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The Modification of Second Derivative Linear Multistep Ordinary Differential Equation for solving stiffly Differential Equation

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Abstract: *This research discussed the modification of second derivative linear multistep method (LMM) using Enright's approach,which focused in solving second order ordinary differential equations (ODEs).The newly constructed method satisfied the basic requirements for the analysis of Linear Multistep methods (LMM). The methods displayed better accuracy when implemented with numerical examples than the existing method with which we compared our results.*

Keywords: *Modification, Stiffly differential equation, LMM, second derivative.*

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I. Introduction

In the early 1950s, as a result of some pioneering work by C. F. Curtiss and J. O. Hirschfelder, (1952), A. M. Stuart and A. R. Humphries, (1996) it was realized that there was an important class of ordinary differential equations (ODEs), which have become known as stiff equations, which presented a severe challenge to numerical methods that existed at that time. Since then an enormous amount of effort has gone into the analysis of stiff problems and, as a result, a great many numerical methods have been proposed for their solution. More recently, however, there have been some strong indications that the theory which underpins stiff computation is now quite well understood, and, in particular, the excellent text of Hairer&Wanner, (1996) has helped put this theory on a firm basis. As a result of this, some powerful codes have now been developed and these can solve quite difficult problems in a routine and reliable way.

The main purpose of this research is to outline some of the important theory behind stiff computation and to direct users of numerical software to those codes which are most likely to be effective for their particular problem. In sciences and engineering, mathematical models are formulated to aid in the understanding of physical phenomena. The formulated model often yields an equation that contains the derivatives of an unknown function. Such an equation is referred to as Differential equation.

Interestingly, differential equations arising from the modeling of physical phenomena often do not have exact solutions. Hence, the development of numerical methods to obtain approximate solutions becomes necessary. To that extent, several numerical methods such as finite difference methods, finite element methods and finite volume methods, among others, have been developed based on the nature and type of the differential equation to be solved.

A differential equation can be classified into Ordinary Differential Equation (ODE), Partial Differential Equation (PDE), Stochastic Differential Equation (SDE), Impulsive Differential Equation (IDE), Delay Differential Equation (DDE), etc. Stuart and Humphries,(1996).

In recent times, the integration of Ordinary Differential Equations (ODEs) is investigated using some kind of block methods. This research discusses the formation of implicit Linear Multistep Method (LMM) for numerical integration of general second order ODEs which arise frequently in the area of science and engineering especially mechanical system, control theory and celestial mechanics, Y. Skwame, J. Sunday and J. Sabo, (2018). In this research, the system of second-order ODEs of the following form

$$y''= f(x, y', y), \quad y(a)= y_0, \quad y'(a)=\eta_0, \quad x\in[a, b] \tag{1.1}$$

shall be considered. Many scholars have been developed numerical methods for solving equation of the form (1.1) directly. These techniques have been introduced in many literature such as Hairer, E., Nørsett, S. P. &Wanner, G., (1987), M. Alkasassbeh and Zurni O, (2017), Y. Skwame, J. Sabo, P. Tumba and T. Y. kyagya, (2017), James, A., Adesanya, A. and Joshua, S., (2013), P. Tumba, J. Sabo and M. Hamadina, (2018), Y.

Skwame, J. Sabo, J. M. Althemai, P. Tumba, (2018)and others. This mathematical modeling is the art of translating problem from an application area into tractable mathematical formulations whose theoretical and numerical analysis provides insight, answers and guidance useful for the originating application Kuboye, J. O. and Omar, Z., (2015).
The aim of this research is to develop a new numerical method for solving systems of second-order stiff ODEs of the form (1.1).

This research is organized as follows: in the coming section, we carried out the derivation of the method, where we considered two-step with a single off-grid points through interpolation and collocation method approach. The details of the analysis of the method which include order, error constant, consistency and zero stability were discussed in Section three. In the fourth section, some numerical problems were solved and the performance of the developed method was compared with those of the existing methods, Y. Skwame; G. M. Kumleng and I. A. Bakari,(2017), Y. Skwame,(2018), Y. Skwame, J. Sunday and J. Sabo,(2018) and J. Sabo, T. Y. Kyagya, A. A. Bumbur, (2018). Finally, the conclusion was drawn in section five.

