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GCD Properties of the Altered Pell And Pell Lucas Numbers

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Abstract : *In this study, we establish new sequences that are obtained by altering the Pell and Pell Lucas sequences. Unlike other altered sequences in the literature, these new altered sequences depend on two integer parameters. Further, the greatest common divisors properties of these altered sequences are investigated.*
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I. Introduction

The Pell $\{P_n\}_{n\geq 0}$ and Pell Lucas $\{Q_n\}_{n\geq 0}$ sequences are generated with initial conditions $P_0 = 0$, $P_1 = 1$ and $Q_0 = 2$, $Q_1 = 2$ by using the recurrence formula X_n as

$$X_n = 2X_{n-1} + X_{n-2}, n \geq 2 . \quad (1)$$

Also, in [3], it is known that there is a sequence $\{q_n\}_{n\geq 0}$ holding identities $Q_n = 2q_n$ and $q_n = P_n + P_{n-1}$, the numbers q_n can be defined with initial conditions $q_0 = q_1 = 1$ by recurrence formula X_n in (1). The Pell and Pell Lucas numbers are extended to negative subscripts:

$$P_{-n} = (-1)^{n-1} P_n \text{ and } Q_{-n} = (-1)^n Q_n .$$

In the studies [2] and [4], the authors noted that the successive Fibonacci numbers are relatively prime, $\gcd(F_n, F_{n+1}) = (F_n, F_{n+1}) = 1$, where $\gcd(a,b) = (a,b)$ represent the greatest common divisor of a and b, but the altered Fibonacci numbers, $G_n = F_n + (-1)^n$ and $H_n = F_n - (-1)^n$, are not relatively prime for the theirs successive terms; such that $(G_{4n}, G_{4n+1}) = L_{2n+1}$, $(G_{4n+2}, G_{4n+3}) = F_{2n+2}$, $(H_{4n}, H_{4n+1}) = F_{2n+1}$ and $(H_{4n+2}, H_{4n+3}) = L_{2n+2}$. On the other hand, in [1], K. Chen defined the sequence $\{f_n(a)\}_{n\geq 0} = \{\gcd(F_n + a, F_{n+1} + a)\}_{n\geq 0}$, proved that $\{f_n(a)\}_{n\geq 0}$ is bounded if $a \neq \pm 1$. The author also derived the greatest common divisor sequences $\{\gcd(L_{4n+k-1} + 1, L_{4n+k} + 1)\}_{n\geq 0}$, $k = 0,1,2,3$ have got some values for any integer n, and mentioned that the sequences $(L_{4n+k-1} + 1, L_{4n+k} + 1)$, $k = 0,1,2,3$ are the constant sequences.

It is well known that the Pell numbers have gcd property $(P_m, P_n) = P_{(m,n)}$, and let $m = 2^a m'$, $n = 2^b n'$, with m' and n' odd integers and $a, b \geq 0$. Then, $(Q_m, Q_n) = Q_{(m,n)}$ if $a = b$ and it is 2 otherwise, and also $(P_m, Q_n) = Q_{(m,n)}$ if $a > b$, its value is 1 or 2 if $a \leq b$ [5]. Any two successive terms of the Pell sequence are coprime, but this value is 2 for the Pell Lucas sequences.

Our aim in this study is to define several altered Pell and Pell Lucas sequences which depends not only on one integer parameter, and is defined in (2) and (3). Now, we consider the sequences $\{P_n + (-1)^n a\}$ and $\{P_n + (-1)^n b\}$ as altered Pell sequences, and also the sequences $\{Q_n + (-1)^n c\}$ and $\{Q_n + (-1)^n d\}$ as altered Pell Lucas sequences according to whether n is even or not:

Definition 1. The altered Pell sequences $\{E_n\}_{n\geq1}$ and $\{K_n\}_{n\geq1}$ are defined by

$$E_n=\begin{cases}P_n+2, if\ n\,is\,even\\ P_n-1,\ otherwise\end{cases} \text{ and } K_n=\begin{cases}P_n-2, if\ n\,is\,even\\ P_n+1,\ otherwise\end{cases} \quad (2)$$

where P_n is the n^{th} Pell number, respectively.

