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A new approach for Derivation of Quadratic and Family of Roots Finding Methods

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Abstract*There are many methods for solving a nonlinear algebraic equation. Here a recurrence iteration formula for two-roots finding is derived based on the quadratic expansion of Taylor series. The general formula of quadratic equation is obtained using the derived formula. A family of iteration functions is derived from the derived formula. This family includes the Newton,Patrik, Halley, and Schroder'smethods.All methods of the family are cubically convergent for a simple root(except Newton's which is quadratically convergent).A simple general formula is derived and proved to be one of the familyof Halley-like method.*
Keywords*Simple roots, Nonlinear equations, Halley method, Taylor expansion*

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I. Introduction

The root-finding problem is one of the most important computational problems. It arises ina wide variety of practical applications in physics, chemistry, biosciences, engineering,etc.

In 1694 Halley [1] developed the third order method given by

$$x_{n+1} = x_n - \frac{f_n}{f_n' - \frac{f_n f_n''}{2f_n'}}$$

Here, and in the following, we denote $f_n = f(x_n)$ and similarly for the derivatives.

Since the method requires the evaluation of the function and its first and second derivatives, then we can say that theefficiency index (see Traub [2]) is $E = p^{1/d} = 3^{(1/3)} = 1.44$, which is higher than Newton's efficiency index of 1.4142. Thisis assuming that the cost of the derivatives is the same as the function.

Wynn [3] noted that methods using second derivatives are very useful for evaluating zeros of functions satisfying asecond order ordinary differential equation (e.g., Bessel's functions). In such cases the evaluation of second derivatives istrivial and thus the increase in efficiency.

Hansen, Eldon, and Merrell Patrick [2] derived a one parameter family of iteration functions forroot finding. This family includes the Laguerre, Halley, Ostrowski, and Eulermethods and, as a limiting case, Newton's method.

Several new methods for solving one nonlinear equation are developed by BenyNeta[3]. Most of the methods are of order three and they require the knowledge of $f, f'\ and\ f''$. See also Scavo, T. R., and J. B. Thoo[4], Thoo, J. B[5] and Neta, Beny, and Melvin Scott[6]

In this paper a new approach of a family of iteration functions forroot finding is derived based on quadratic expansion of Taylor series. This family includes the Newton, Patrik, Halley, and Schroder methods.All the methods of the family are cubically convergent for a simple root(except Newton's which is quadratically convergent). Also, the general formula of quadratic equation is obtained using the derived formula.

II. Basic Fundamentals

In this section the basic fundamentals of Taylor's theorem as well as the forward, backward and central difference approximations of higher order derivative are reviewed.

Taylor's Theorem: If f is a function continuous and n times differentiable inan interval $[x, x + h]$, then there exists some point in this interval, denoted by$x + \lambda h$ for some $\lambda \in [0, 1]$, such that

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2}f''(x) + \cdots \quad + \frac{h^{n-1}}{(n-1)!}f^{n-1}(x) + \frac{h^n}{n!}f^n(x+\lambda h) \qquad (1)$$

If *f* is a so-called analytic function of which the derivatives of all orders exist,then one may consider increasing the value of *n* indefinitely. Thus, if thecondition holds that

$$\lim_{n\to\infty} \frac{h^n}{n!} f^n(x) = 0 \qquad (2)$$

which is to say that the terms of the series converge to zero as their orderincreases, then an infinite-order Taylor-series expansion is available in the formof

$$f(x+h) = \sum_{j=0}^{\infty} \frac{h^j}{j!} f^j(x) \qquad (3)$$

This is obtained simply by extending indefinitely the expression from Taylor'sTheorem. In interpreting the summary notation for the expansion, one mustbe aware of the convention that 0! = 1.

III. Derivation of the proposed Equation

In this section the proposed equation for roots finding is derived based on Taylor's theorem. Write down Taylor's expansion of a function*f* up to the third term., such that

$$f(x+\Delta x) = f(x) + \Delta x f'(x) + \frac{\Delta x^2}{2} f''(x) + O(\Delta x^3)$$

Where $\Delta x = x_{k+1} - x_k$

Here, and in the following, we denote $f(x+\Delta x)$ by $f(x_{k+1})$and similarly for the derivatives.

$$f(x_{k+1}) = f(x_k) + \Delta x f'(x_k) + \frac{\Delta x^2}{2} f''(x_k)$$

If x_{k+1} is the root of function $f(x)$ so; $f(x_{k+1}) = 0$so;

$$\Delta x^2 + 2\frac{f'(x_k)}{f''(x_k)}\Delta x + 2\frac{f(x_{k+1})}{f''(x_k)} = 0$$

Rearrange this equation;

$$\left(\Delta x + \frac{f'(x_k)}{f''(x_k)}\right)^2 + 2\frac{f(x_{k+1})}{f''(x_k)} - \left(\frac{f'(x_k)}{f''(x_k)}\right)^2 = 0$$

$$\Delta x = \pm\sqrt{\left(\frac{f'(x_k)}{f''(x_k)}\right)^2 - 2\frac{f(x_{k+1})}{f''(x_k)}} - \frac{f'(x_k)}{f''(x_k)}$$

The recurrence formula can be written as:

$$x_{k+1} = x_k - \frac{f'(x_k)}{f''(x_k)} \pm \sqrt{\left(\frac{f'(x_k)}{f''(x_k)}\right)^2 - 2\frac{f(x_k)}{f''(x_k)}}$$

