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# Estimates in the Operator Norm

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***Abstract*:** *In this paper, we will obtain estimates of the distance between the q-k-eigenvalues of two q-k-normal matrices A and B interms of $\|A-B\|$. Apart from the optimal matching distances.*

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## I. Introduction

In this paper, we will obtain estimates of the distance between the q-k-eigenvalues of two q-k-normal matrices A and B interms of $\|A-B\|$. Apart the optimal matching distances $s(L,M)$ and $h(L,M)$. Note that $s(L,M)$ is the smallest number $\delta$ such that every element of $L$ is within a distance $\delta$ of some element of $M$; and $h(L,M)$ is the smallest number $\delta$ for which this as well as the symmetric assertion with $L$ and $M$ interchanged, is true.

We will use the notation $\sigma(A)$ for both the subset of the quaternion plane that consists of all the q-k-eigenvalues on $n\times n$ matrix $A$, and for the unordered $n$-tupile whose entries are the q-k-eigenvalues of $A$ counted with multiplicity. Since we will be taking of the distances $s(\sigma(A),\sigma(B)), h(\sigma(A),\sigma(B))$ and $d(\sigma(A),\sigma(B))$, it will be clear which of the two objects is being represented by $\sigma(A)$.

## II. Definitions And Some Theorems

**Definition 2.1:**

If L, M are closed subsets of a quaternion space $H_n$

let $s(L,M)=\sup\limits_{\lambda\in L} dist(\lambda,M)=\sup\limits_{\lambda\in L}\inf\limits_{\mu\in M}|\lambda-\mu|$

**Definition 2.2:**

The Housdorff distance between L and M is defined as

$$h(L,M)=\max(s(L,M),s(M,L))$$

**Definition 2.3:**

The $d(\sigma(A),\sigma(B))$ is defined as $d(\sigma(A),\sigma(B))\le\|A-B\|$ if either A and B are both q-k-Hermitian or one is q-k-Hermitian and other q-k-Skew-Hermitian.

**Theorem 2.4:**

Let $A$ be q-k-normal and $B$ an arbitrary matrix of same order of $A$. Then

$s(\sigma(B),\sigma(A))\le\|A-B\|$

**Proof:**

Let $\delta=\|A-B\|$. For proving the theorem, we have to show that if $\beta$ is any q-k-eigenvalues of $B$, then $\beta$ is within a distance $\delta$ of some q-k-eigenvalue $\alpha_j$ of $A$.

By applying a translation, we assume that $\beta=0$. If none of the $\alpha_j$ is within a distance $\delta$ of this, then $A^{-1}$ exists.

Since $A$ is q-k-normal.

Therefore , $\left\|A^{-1}\right\| = \dfrac{1}{\max\left|\alpha_j\right|} < \dfrac{1}{\delta}.$

Hence, $\left\|A^{-1}(B-A)\right\| \le \left\|A^{-1}\right\|\left\|B-A\right\|.$

$$< \frac{1}{\delta}\delta$$
$$= 1$$

Since $B = A(I + A^{-1}(B-A))$, This show that $B$ is invertible. Then but $B$ could not have a zero q-k-eigenvalue.
Hence proved.

**Corollary 2.5:**

If $A$ and $B$ are $n \times n$ q-k-normal matrices then $h(\sigma(A),\sigma(B)) \le \left\|A-B\right\|$.

**Proof:**

Since $A$ and $B$ are q-k-normal matrices of order $n \times n$.

Let $\sigma(A)$ and $\sigma(B)$ be set of all q-k-eigenvalues of $A$ and $B$ respectively.

$$s(\sigma(A),\sigma(B)) \le \left\|A-B\right\| \quad (1)$$

and $h(\sigma(A),\sigma(B)) = \max(s(\sigma(A),\sigma(B)), s(\sigma(B),\sigma(A)))$.

From these two, one can conclude that $h(\sigma(A),\sigma(B)) \le \left\|A-B\right\|.$

**Remark 2.6:**

For $n = 2$, the corollary 2.5 will lead to $d(\sigma(A),\sigma(B)) \le \left\|A-B\right\|$.

**Theorem 2.7:**

For any two q-k-unitary matrices $d(\sigma(A),\sigma(B)) \le \left\|A-B\right\|.$

**Proof:**

The proof will use the marriage theorem and above, Let $\{\lambda_1,\lambda_2,...,\lambda_n\}$ and $\{\mu_1,\mu_2,...,\mu_n\}$ be the q-k-eigenvalues of $A$ and $B$ respectively.

Let $\Lambda$ be any subset of $\{\lambda_1,\lambda_2,...,\lambda_n\}$.

Let $\mu(\Lambda) = \{\mu_j : \left|\mu_j - \lambda_i\right| \le \delta and \lambda_i \in \Lambda\}.$

By the marriage theorem, the assertion would be proved if we show that $\left|\mu(\Lambda)\right| \ge \left|\Lambda\right|$.

Let $I(\Lambda)$ be the set at all points on the unit ball T that are within distance of some point of $\Lambda$. Then $\mu(\Lambda)$ contains exactly those $\mu_j$ that lie in $I(\Lambda)$.