Definition 2. The altered Pell Lucas sequences $\{E'_n\}_{n\geq1}$ and $\{K'_n\}_{n\geq1}$ are defined by

$$E'_n=\begin{cases}Q_n+6, if\ n\,is\,even\\ Q_n-2,\ otherwise\end{cases} \text{ and } K'_n=\begin{cases}Q_n-6, if\ n\,is\,even\\ Q_n+2,\ otherwise\end{cases} \quad (3)$$

where Q_n is the n^{th} Pell Lucas number, respectively.

II. The Altered Pell Sequences

In this section, some results concerning the greatest common divisors of the altered Pell sequences, $\{E_n\}_{n\geq1}$ and $\{K_n\}_{n\geq1}$, are established. Now, some interesting observations in the following table can be made;

n	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*	*11*	*12*
E_n	*0*	*4*	*4*	*14*	*28*	*72*	*168*	*410*	*984*	*2380*	*5740*	*13862*

It is well known that every even numbered Pell number has even parity. Due to definition of the number E_n and Pell recurrence relation, the number E_n has even parity. Hence, any two successive terms of the altered Pell sequence are not coprime, i.e., $(E_n, E_{n+1})\neq1$(see Table 1).

n	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*	*11*	*12*	*13*
(E_n,E_{n+1})	*4*	*4*	*2*	*14*	*4*	*24*	*2*	*82*	*4*	*140*	*2*	*478*	*4*

Table 1

It is seen that (E_{2k},E_{2k+1}) are 2 times the $(k+1)^{th}$ Pell numbers $2P_{k+1}$ for $k=1,3,5,...$ entries, and are the $(k+1)^{th}$ Pell-Lucas numbers Q_{k+1} for $k=2,4,6,...$ entries. Also, (E_{2k+1},E_{2k+2}) are 2 for $k=1,3,5,...$ entries, and are 4 for $k=2,4,6,...$ entries. In addition they can be rewritten $(E_{4k-2},E_{4k-1})=2P_{2k}$, $(E_{4k},E_{4k+1})=Q_{2k+1}=2q_{2k+1}$, $(E_{4k-1},E_{4k})=2$ and $(E_{4k+1},E_{4k+2})=4$ for $k\geq1$. In order to prove our results, a few lemmas are given below.

Lemma 1. In [3], for the Pell and Pell Lucas numbers, the next two identities are valid;

$$P_{m+n}+P_{m-n}=\begin{cases}P_mQ_n\ if\ \ n\ is\ even\\ P_nQ_m\ \ otherwise\end{cases}, \quad (4)$$

$$P_{m+n}-P_{m-n}=\begin{cases}P_nQ_m\ if\ \ n\ is\ even\\ P_mQ_n\ \ otherwise\end{cases}. \quad (5)$$

Lemma 2. In [3], for any integers a, b and c,

$$(a,b)=(a,b+ac). \quad (6)$$

Lemma 3. For any two integers m and n,

$$(P_n-P_m-2P_{m-1},P_{n+1}+P_{m-1}+2P_{m-2})=(P_{n-2}-P_{m+2}-2P_{m+1},P_{n-1}+P_{m+1}+2P_m). \quad (7)$$

Proof By using (1) for the number P_n and (6), we have

$$(P_n-P_m-2P_{m-1},P_{n+1}+P_{m-1}+2P_{m-2})=(P_n-P_m-2P_{m-1},P_{n+1}-2P_n+P_{m-1}+2P_m+2P_{m-2}+4P_{m-1})$$
$$=(P_n-2P_{n-1}-P_m-P_{m+1}-2P_{m-1}-4P_m,P_{n-1}+P_{m+1}+2P_m).$$

Lemma 4. For any two integers m and n,

$$(P_n-1,P_{n+1}+2)=(P_{n-2m}-P_{2m+1}-2P_{2m},P_{n-2m+1}+P_{2m}+2P_{2m-1}), \quad (8)$$

$$(P_n+1,P_{n+1}-2)=(P_{n-2m}+P_{2m+1}+2P_{2m},P_{n-2m+1}-P_{2m}-2P_{2m-1}). \quad (9)$$

Proof By using (6), we simplify alike

$$(P_n-1,P_{n+1}+2)=(P_n-P_1,P_{n+1}-2P_n+P_2+2P_1)$$
$$=(P_{n-2}-2P_2-5P_1,P_{n-1}+P_2+2P_1).$$

Since $P_{-1}=1$, $P_0=0$ and $P_1=1$, we rewrite

$$\left(P_n-1,P_{n+1}+2\right)=\left(P_n-P_1-2P_0,P_{n+1}+P_0+2P_{-1}\right),$$

and applying m times of (7) to $\left(P_n-1,P_{n+1}+2\right)$ gives desired result. Proof of (9) are made with similar above mentioned way.