Application

1- Quadratic formula

Les us using this equation to obtain the roots of

$f(x) = a\,x^2 + b\,x + c = 0$

$f'(x) = 2a\,x + b$

$f''(x) = 2a$

$$x_{k+1} = x_k - \frac{2a\, x_k + b}{2a} \pm \sqrt{\left(\frac{2a\, x_k + b}{2a}\right)^2 - 2\,\frac{a x_k^2 + b\, x_k + c}{2a}}$$

$$x_{k+1} = -\frac{b}{2a} \pm \sqrt{\frac{4a^2\, x_k^2 + 4ab\, x_k + b^2 - 4a^2\, x_k^2 - 4ab\, x_k - 4ac}{4a^2}}$$

$$x_{k+1} = -\frac{b}{2a} \pm \sqrt{\frac{b^2 - 4ac}{4a^2}}$$

2- Newton's formula

After the square-root in the recurrence formula is linearized

$$\frac{f'(x_k)}{f''(x_k)}\sqrt{1 - 2\frac{f(x_k)f''(x_k)}{\left(f'(x_k)\right)^2}} = \frac{f'(x_k)}{f''(x_k)}\left(1 - \frac{f(x_k)f''(x_k)}{\left(f'(x_k)\right)^2}\right) = \frac{f'(x_k)}{f''(x_k)} - \frac{f(x_k)}{f'(x_k)}$$

Substituting this result in the recurrence formula with +ve sign, the recurrence formula becomes

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$

Which is the Newton formula.

3- Patrik'sformula

The recurrence formula is rewritten as:

$$x_{k+1} = x_k - \left(\frac{f'(x_k)}{f''(x_k)} \mp \sqrt{\left(\frac{f'(x_k)}{f''(x_k)}\right)^2 - 2\frac{f(x_k)}{f''(x_k)}}\right)$$

then multiply by

$$\frac{\left(\frac{f'(x_k)}{f''(x_k)} \pm \sqrt{\left(\frac{f'(x_k)}{f''(x_k)}\right)^2 - 2\frac{f(x_k)}{f''(x_k)}}\right)}{\left(\frac{f'(x_k)}{f''(x_k)} \pm \sqrt{\left(\frac{f'(x_k)}{f''(x_k)}\right)^2 - 2\frac{f(x_k)}{f''(x_k)}}\right)}$$

The recurrence formula become

$$x_{k+1} = x_k - \frac{2f(x_k)}{f'(x_k) \pm \sqrt{(f'(x_k))^2 - 2f(x_k)f''(x_k)}}$$

Which is the Patrik formula with α=1

4- Halley's formula

After the positive square-root in Patrik's formula is linearized, the resulting formula is

$$x_{k+1} = x_k - \frac{2f(x_k)f'(x_k)}{2\,(f'(x_k))^2 - f(x_k)f''(x_k)}$$

which is the Halley's formula.

5- Schrode'sformula

If return back to the derived formula with multiple roots and multiply the second term by

$$\frac{\sqrt{\left(\frac{f'(x_k)}{f''(x_k)}\right)^2 - 2\frac{f(x_k)}{f''(x_k)}}}{\sqrt{\left(\frac{f'(x_k)}{f''(x_k)}\right)^2 - 2\frac{f(x_k)}{f''(x_k)}}}$$

The resulting formula is

$$x_{k+1} = x_k - \frac{f'(x_k)}{f''(x_k)} \pm \frac{\left(\frac{f'(x_k)}{f''(x_k)}\right)^2 - 2\frac{f(x_k)}{f''(x_k)}}{\frac{f'(x_k)}{f''(x_k)}\sqrt{1 - \frac{2f(x_k)f''(x_k)}{(f'(x_k))^2}}}$$

After linearizing the square-root in the denominator, the recurrence formula is

$$x_{k+1} = x_k - \frac{f'(x_k)}{f''(x_k)} \pm \frac{\left(\frac{f'(x_k)}{f''(x_k)}\right)^2 - 2\frac{f(x_k)}{f''(x_k)}}{\frac{f'(x_k)}{f''(x_k)}\left(1 - \frac{f(x_k)f''(x_k)}{\left(f'(x_k)\right)^2}\right)}$$

After simplification with taking the third term with positive sign, the recurrence formula becomes:

$$x_{k+1} = x_k - \frac{f(x_k)f'(x_k)}{\left(f'(x_k)\right)^2 - f(x_k)f''(x_k)}$$

which is the Schroder's method

6- General formula

A general form for Newton, Halley and Schroder methods can be driven by applyingNewton's formula for the function $\frac{f(x_k)}{(f'(x_k))^{\alpha}}$

The general Newton' formula can be written as

$$x_{k+1} = x_k - \frac{\frac{f(x_k)}{(f'(x_k))^{\alpha}}}{\left(\frac{f(x_k)}{(f'(x_k))^{\alpha}}\right)'}$$

where;

$$\left(\frac{f(x_k)}{(f'(x_k))^{\alpha}}\right)' = \frac{(f'(x_k))^{\alpha} f'(x_k) - \alpha f(x_k)(f'(x_k))^{\alpha-1} f''(x_k)}{(f'(x_k))^{2\alpha}}$$

After simplifications the general form of the Newton's method is

$$x_{k+1} = x_k - \frac{f(x_k)f'(x_k)}{\left(f'(x_k)\right)^2 - \alpha f(x_k)f''(x_k)}$$

For $\alpha = 0$ the general formula becomes Newton's formula
For $\alpha = \frac{1}{2}$ the general formula becomes Halley's formula
For α=1 the general formula becomes Schroder formula

IV. Conclusion

In the present study, a new approach of deriving a family of iteration functions for root finding based on quadratic expansion of Taylor seriesis presented. This family includes the Newton Patrik, Halley, and Schroder methods.
The general formula of quadratic equation is obtained using the derived formula.

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