II. Theoretical Procedure

Reactions in physical systems often transform into system of ODE, which some class of these system are called Stiff system. The numerical methods for obtaining solutions to class of problems are one-step method and Multistep method (MM), Adeniran A. O., Odejide S. A., Ogundare B. S., (2015).The second derivative multistep methods are derived using interpolation and collocation technique as discussed in W. H. Enright, T. E. Hull, (1975),W. H. Enright and J. D. Pryce, (1987),Ehigie J. O., Jator S. N., Sofoluwe A. B., Okunuga S. A., (2014).
Consider the initial value problem of the form

$$y' = f(x, y), \quad y(a) = y_0, \ x \in [a, b] \tag{2.1}$$

The general second derivative formula for solving equation (2.1) using $k - step$ second derivative linear multistep method is of the form

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h \sum_{j=0}^{k} \beta_j f_{n+j} + h^2 \sum_{j=0}^{k} \delta_j g_{n+j} \tag{2.2}$$

Where

$$y_{n+j} = y(x_{n+jh})$$

$$f_{n+j} = f(x_{n+jh}, \quad y(x_{n+jh}))$$

$$g_{n+j} = df(x_n, \quad y(x))$$

$$g_{n+j} = \left.\frac{df(x_n, \quad y(x))}{dx}\right\rangle_{y=y_{n+j}}^{x=y_{n+j}}$$

x_n is a discrete point at $x \quad and \quad \alpha_j, \quad \beta_j, \quad \gamma_j$ are coefficients to be determined. To obtained the method of the form (2.2), $y(x)$ is approximated by a basis polynomial of the form

$$y(x) = \sum_{j=0}^{m} \alpha_j \left(\frac{x - x_n}{h} \right)^j \tag{2.3}$$

equation (2.3) will be used for the derivation of the main and complementary methods for the class of continuous second derivatives multistep method of W. H. Enright, (1974) which is a special case of (2.3). Interpolating $y(x)$ at point x_n, collocating $y'(x)$ at point x_n and collocating $y''(x)$ at point $x = x_n, x_{n+1}, x_{n+\frac{7}{4}}, x_{n+2}$.

$$\left.\begin{aligned} y'(x) &= f_{n+j} \\ y''(x) &= f_{n+j} \end{aligned}\right\rangle j = 0, 1, 2, \ldots, k$$

The system of equations generated are solved to obtained the coefficients of α_j, $j = 0, 1, 2, ..., k+2$ which are used to generate the continuous multistep method of Enright of the form

$$y(x) = y_{n+k-1} + h\sum_{j=0}^{k}\beta_j f_{n+j} + h^2\delta_k g_{n+k} \qquad (2.4)$$

evaluating (2.4) $x = x_{n+k}$ yields the second derivative multistep method of Enright, evaluating at $x = x_{n+j}$, $j = 0, 1, 2, \ldots, k-2$ gives $(k-1)$methods, which will be called complementary methods to complete the k block for the system. The Enright's method so obtained is of the form

$$y_{n+k} = y_{n+k-1} + h\sum_{j=0}^{k}\beta_j f_{n+j} + h^2\delta_k g_{n+k} \qquad (2.5)$$

To derive the continuous second derivative multistep method of Enright, Let the basis function $y(x)$ be

$$y''(x) = \sum_{j=0}^{k}\left(\frac{x-x_n}{h}\right)^j \qquad (2.6)$$

We interpolate (2.6) at point $x = x_n$ collocate $y'(x)$ at point $x = x_n, x_{n+1}, x_{n+\frac{7}{4}}, x_{n+2}$, collocate $y''(x)$ at point $x = x_n, x_{n+1}, x_{n+\frac{7}{4}}, x_{n+2}$, we obtain a system of equation represented in matrix form

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 0 & 1 & \frac{7}{2} & \frac{147}{16} & \frac{343}{16} & \frac{12005}{256} & \frac{50421}{512} & \frac{823543}{4096} & \frac{823543}{2048} \\ 0 & 1 & 4 & 12 & 32 & 80 & 192 & 448 & 1024 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 6 & 12 & 20 & 30 & 42 & 56 \\ 0 & 0 & 2 & \frac{21}{2} & \frac{147}{4} & \frac{1715}{16} & \frac{36015}{128} & \frac{352947}{512} & \frac{823543}{512} \\ 0 & 0 & 2 & 12 & 48 & 160 & 480 & 1344 & 3584 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \\ a_8 \end{pmatrix} = \begin{pmatrix} y_n \\ hf_n \\ hf_{n+1} \\ hf_{n+\frac{7}{4}} \\ hf_{n+2} \\ h^2 f_n \\ h^2 f_{n+1} \\ h^2 f_{n+\frac{7}{4}} \\ h^2 f_{n+2} \end{pmatrix} \qquad (2.7)$$

