X)) = \frac{n^2-n}{2}$.

The number of complex spaces in assertion (ii) of corollary 1 can be always reduced to one. As a matter of fact, this is not true, [see example 4 in 2].

Theorem 3. Let X be an extremely non-complex Banach space. Then

$$n(X) \geq \sqrt{1 + e^{-2}} - 1.$$

The numerical index of any extremely non-complex Banach space is positive, this result was introduced by M.Martin and J.Meri in [3], also said in theorem 2 Let$X$ be an extremely non-complex Banach space of dimension greater than one. Then, the infimum of the diameter of the slices of B_X is positive. The prove of this theorem is depending on the following assumption:

$B_X$is unit ball and S_X is unit sphere with diameter d without loss of generality that $d < 1$, $x^* \in S_{X^*}$, $0 < \alpha \leq 1, S = S(B_X, x^*, \alpha)$, Fixed $0 < \varepsilon < min\{1-d, \alpha\}$, pick $y_0 \in S_X$ such that $x^*(y_0) > 1 - \varepsilon$.(then the following equation is satisfy : $|| x + ty_0 || > (1 - \varepsilon - d)(|| x ||+|t|)$)

holds for every $x \in \ker\ x^*$ and $t \in \mathbb{R}$. And when take $\varepsilon \to 0$ the last equation in the form :

$$2 - d \leq \left(1 + \frac{\sqrt{2}-1}{1-d}\right)\frac{1}{1-d}$$

which obviously implies that d cannot be arbitrarily close to 0.
The above proof only use the fact that the norm equality $\|Id + T^2\| = 1 + \|T^2\|$ holds for every rank-one operator on the space. Therefore, for every Banach space Xsatisfying this condition, the infimum of the diameter of the slices of B_X is positive.

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Nahid H. K. Abdelgader. Numerical Index and Extremely Non-Complex Banat Spaces" IOSR Journal of Mathematics (IOSR-JM) 14.3 (2018) PP: 19-20.