Theorem 1. Let E_n, $n\geq 1$ be the altered Pell numbers given in (2), then

$$\left(E_{2k},E_{2k+1}\right)=\begin{cases}2P_{k+1}, & \textit{for odd } k\\ Q_{k+1}, & \textit{for even } k\end{cases} \text{ and } \left(E_{2k+1},E_{2k+2}\right)=\begin{cases}2, & \textit{for odd } k\\ 4, & \textit{for even } k\end{cases}.$$

Proof From the Lemma 1, substitute $k+1$ for m and $k-1$ for n in (4) for E_{2k}, and for E_{2k+1}, $k+1$ for m and k for n in (5), then we rewrite as

$$P_{(k+1)+(k-1)}+P_{(k+1)-(k-1)}=\begin{cases}P_{k+1}Q_{k-1} & \textit{if } k-1 \textit{ is even}\\ P_{k-1}Q_{k+1} & \textit{otherwise}\end{cases},$$

$$P_{(k+1)+k}-P_{(k+1)-k}=\begin{cases}P_kQ_{k+1} & \textit{if } k \textit{ is even}\\ P_{k+1}Q_k & \textit{otherwise}\end{cases}.$$

The first equation is valid from $\left(P_k,P_{k-1}\right)=1$ and $\left(Q_k,Q_{k-1}\right)=2$.

Since $\left(E_{2k+1},E_{2k+2}\right)=\left(P_{2k+1}-1,P_{2k+2}+2\right)$, let $n=2k+1$ and $m=\frac{k}{2}$ for k even in (8):

$$\begin{aligned}\left(P_{2k+1}-1,P_{2k+2}+2\right)&=\left(-2P_k,P_{k+2}+P_k+2P_{k-1}\right)\\&=\left(-2P_k,Q_{k+1}+2P_{k-1}\right)\\&=2\left(-P_k,q_{k+1}+P_{k-1}\right).\end{aligned}$$

Due to $\left(-P_k,q_{k+1}+P_{k-1}\right)=\left(-P_k,q_{k+1}+P_{k-1}+2P_k\right)$ for $P_{k+1}+q_{k+1}=P_{k+2}$, it gives

$$\begin{aligned}\left(P_{2k+1}-1,P_{2k+2}+2\right)&=2\left(-P_k,P_{k+2}\right)\\&=2\left(-P_k,2P_{k+1}+P_k\right)\\&=2\left(-P_k,2P_{k+1}\right).\end{aligned}$$

If k is even, then $\left(P_k,2\right)=2$, it follows $\left(E_{2k+1},E_{2k+2}\right)=4$.

Let $n=2k+1$ and $m=\frac{k+1}{2}$ for k odd in (8), we obtain

$$\begin{aligned}\left(P_{2k+1}-1,P_{2k+1+1}+2\right)&=\left(-4P_{k+1},Q_{k+1}\right)\\&=2\left(-2P_{k+1},q_{k+1}\right).\end{aligned}$$

Since $\left(2,q_{k+1}\right)=1$ and $\left(-P_{k+1},q_{k+1}\right)=1$, it completes the proof of the Theorem 1.