Applying the Gaussian elimination method on Equation (2.7) gives the coefficient a_i's, *for* $i = 0\,(1)\,9$.
These values are then substituted into Equation (2.6) to give the implicit continuous hybrid method of the form:

$$^j y(x) = \sum_{i=0,} {}^j\alpha_i(x)^j y_{n+i} + h \sum_{i=0,1,\frac{7}{4},2} {}^j\beta_i(x)^j f_{n+i}, \quad j = 1, \cdots, m. \qquad (2.8)$$

Differentiating (2.8) once yields:

$$^j y''(x) = \sum_{i=0,1,\frac{7}{4},2} \frac{d}{dx}{}^j\beta_i(x)^j f_{n+i} + \sum_{i=0}^{2}\frac{d}{dx}{}^j\beta_i(x)^j f_{n+i}, \quad j = 1, \cdots, m. \qquad (2.9)$$

which give the continuous schemes as

$$y(x) = h\left(\beta_0(x)f_n + \beta_1(x)f_{n+1} + \beta_{\frac{7}{4}}(x)f_{n+\frac{7}{4}} + \beta_2(x)f_{n+2}\right) +$$
$$h^2\left(\gamma_0(x)g_n + \gamma_1(x)g_{n+1} + \gamma_{\frac{7}{4}}(x)g_{n+\frac{7}{4}} + \gamma_2(x)g_{n+2}\right) \quad (2.10)$$

Where

$$\alpha_0 = 0$$
$$\beta_0 = th - \frac{1}{115248}t^3h\left(390236 + 657699t - 251724t^2 + 225862t^3 - 51552t^4 + 4872t^5\right)$$
$$\beta_1 = \frac{1}{567}t^3h\left(-6860h + 19257th^2 - 21189t^2 + 11536t^3 - 3120t^4 + 336t^5\right)$$
$$\beta_{\frac{7}{4}} = \frac{4096}{194481}t^3h\left(-3724 + 10227t - 11697t^2 + 6797t^3 - 1983t^4 + 231t^5\right)$$
$$\beta_2 = \frac{1}{336}t^3h\left(31556 - 85701t + 96852t^2 - 55594t^3 + 16032t^4 - 1848t^5\right)$$
$$\gamma_0 = \frac{1}{82320}t^2h^2\begin{pmatrix}41160 - 113680t + 144165t^2 - 101976t^3 + 41510t^4 - 9120t^5 \\ +840t^6\end{pmatrix}$$
$$\gamma_1 = \frac{1}{1890}t^3h^2\left(-13720 + 32340t - 31794t^2 + 15995t^3 - 4080t^4 + 420t^5\right)$$
$$\gamma_{\frac{7}{4}} = \frac{512}{46305}t^3h^2\left(-1960 + 5250t - 5838t^2 + 3290t^3 - 930t^4 + 105t^5\right)$$
$$\gamma_2 = \frac{1}{1680}t^3h^2\left(-13720 + 37485t - 42672t^2 + 24710t^3 - 72000t^4 + 840t^5\right) \quad (2.11)$$

III. Analysis of the Method

In this section, the basic properties of the method derived shall be analyzed.

3.1 Order and error Constants of the Method

Given linear difference operator

$$\ell[y(x),h] = \sum_{j=0}^{k}\left[(a_j y(x+jh) - h\beta_j y'(x+jh)\right]$$

And using Taylor expansion about the point x, we get

$$\ell[y(x),h] = c_0 y(x) + c_1 h y^l(x) + \cdots + c_q h^q y^q(x) + \cdots \quad (3.1)$$

Where

$$c_0 = \alpha_0 + \alpha_1 + \quad \cdots + \alpha_k$$
$$c_1 = \alpha_1 + 2\alpha_2 + \quad \cdots + k\alpha_k - (\beta_0 + \beta_1 \cdots \beta_k)$$
$$\vdots$$
$$c_q = \tfrac{1}{q!}(\alpha_1 + 2^q\alpha_2 + \cdots + k^q\alpha_k) - \tfrac{1}{(q-1)!}(\beta_1 + 2^{(q-1)}\beta_2 + \cdots + k^{(q-1)}\beta_k), \quad q = 2,3,\cdots$$

Such that when $c_0 = c_1 = \cdots = c_p = 0$ *and* $c_{p+1} \neq 0$ of (3.1) then

c_{p+1} is the Error constant, and p is the order of LMM, J. C. Butcher, (2009).

