IOSR Journal of Mathematics (IOSR-JM)
e-ISSN: 2278-5728, p-ISSN: 2319-765X. Volume 14, Issue 5 Ver. I (Sep - Oct 2018), PP 41-47
www.iosrjournals.org

Infinite Series Proof based on Division and Geometry

Rachit Surana
(Maths, La Martiniere for Boys, India)
Corresponding Author: Rachit Surana

Abstract: *The Infinite Series is ubiquitous in mathematics. It pops up everywhere- Euler's number, trigonometric ratios, logarithms etc. One of the most important infinite series is the Geometric Series. Traditionally its calculation requires concepts of Algebra, Limits, and Convergence. However, today I will show an elegant method based on the simple concepts of division and geometry to help you visualise it better.*

Date of Submission: 10-09-2018 Date of acceptance: 28-09-2018

I. Introduction

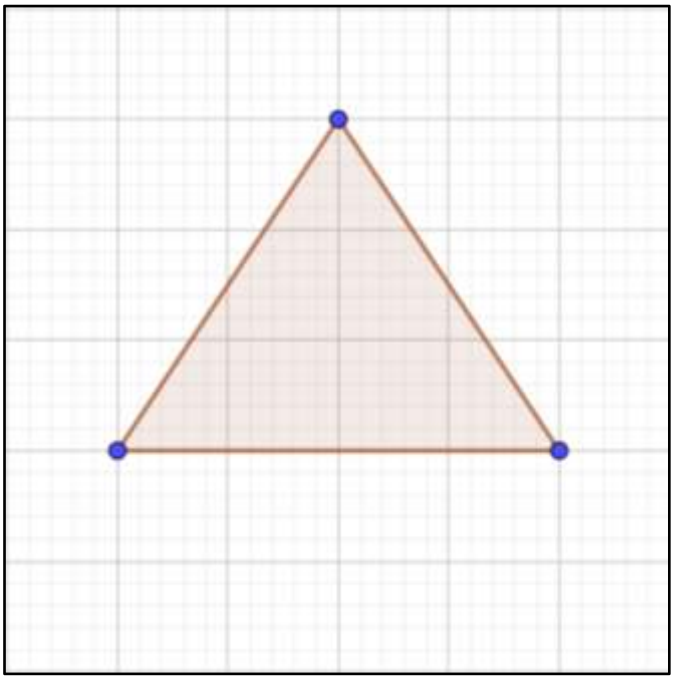
The sum of an Infinite Geometric Series is calculated using concepts of Algebra, Limits and Convergence. However, I will show you a simple and elegant proof based on elementary concepts of division and geometry.

The Interpretation:

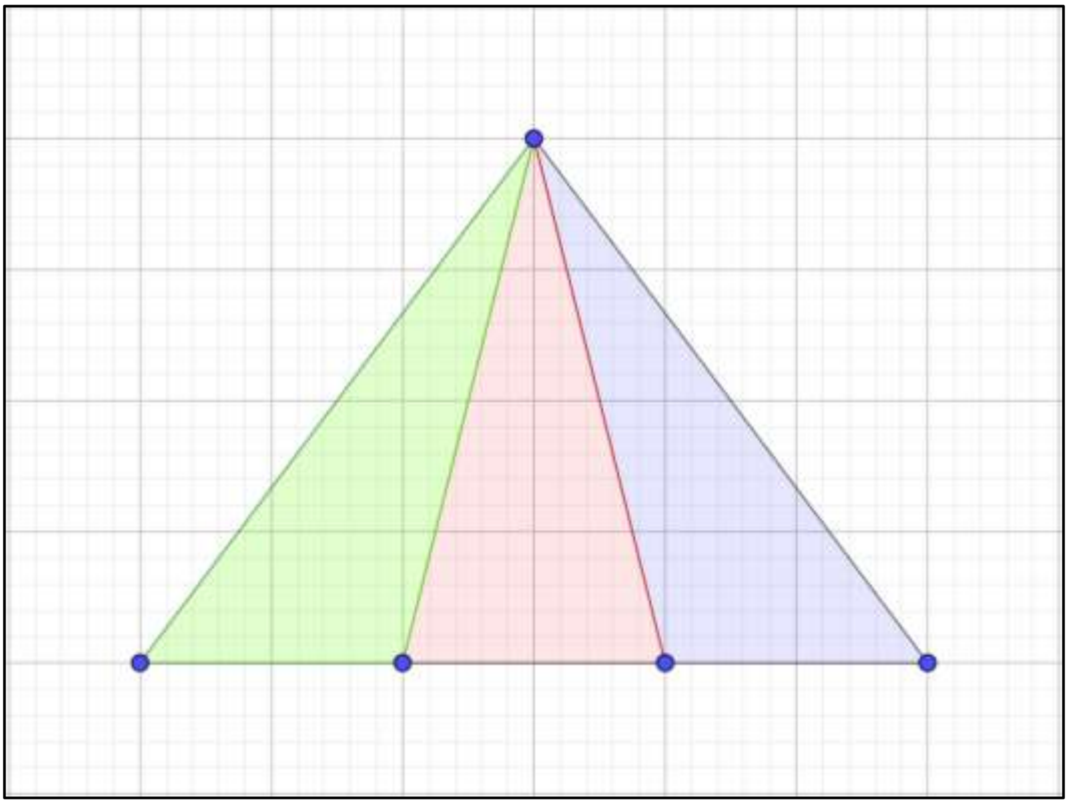
We start with an example:

$$S = \frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \cdots \infty$$

Now, consider a triangle with area A = 1 square unit.

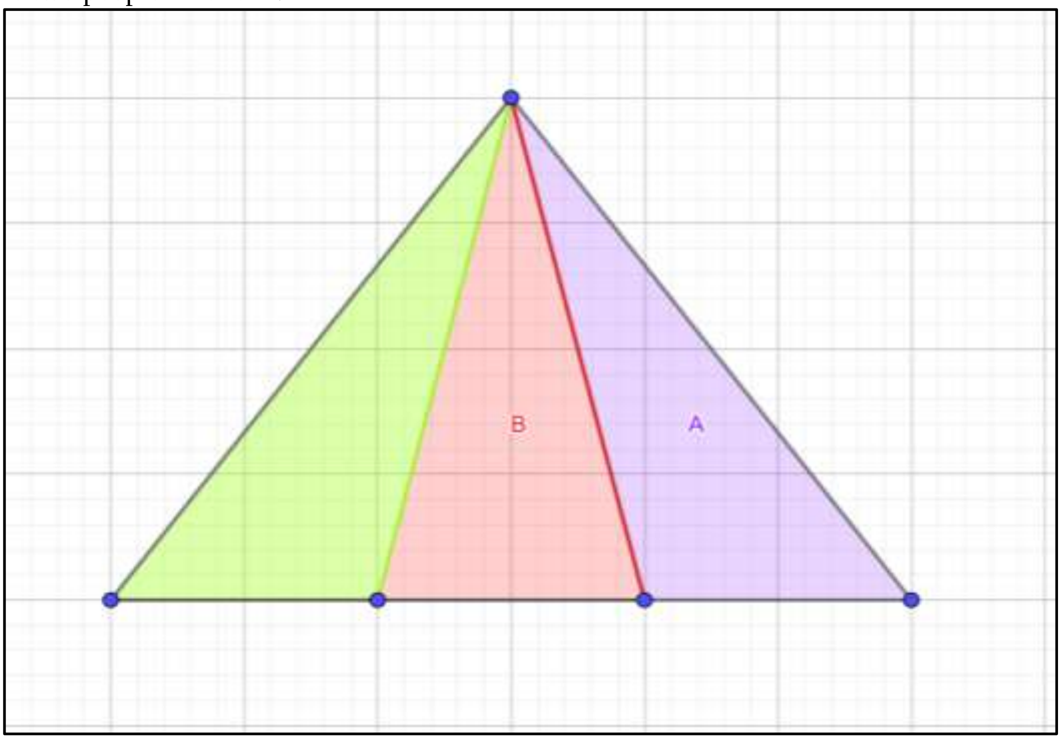


Divide this triangle into 3 equal parts:



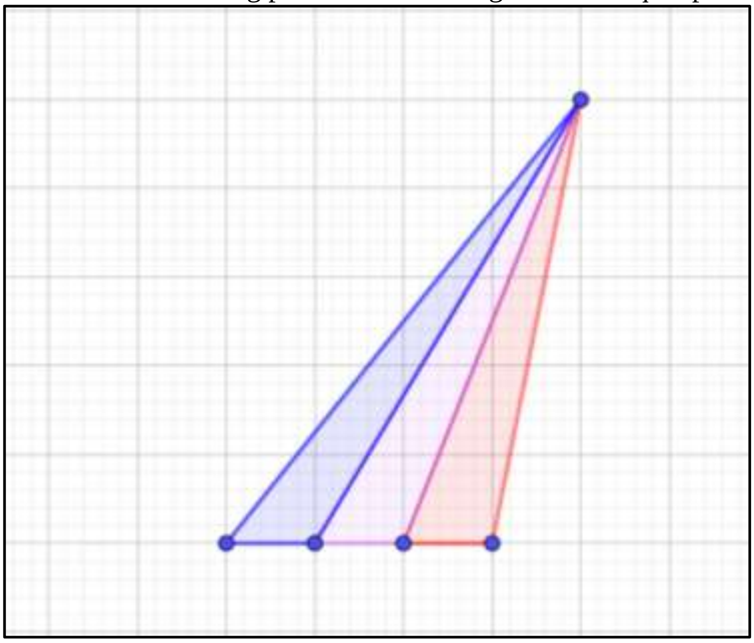
(Note: I have magnified the triangle for greater clarity).

Imagine two people- A and B.



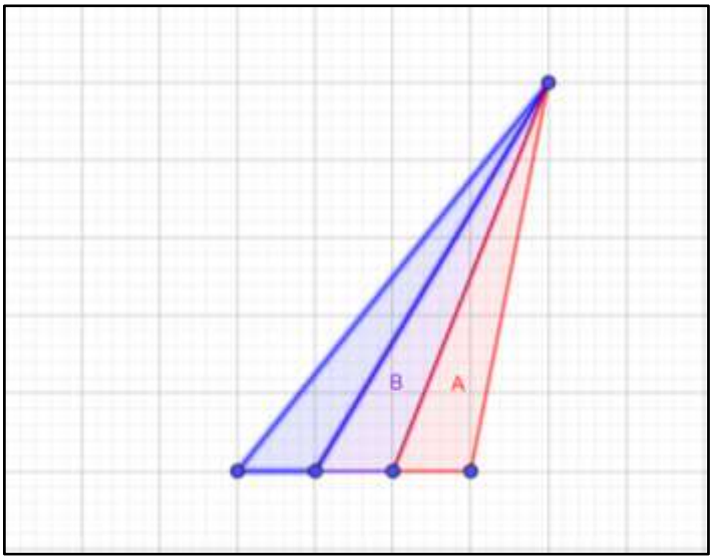
We give one part to A and the other part to B.

We take the remaining part and divide it again into 3 equal parts.



(Note: I have magnified and re divided the green part of the triangle).

Again, we give one part to A and the other part to B.



If we keep doing this successively, up to Infinity, we will have used up the entire triangle.

But, at every step we give equal parts to A and B. Hence, the triangle is divided equally between A and B and both of them receive an equal share. Therefore, the area of triangle received by both A and B will be equal to
1/2 square units.