Theorem 2. The following identities are valid;

$$\begin{gathered}P_{4k}+2=P_{2k-1}Q_{2k+1},\quad P_{4k}-2=P_{2k+1}Q_{2k-1},\\ P_{4k+1}+1=P_{2k+1}Q_{2k},\quad P_{4k+1}-1=P_{2k}Q_{2k+1},\\ P_{4k+2}+2=P_{2k+2}Q_{2k},\quad P_{4k+2}-2=P_{2k}Q_{2k+2},\\ P_{4k+3}+1=P_{2k+1}Q_{2k+2},\quad P_{4k+3}-1=P_{2k+2}Q_{2k+1}.\end{gathered}$$

Proof From the Lemma 1, substitute firstly, $2k+1$ for m and $2k$ for n, and secondly, $2k+1$ for m and $2k-1$ for n in (4), then

$$P_{(2k+1)+2k}+P_{(2k+1)-2k}=P_{4k+1}+1=P_{2k+1}Q_{2k},$$

$$P_{(2k+1)+(2k-1)}+P_2=P_{4k}+2=P_{2k-1}Q_{2k+1}.$$

Also, similar applications of values given for m and n give the remaining other identities.

Theorem 3. Let E_n be the altered Pell number given in (2), then

$$\begin{gathered}\left(E_{4k},E_{4k+1}\right)=Q_{2k+1},\quad \left(E_{4k+1},E_{4k+2}\right)=4,\\ \left(E_{4k+2},E_{4k+3}\right)=2P_{2k+2},\quad \left(E_{4k-1},E_{4k}\right)=2.\end{gathered}$$

Proof From the Theorem 2, and the number E_n given in (2), we write

$$\left(E_{4k},E_{4k+1}\right)=\left(P_{4k}+2,P_{4k+1}-1\right)=\left(P_{2k-1}Q_{2k+1},P_{2k}Q_{2k+1}\right)=Q_{2k+1}\left(P_{2k-1},P_{2k}\right),$$

$$\left(E_{4k+2},E_{4k+3}\right)=\left(P_{4k+2}+2,P_{4k+3}-1\right)=\left(P_{2k+2}Q_{2k},P_{2k+2}Q_{2k+1}\right)=P_{2k+2}\left(Q_{2k},Q_{2k+1}\right).$$

Now, substitute $n=4k+1$ and $m=k$ in (8) for $(E_{4k+1},E_{4k+2})=(P_{4k+1}-1,P_{4k+2}+2)$, then

$$\begin{aligned}(P_{4k+1}-1,P_{4k+2}+2)&=(-2P_{2k},P_{2k+2}+P_{2k}+2P_{2k-1})\\&=2(-P_{2k},q_{2k+1}+P_{2k-1}+P_{2k})\\&=2(-P_{2k},q_{2k+1}+q_{2k})\\&=2(-P_{2k},2P_{2k+1}).\end{aligned}$$

Because $(P_{2k},2)=2$ and $(P_{2k},P_{2k+1})=1$ for integer k, it gives $(E_{4k+1},E_{4k+2})=4$.

Since $(E_{4k-1},E_{4k})=(P_{4k-1}-1,P_{4k}+2)$, applying $n=4k-1$ and $m=k$ in (8):

$$\begin{aligned}(P_{4k-1}-1,P_{4k}+2)&=(-4P_{2k},2P_{2k}+2P_{2k-1})\\&=2(-2P_{2k},q_{2k}).\end{aligned}$$

It is obtained that $(E_{4k-1},E_{4k})=2$ for $(2,q_{2k})=1$ and $(P_{2k},q_{2k})=1$.

Let us make an investigation on the altered Pell numbers K_n defined by (2) in the following table

n	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*	*11*	*12*
K_n	*2*	*0*	*6*	*10*	*30*	*68*	*170*	*406*	*986*	*2376*	*5742*	*13858*

Notice that there are some constant sequences $(K_1,K_2)=2$, $(K_3,K_4)=2$, $(K_5,K_6)=2$, ... Then, $(K_{2k-1},K_{2k})=2$ for $k=1,2,3,...$. Since it is seen that the $(K_{2k},K_{2k+1})=Q_{k+1}$ for $k=1,3,5,...$ and $(K_{2k},K_{2k+1})=2P_{k+1}$ for $k=2,4,6,...$. In addition they can be rewritten $(K_{4k-2},K_{4k-1})=Q_{2k}=2q_{2k}$ and $(K_{4k},K_{4k+1})=2P_{2k+1}$, $k\geq 1$.