According toLambert, (1973), the order of the new method in Equation (2.11) is obtained by using the Taylor series and it is found that the developed method is of uniform order eight, with an error constants vector given by,

$$C_9 = \left[3.2429\times10^{-6}, \quad 3.3834\times10^{-6}, \quad 3.3856\times10^{-6}\right]^T$$

3.2 Consistency

Definition 3.1: The hybrid block method (2.11) is said to be consistent if it has an order more than or equal to one i.e. $P \geq 1$. Therefore, the method is consistent, E. Suli and D. F. Mayers (2003).

3.3 Zero Stability

Definition 3.2: The hybrid block method (2.11) said to be zero stable if the first characteristic polynomial $\pi(r)$ having roots such that $|r_z| \leq 1 \quad and \quad if \quad |r_z| = 1$, then the multiplicity of r_z must not greater than two, G. Dahlquist, (1956).In order to find the zero-stability of hybrid block method (2.11), we only consider the first characteristic polynomial of the method according to definition (3.2) as follows

$$\prod(r) = \left| r\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right| = r^3(z-1)$$

Which implies $r = 0, 0, 0, 1$. Hence the method is zero-stable since $|r_z| \leq 1 \quad and \quad if \quad |r_z| = 1$.

3.4 Convergence

Theorem (3.1): Consistency and zero stability are sufficient condition for linear multistep method to be convergent. Since the method (2.11) is consistent and zero stable, it implies the method is convergent for all point, C. Baker, G. Monegato, J. Pryce and G. V. Bergh, (2001) and G. Dahlquist (1956).

3.5. Regions of Absolute Stability

Definition 3.3: The Region of Absolute Stability of the new method (2.11) is the set of all points $z \in Z$ such that all roots of characteristic equation are of absolute value less than one, Lambert, (1973). According to G. Dahlquist (1956) and J. D. Lambert, (1991) the absolute stability region of the new method is shown in the figure below.

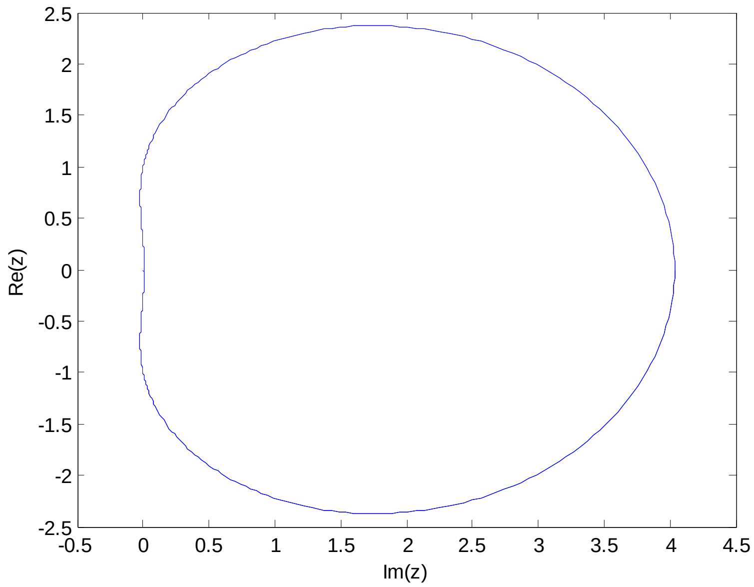

Figure 3.1: The Absolute Stability Region.

IV. The Implementation of Method

In this section, the efficiency and the performance of the method is investigated with three systems ODEs. The problems consideredwill be compare with existing onceand the results obtained in tables 4.1, 4.2 and 4.3 shows the comparison of the results obtained by the new methods with that of the existing method,Y. Skwame; G. M. Kumleng and I. A. Bakari, (2017), Y. Skwame, (2018),Y. Skwame, J. Sunday and J. Sabo, (2018) and J. Sabo, D. Raymond A. A. Bumburand T. Y. Kyagya(2018).