Applying the traditional formula:

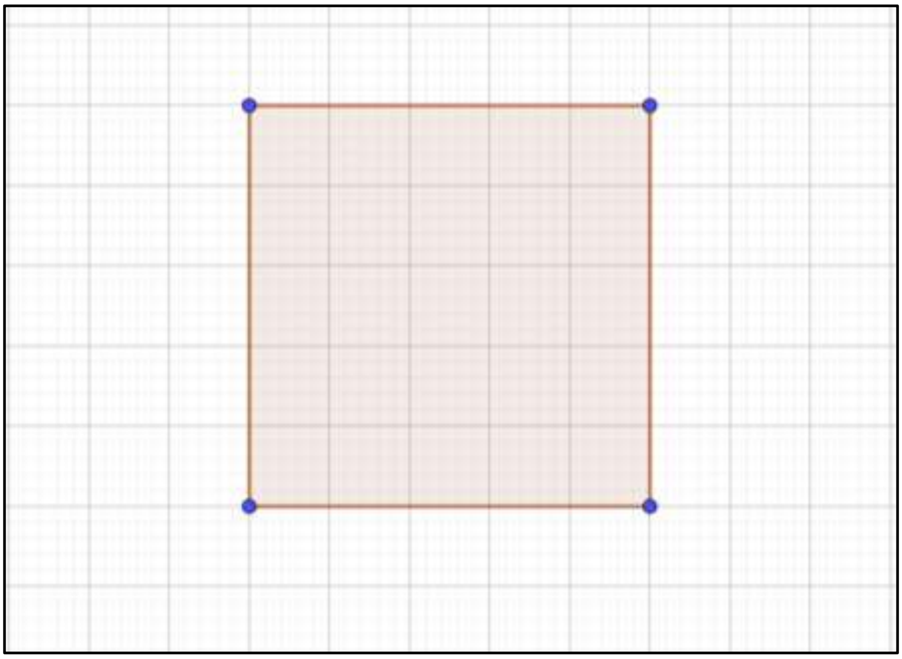
$$S = \frac{a(1 - r^n)}{1 - r}$$

Here,
$a = \frac{1}{3}$, $r = \frac{1}{3}$, $n \to \infty$
Hence,
S= 1/2

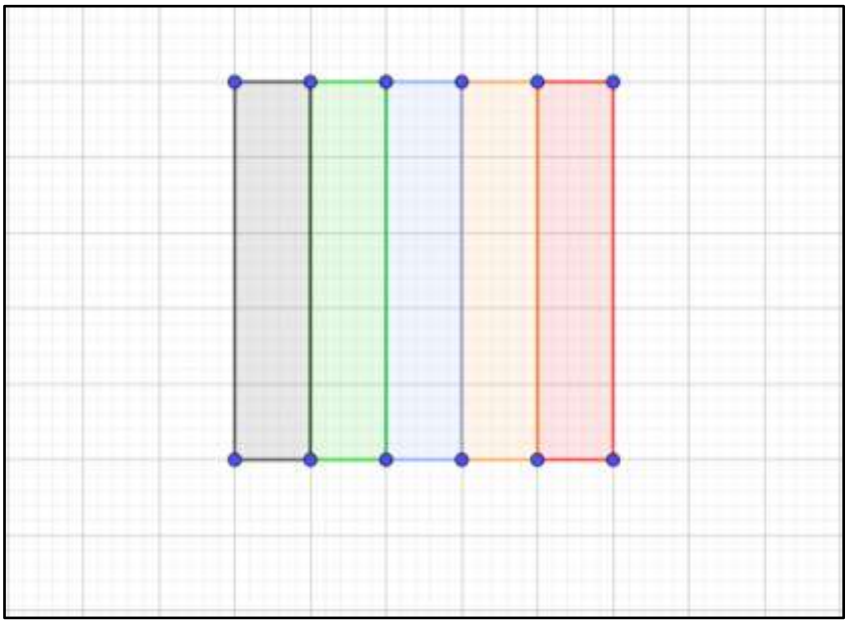
Let us take another example:

$$S = \frac{1}{5} + \frac{1}{5^2} + \frac{1}{5^3} + \cdots \infty$$

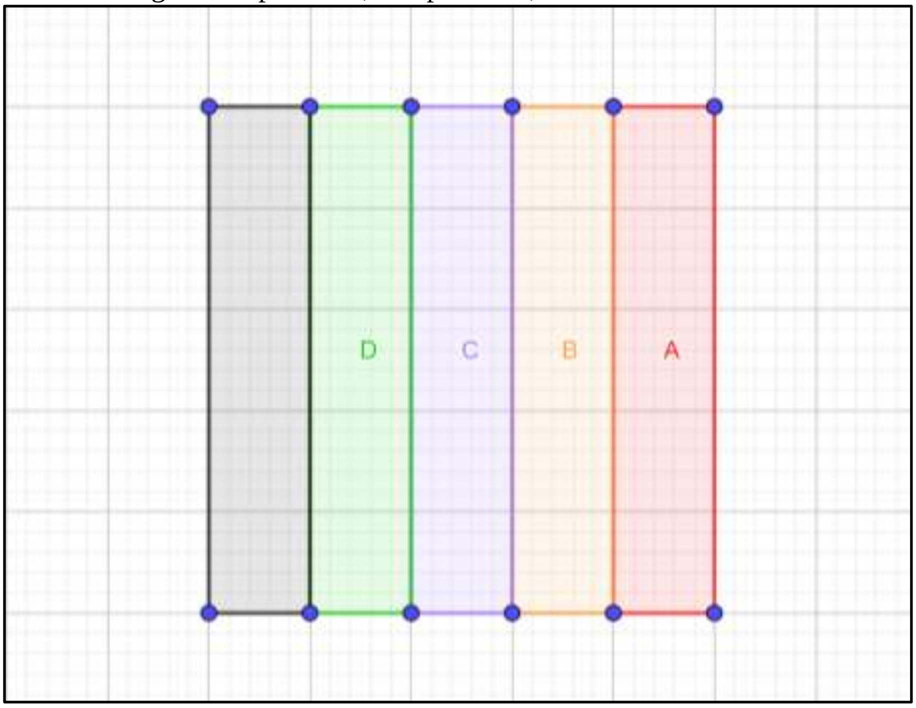
Now, consider (for the sake of variety) a rectangle of area A= 1 square unit.



Divide it into 5 equal parts.