Theorem 4. Let K_n, $n\geq 1$ be an altered Pell number given in (2), then

$$(K_{2k},K_{2k+1})=\begin{cases}2P_{k+1}, & \text{for even } k\\ Q_{k+1}, & \text{for odd } k\end{cases} \text{ and } (K_{2k+1},K_{2k+2})=2.$$

Proof To obtain K_{2k}, and K_{2k+1}, substitute $k+1$ for m and k for n in (4) and $k+1$ for m and $k-1$ for n in (5), we rewrite as the first equation,

$$P_{(k+1)+k}+P_{(k+1)-k}=\begin{cases}P_{k+1}Q_k & \text{if } k \text{ is even}\\ P_kQ_{k+1} & \text{otherwise}\end{cases},$$

$$P_{(k+1)+(k-1)}-P_{(k+1)-(k-1)}=\begin{cases}P_{k-1}Q_{k+1} & \text{if } k-1 \text{ is even}\\ P_{k+1}Q_{k-1} & \text{otherwise}\end{cases}.$$

Since $(K_{2k+1},K_{2k+2})=(P_{2k+1}+1,P_{2k+2}-2)$, let $n=2k+1$ and $m=\frac{k}{2}$ for k even or $m=\frac{k+1}{2}$ for k odd in (9), we have desired result with exactly the same way in the proof of the Theorem 1.

Theorem 5. Let. K_n, $n\geq 1$ be the altered Pell number given in (2), then

$$(K_{4k},K_{4k+1})=2P_{2k+1},\ (K_{4k+1},K_{4k+2})=2,$$
$$(K_{4k+2},K_{4k+3})=Q_{2k+2},\ (K_{4k-1},K_{4k})=2.$$

Proof By using the Theorem 2 and the number K_n given in (2), we have $(K_{4k},K_{4k+1})=2P_{2k+1}$ and $(K_{4k+2},K_{4k+3})=Q_{2k+2}$ with the same way in the proof of the Theorem 3.

Also, since $(K_{4k+1},K_{4k+2})=(P_{4k+1}+1,P_{4k+2}-2)$, let $n=4k+1$ and $m=k$ in (9), and since $(K_{4k-1},K_{4k})=(P_{4k-1}+1,P_{4k}-2)$, let $n=4k-1$ and $m=k$ in (9), they are valid.

Now, we consider the numbers $P^{+}_{n,3}=(E_n,E_{n+3})$ and $P^{-}_{n,3}=(K_n,K_{n+3})$, and generalize some classes of the altered Pell numbers

$$P^{+}_{2k,3}=(P_{2k}+2,P_{2k+3}-1) \text{ and } P^{+}_{2k+1,3}=(P_{2k+1}-1,P_{2k+4}+2), \tag{10}$$

$$P^{-}_{2k,3}=(P_{2k}-2,P_{2k+3}+1) \text{ and } P^{-}_{2k+1,3}=(P_{2k+1}+1,P_{2k+4}-2), \tag{11}$$

where the P_n is the n^{th} Pell number.

Theorem 6. The next four equations are valid;

$$P^{+}_{4k,3}=\left(E_{4k},E_{4k+3}\right)=\begin{cases}5Q_{2k+1}, k\equiv 2\bmod(3)\\ Q_{2k+1}, \quad otherwise\end{cases},$$

$$P^{+}_{4k+2,3}=\left(E_{4k+2},E_{4k+5}\right)=2P_{2k+2},$$

$$P^{-}_{4k,3}=\left(K_{4k},K_{4k+3}\right)=2P_{2k+1},$$

$$P^{-}_{4k+2,3}=\left(K_{4k+2},K_{4k+5}\right)=\begin{cases}5Q_{2k+2}, k\equiv 0\bmod(3)\\ Q_{2k+2}, \quad otherwise\end{cases},$$

where the P_n and Q_n are the n^{th} Pell and Pell Lucas numbers, respectively.