Problem 4.1
Consider the stiffly problem,

$$y_1' = 198y_1 + 199y_2 \quad y_1(0) = 1$$

$$y_2' = -398y_1 - 399y_2 \quad y_2(0) = -1, \quad h = 0.1$$

With Exact Solution

$$y_1(x) = e^{-x}$$

$$y_2(x) = -e^{-x}$$

$$x \in [0,1]$$

Source, Skwame, *et-al* (2017).

Table 4.1: Comparison of results of the proposed method with that of Skwame, *et-al*(2017).

X	*Error in Skwame, , et-al(2017)*				*Errors in New method,*	
	$K = 2, \quad p = 6$		$K = 3, \quad p = 7$		$K = 2, \quad p = 8$	
	$y_1(x)$	$y_2(x)$	$y_1(x)$	$y_2(x)$	$y_1(x)$	$y_2(x)$
0.1	3.61×10^{-7}	3.60×10^{-7}	2.60×10^{-6}	2.60×10^{-6}	2.90×10^{-7}	2.92×10^{-7}
0.2	3.21×10^{-7}	3.30×10^{-7}	2.42×10^{-6}	2.42×10^{-6}	3.59×10^{-8}	7.60×10^{-9}
0.3	6.28×10^{-7}	3.27×10^{-7}	2.18×10^{-6}	2.18×10^{-6}	2.92×10^{-7}	2.94×10^{-7}
0.4	5.65×10^{-7}	5.65×10^{-7}	3.90×10^{-6}	3.90×10^{-6}	8.19×10^{-8}	5.67×10^{-8}
0.5	6.69×10^{-7}	6.68×10^{-7}	3.58×10^{-6}	3.58×10^{-6}	2.89×10^{-7}	2.91×10^{-7}
0.6	6.03×10^{-7}	6.02×10^{-7}	3.23×10^{-6}	3.23×10^{-6}	1.12×10^{-7}	9.11×10^{-7}
0.7	5.92×10^{-7}	5.92×10^{-7}	4.35×10^{-6}	4.35×10^{-6}	2.77×10^{-7}	2.79×10^{-7}
0.8	5.36×10^{-7}	5.37×10^{-7}	3.97×10^{-6}	3.97×10^{-6}	1.28×10^{-7}	1.12×10^{-7}
0.9	7.38×10^{-7}	7.38×10^{-7}	3.59×10^{-6}	3.59×10^{-6}	2.60×10^{-7}	2.61×10^{-7}
1.0	6.70×10^{-7}	6.70×10^{-7}	4.31×10^{-6}	4.30×10^{-6}	1.35×10^{-7}	1.21×10^{-7}

Problem 4.2
Consider the stiffly problem,

$$y_1' = 998y_1 + 1998y_2 \quad y_1(0) = 1$$

$$y_2' = -999y_1 - 1999y_2 \quad y_2(0) = 0, \quad h = 0.1$$

With Exact Solution

$$y_1(x) = 2e^{-x} - e^{-1000x}$$

$$y_2(x) = -e^{-x} - e^{-1000x}$$

$$x \in [0,1]$$

Source, Skwame, *et-al* (2017), (2018) and *et-al* (2018).

Table 4.2: Comparison of results of the proposed method with that of Skwame, *et-al* (2017), (2018) and *et-al* (2018).