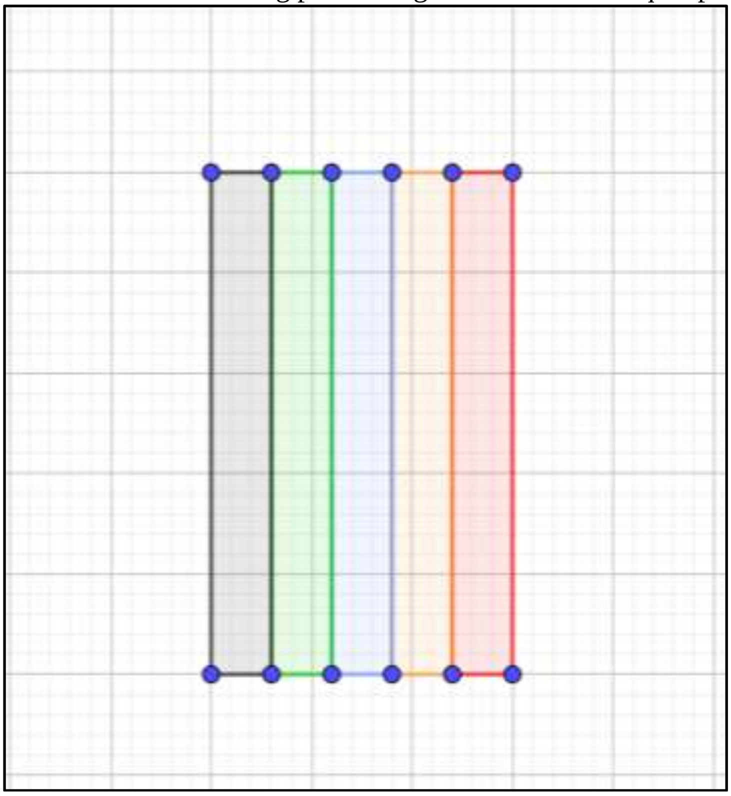


Imagine 4 people A, B,C and D.
We give one part to A, one part to B, one to C and one to D.



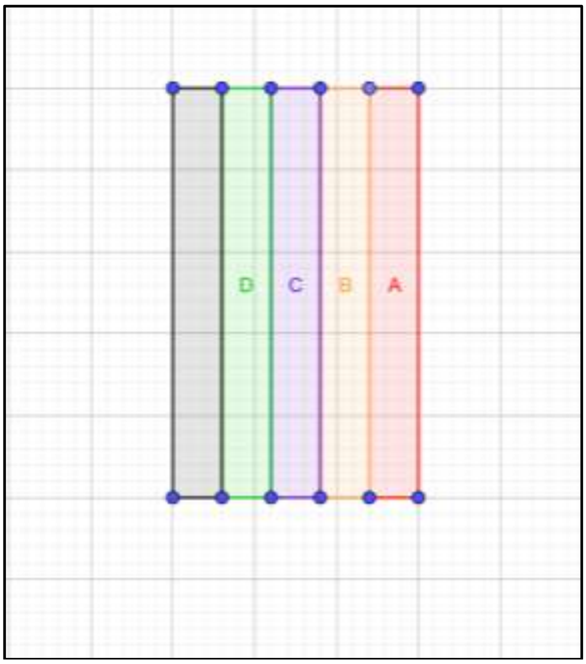
(Note the square has been magnified for clarity).

We now take the remaining part and again divide into 5 equal parts.



(Note: I have magnified and re divided the black part of the square).

Give one part to A, one part to B, one to C and one to D.



If we keep doing this successively up to infinity, we will have used up the entire square.
But, at each step the division between the four people is made equally and each person receives an equal share.
Thus, at the end, the part of the square received by each person is
1/4 square units.

Applying the traditional formula:

$$S = \frac{a(1 - r^n)}{1 - r}$$

Here,
$a = \frac{1}{5}$, $r = \frac{1}{5}$, $n \to \infty$
Hence,
S= 1/4

Explanation
To understand why we are getting the same answer, let us reflect on the concept of division.
Referring to the first example,
When we divide the triangle into 3 parts, we give A 1/3rd of the triangle.
Then when we divide the remaining part again into 3 parts, we get $\frac{1}{3} \times \frac{1}{3} = \frac{1}{3^2}$th of the triangle.
In this way each division represents a term of the series.
Ultimately, the entire triangle is divided equally between two people, each person thus receiving 1/2 of the triangle, which is the value of the series.

Generalisation
From the above statements, we prove that the value of the infinite GP,

$$S = \frac{1}{x} + \frac{1}{x^2} + \frac{1}{x^3} + \cdots \infty$$

, is equal to the share of each person when 1 is divided between (x-1) people equally.
The concept works even if x itself is a fraction. (However, it is difficult to visualise in that case).

II. Conclusion

The above mentioned explores a non-traditional approach to solving infinite series, without using traditional algebra and convergence. It gives us a way of visualising the infinite series and then using that visualisation cleverly to solve the series. This insight of using geometric shapes and elementary concept of division can be applied in other infinite series. Using different non-traditional approaches to prove a theorem has always been an important part of the development and evolution of Mathematics.

Rachit Surana "Infinite Series Proof based on Division and Geometry" IOSR Journal of Mathematics (IOSR-JM) 14.5(2018): 41-47.