Proof By using the Theorem 2 and $\left(E_{4k},E_{4k+3}\right)=\left(P_{4k}+2,P_{4k+3}-1\right)$ from the definition of the E_n given in (10), we have

$$\left(P_{4k}+2,P_{4k+3}-1\right)=Q_{2k+1}\left(P_{2k+2},P_{2k-1}\right).$$

Since $\left(P_{qn+r},P_n\right)=\left(P_n,P_r\right)$ for integers q, r and n, it gives

$$\left(E_{4k},E_{4k+3}\right)=Q_{2k+1}\left(P_{2k-1},P_3\right),$$

$$P_{(2k-1,3)}=\begin{cases}P_3, k\equiv 2\bmod(3)\\ P_1, \quad otherwise\end{cases}.$$

Using $\left(E_{4k+2},E_{4k+5}\right)=\left(P_{4k+2}+2,P_{4k+5}-1\right)$ and the Theorem 2 gives

$$\left(P_{4k+2}+2,P_{4k+5}-1\right)=P_{2k+2}\left(Q_{2k},Q_{2k+1}\right).$$

Applying the Theorem 2 and $\left(K_{4k},K_{4k+3}\right)=\left(P_{4k}-2,P_{4k+3}+1\right)$ from the definition of the K_n given in (11), we obtain

$$\begin{aligned}\left(P_{4k}-2,P_{4k+3}+1\right)&=P_{2k+1}\left(Q_{2k-1},Q_{2k+2}\right)\\ &=2P_{2k+1}\left(q_{2k-1},5q_{2k}\right).\end{aligned}$$

Due to $\left(q_{2k-1},q_{2k}\right)=1$ and $\left(q_{2k-1},P_3\right)=1$ for integer k, it gives $\left(K_{4k},K_{4k+3}\right)=2P_{2k+1}$.

By using $\left(K_{4k+2},K_{4k+5}\right)=\left(P_{4k+2}-2,P_{4k+5}+1\right)$ and the Theorem 2, we write

$$\begin{aligned}\left(P_{4k+2}-2,P_{4k+5}+1\right)&=Q_{2k+2}\left(P_{2k},P_{2k+3}\right)\\ &=Q_{2k+2}\left(P_{2k},P_3\right),\end{aligned}$$

$$P_{(2k,3)}=\begin{cases}P_3, k\equiv 0\bmod(3)\\ P_1, \quad otherwise\end{cases}.$$

III. The Altered Pell Lucas Sequences

We make some observations the numbers E'_n defined with equation (3) in the Table 2

n	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*	*11*	*12*
E'_n	*0*	*12*	*12*	*40*	*80*	*204*	*476*	*1160*	*2784*	*6732*	*16236*	*39208*
(E'_n,E'_{n+1})	12	$2Q_2$	4	$8P_3$	4	$2Q_4$	4	$8P_5$	12	$2Q_6$	4	$8P_7$

Table 2

Any two successive terms of the altered Pell Lucas numbers E'_n are not coprime. It is seen that $(E'_{2k},E'_{2k+1})=2Q_{k+1}$ for $k=1,3,5,...$ and $(E'_{2k},E'_{2k+1})=8P_{k+1}$ for $k=2,4,6,...$. The other entries are noticed that $(E'_{2k-1},E'_{2k})=12$ if $k\equiv 1\left(\bmod 4\right)$ and otherwise, $(E'_{2k-1},E'_{2k})=4$ for $k\in\left\{\bar{0},\bar{2},\bar{3}\right\}$, where $\bar{x}$ denotes the equivalence class of x modulo 4. In addition they can be rewritten $\left(E'_{4k-2},E'_{4k-1}\right)=2Q_{2k}=4q_{2k}$ and $\left(E'_{4k},E'_{4k+1}\right)=8P_{2k+1}$, $k\geq 1$.

To establish similar results for the altered Pell numbers to the numbers E_n and K_n, we need the following Lemmas.

Lemma 5. [3] The numbers P_n and Q_n are the n^{th} Pell and Pell Lucas numbers, respectively. The next two identities are valid;

$$Q_{m+n}+Q_{m-n}=\begin{cases}8P_mP_n \ \textit{if } \ n \textit{ is odd}\\ Q_nQ_m \ \ \textit{otherwise}\end{cases}, \tag{12}$$

$$Q_{m+n}-Q_{m-n}=\begin{cases}Q_nQ_m \ \textit{if } \ n \textit{ is odd}\\ 8P_mP_n \ \ \textit{otherwise}\end{cases}. \tag{13}$$