X	*Error in Skwame, et-al, (2017)*		*Error in Skwame, (2018)*		*Error in Skwame, et-al, (2018)*		*Error in new Method*	
	$K=2, \ p=6$		$K=1, \ p=8$		$K=2, \ p=10$		$K=2, \ p=8$	
	$y_1(x)$	$y_2(x)$	$y_1(x)$	$y_2(x)$	$y_1(x)$	$y_2(x)$	$y_1(x)$	$y_2(x)$
0.1	2.43×10^{-2}	243×10^{-2}	2.98×10^{-1}	2.98×10^{-1}	3.00×10^{-1}	3.47×10^{-3}	4.38×10^{-2}	4.38×10^{-2}
0.2	3.87×10^{-2}	3.81×10^{-2}	8.90×10^{-2}	8.90×10^{-2}	3.17×10^{-2}	3.17×10^{-2}	1.29×10^{-2}	1.29×10^{-2}
0.3	9.31×10^{-4}	9.85×10^{-4}	265×10^{-2}	2.65×10^{-2}	1.67×10^{-4}	1.13×10^{-4}	5.73×10^{-4}	5.69×10^{-4}
0.4	1.51×10^{-5}	1.51×10^{-3}	7.91×10^{-3}	7.91×10^{-3}	1.01×10^{-3}	1.01×10^{-5}	1.62×10^{-4}	1.65×10^{-4}
0.5	2.32×10^{-5}	2.20×10^{-5}	2.35×10^{-3}	2.36×10^{-3}	1.17×10^{-5}	7.63×10^{-6}	1.13×10^{-5}	9.29×10^{-6}
0.6	6.99×10^{-5}	7.14×10^{-5}	6.97×10^{-4}	7.01×10^{-4}	3.91×10^{-5}	3.56×10^{-5}	3.74×10^{-6}	8.23×10^{-7}
0.7	2.15×10^{-5}	122×10^{-5}	203×10^{-4}	2.06×10^{-4}	1.94×10^{-4}	4.58×10^{-6}	1.60×10^{-6}	8.45×10^{-7}
0.8	2.34×10^{-5}	146×10^{-5}	5.52×10^{-5}	5.89×10^{-5}	8.92×10^{-6}	4.73×10^{-2}	6.38×10^{-6}	0
0.9	2.17×10^{-5}	160×10^{-5}	1.11×10^{-5}	1.49×10^{-5}	2.04×10^{-6}	4.54×10^{-6}	1.99×10^{-7}	9.95×10^{-8}
1.0	1.97×10^{-5}	148×10^{-5}	210×10^{-6}	1.74×10^{-6}	1.02×10^{-5}	4.16×10^{-6}	6.54×10^{-6}	3.29×10^{-6}

Problem 4.3
Consider the stiffly problem,

$$y_1^{'} = -100y_1 + 9.901y_2; \quad y_1(0) = 1$$

$$y_2^{'} = 0.1y_1 - y_2; \quad y_2(0) = 10, \quad h = 0.1$$

With Exact Solution

$$y_1(x) = e^{-0.99x}$$

$$y_2(x) = 10e^{-0.99x}$$

$$x \in [0, 1]$$

Source, Sabo, *et-a.*, *(2018).*

Table 4.3: Comparison of results of the proposed method with that of Sabo, *et-al.*, (2018)

X	*Error in Sabo, et-a., (2018)*		*Error in new Method*	
	$K=1, \quad p=6$		$K=2, \quad p=8$	
	$y_1(x)$	$y_2(x)$	$y_1(x)$	$y_2(x)$
0.1	5.00×10^{-10}	4.00×10^{-9}	9.60×10^{-9}	5.00×10^{-9}
0.2	1.50×10^{-9}	7.00×10^{-9}	1.87×10^{-8}	0
0.3	1.50×10^{-9}	9.00×10^{-9}	7.50×10^{-9}	5.00×10^{-9}
0.4	2.20×10^{-9}	1.30×10^{-8}	1.55×10^{-8}	3.00×10^{-9}
0.5	2.20×10^{-9}	1.50×10^{-8}	6.40×10^{-9}	2.20×10^{-9}
0.6	2.30×10^{-9}	1.30×10^{-8}	1.22×10^{-8}	2.00×10^{-9}
0.7	1.30×10^{-9}	1.20×10^{-8}	5.10×10^{-9}	2.00×10^{-9}
0.8	2.10×10^{-8}	1.40×10^{-8}	1.00×10^{-8}	2.00×10^{-9}
0.9	1.70×10^{-9}	1.50×10^{-8}	4.20×10^{-9}	2.00×10^{-9}
1.0	2.40×10^{-9}	1.70×10^{-8}	8.10×10^{-9}	3.00×10^{-9}

V. Conclusion

In this research, we discussed the modification of second derivative linear multistep method (LMM) using Enright method which focusing in solving second order ordinary differential equations (ODEs).The analysis of the method was studied and it was found to be consistent, convergent and zero-stable, with the region of absolute stable within which the method is stable. The newly constructed Enright method was applied to solve three systems of second-order stiffly ordinary differential equations and from the results obtained; it is obvious that the developed method performed better than the existing method. This study concluded that, it has been shown in many literatures; the multistep method is very effective method for solving nonlinear stiff ODEs either initial value problems or boundary value problems. Therefore the general solution of second order Linear Multistep Method (LMM) is a convenient technique for determining the solutions of mathematical modeling since it can approximate the result even though the efficiency is less than the other multistep method.

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