Let $I(\Lambda)$ be written as a disjoint union of arcs $I_1,...,I_r$. For each $k; k < r$, let $J_k$ be the arc contained in $I_k$ all whose points at least distance from the boundary of $I_k$ then $I_k = (J_k)_{\in}$.

We have $\sum_{k=1}^{r} m_A(J_k) \le \sum_{k=1}^{r} m_B(I_k) = m_B(I(\Lambda))$

But all the elements of $\Lambda$ are in some $J_k$.

$$\Rightarrow \left|\Lambda\right| \le \left|\mu(\Lambda)\right|$$

Similarly for, $\mu$ is a subset of $\{\mu_1,\mu_2,...\mu_n\}$.

$$\left|\mu\right| \le \left|\Lambda(\mu)\right|$$
$$\left|\wedge - \mu\right| \le \left|\wedge\right| - \left|\mu\right|$$
$$\left|\Lambda - \mu\right| \le \left|\Lambda(\mu) - \mu(\Lambda)\right|$$

$$[\because \lambda_i \in \sigma(A), \mu_j \in \sigma(B)]$$

$$\Rightarrow \max_{1 \le i, j \le n} |\lambda_i - \mu_j| \le \|A - B\|$$

That is, $d(\sigma(A), \sigma(B)) \le \|A - B\|$

Hence proved.

**Remark 2.8:**

There is one difference between theorem 2.7 and most of our earlier results of this type. Now nothing is said about the order in which the q-k-eigenvalues of $A$ and $B$ are arranged for the optimal matching. No canonical order can be prescribed in general.

**Theorem 2.9:**

Let $A$ and $B$ be q-k-normal matrices with q-k-eigenvalues $\{\lambda_1, \lambda_2, ..., \lambda_n\}$ and $\{\mu_1, \mu_2, ..., \mu_n\}$ respectively. Then there exists a permutation $\sigma$ such that

$$\|A - B\| \le \sqrt{2} \max_{1 \le j \le n} |\lambda_j - \mu_{\sigma(j)}| \qquad (2)$$

**Proof:**

Since $A$ and $B$ are q-k-normal matrices. So $A \otimes I$ and $I \otimes B$ are both q-k-normal and commute with each other. Hence $A \otimes I - I \otimes B$ is q-k-normal. The q-k-eigenvalues of this matrix are all the differences $\lambda_i - \mu_j; 1 \le i, j \le n$

Hence $\|A \otimes I - I \otimes B\| = \max_{i, j} |\lambda_i - \mu_j|$

Since q-k-eigenvalues of $B$ are q-k-eigenvalues of $B^T$.

So $\|A \otimes I - I \otimes B^T\| = \max_{i, j} |\lambda_i - \mu_j|$

$$\Rightarrow \|A - B\| = \|A \otimes I - I \otimes B\|$$
$$\le \sqrt{2} \|A \otimes I - I \otimes B^T\|$$

This is equivalent to (2)

Therefore, $\|A - B\| \le \sqrt{2} \max_{1 \le j \le n} |\lambda_j - \mu_{\sigma(j)}|$

Hence proved.

**Remark 2.10:**

This is, in fact, true for all $A, B$ and is proved below.

**Theorem 2.11:**

For all quaternion matrices $A, B$ $\|A - B\| \le 2 \|A \otimes I - I \otimes B^T\|$ (3)

**Proof:**

We have to prove that for all $x, y$ in $H_n$

$$|\langle x, (A - B) y \rangle| \le \sqrt{2} \|A \otimes I - I \otimes B^T\| \, \|x\| \, \|y\|$$

Now, $|\langle x, (A - B) y \rangle| = |\langle x, Ay - By \rangle|$

$$= |x^* Ay - x^* By|$$
$$= |tr(Ayx^* - yx^* B)|$$
$$\le \|Ayx^* - yx^* B\|_1$$

This matrix $Ayx^* - yx^* B$ has rank atmost 2 so, $\|Ayx^* - yx^* B\|_1 \le \sqrt{2} \|Ayx^* - yx^* B\|_2$.

Let $\overline{x}$ be the vector whose components are the conjugates of the components of $x$. Then with respect to the standard basis $e_i \otimes e_j$ of $H_n \otimes H_n$, $(i,j)$-coordinate of the vector $(A \otimes I)(y \otimes \overline{x})$ is $\sum_k a_{ik} y_k \overline{x}_j$ .This is also $(i,j)$-entry of the matrix $Ayx^*$. In the same way, the $(i,j)$-entry of $yx^*B$ is the $(i,j)$-coordinate of the vector $(I \otimes B^T)(y \otimes \overline{x})$.

Thus we have , $$\left\| Ayx^* - yx^*B \right\|_2 = \left\| (A \otimes I - I \otimes B^T)(y \otimes \overline{x}) \right\|$$

$$\leq \left\| A \otimes I - I \otimes B^T \right\| \left\| y \otimes \overline{x} \right\|$$

$$= \left\| A \otimes I - I \otimes B^T \right\| \; \left\| x \right\| \left\| y \right\|$$

Hence proved.

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