Lemma 6. The following identities are valid:

$$Q_{4k}+6=8P_{2k-1}P_{2k+1}, \quad Q_{4k}-6=Q_{2k+1}Q_{2k-1},$$
$$Q_{4k+1}+2=Q_{2k+1}Q_{2k}, \quad Q_{4k+1}-2=8P_{2k}P_{2k+1},$$
$$Q_{4k+2}+6=Q_{2k+2}Q_{2k}, \quad Q_{4k+2}-6=8P_{2k}P_{2k+2},$$
$$Q_{4k+3}+2=8P_{2k+1}P_{2k+2}, Q_{4k+3}-2=Q_{2k+2}Q_{2k+1}.$$

Proof Substitute firstly, $2k+1$ for m and $2k$ for n (13), for value $Q_{4k+1}-2$ and secondly, $2k+1$ for m and $2k-1$ for n in (13), for value $Q_{4k}-6$, we have

$$Q_{(2k+1)+2k}-Q_{(2k+1)-2k}=8P_{2k}P_{2k+1},$$
$$Q_{(2k+1)+(2k-1)}-Q_{(2k+1)-(2k-1)}=Q_{2k+1}Q_{2k-1}.$$

Also, the other equations can be established with above mentioned way.

Lemma 7. For any integers m and n,

$$\left(Q_n-2P_m-6P_{m-1},Q_{n+1}+2P_{m-1}+6P_{m-2}\right)=\left(Q_{n-2}-Q_{m+2}-4P_{m+1},Q_{n-1}+Q_{m+1}+4P_m\right). \tag{14}$$

Proof By applying (6) and the relation $Q_n-2Q_{n-1}=Q_{n-2}$, we have

$$\begin{aligned}\left(Q_n-2P_m-6P_{m-1},Q_{n+1}+2P_{m-1}+6P_{m-2}\right)&=\left(Q_n-2P_m-6P_{m-1},Q_{n+1}-2Q_n+2P_{m-1}+6P_{m-2}+4P_m+12P_{m-1}\right)\\&=\left(Q_n-2Q_{n-1}-2P_m-4P_{m+1}-6P_{m-1}-12P_m,Q_{n-1}+2P_{m+1}+6P_m\right)\\&=\left(Q_{n-2}-2P_{m+2}-6P_{m+1},Q_{n-1}+2P_{m+1}+6P_m\right).\end{aligned}$$

where $2P_m+2P_{m-1}=Q_m$, it completes proof.

Lemma 8. For any integers m and n,

$$\left(Q_n-2,Q_{n+1}+6\right)=\left(Q_{n-2m}-Q_{2m+1}-4P_{2m},Q_{n-2m+1}+Q_{2m}+4P_{2m-1}\right), \tag{15}$$

$$\left(Q_n+2,Q_{n+1}-6\right)=\left(Q_{n-2m}+2P_{2m+1}+6P_{2m},Q_{n-2m+1}-2P_{2m}-6P_{2m-1}\right). \tag{16}$$

Proof By using (6) and $Q_n-2Q_{n-1}=Q_{n-2}$, we simplify alike

$$\begin{aligned}\left(Q_n-2,Q_{n+1}+6\right)&=\left(Q_n-Q_1,Q_{n+1}-2Q_n+Q_2+2Q_1\right)\\&=\left(Q_n-Q_1,Q_{n-1}+Q_2+2Q_1\right)\\&=\left(Q_{n-2}-2Q_2-5Q_1,Q_{n-1}+Q_2+2Q_1\right).\end{aligned}$$

Since $P_{-1}=1$, $P_0=0$ and $P_1=1$, we rewrite

$$\left(Q_n-2,Q_{n+1}+6\right)=\left(Q_n-P_1Q_1-P_0Q_2,Q_{n+1}+P_0Q_1+P_{-1}Q_2\right),$$

and applying m times of (14) to $\left(Q_n-2,Q_{n+1}+6\right)$ gives desired result for $2P_{2m+1}+2P_{2m}=Q_{m+1}$.

Proof of (16) are made with similar above mentioned way.

Theorem 7. If E'_n, $n\geq 1$ is the altered Pell Lucas number given in (3), then

$$\left(E'_{2k},E'_{2k+1}\right)=\begin{cases}2Q_{k+1}, \ \textit{for odd } k\\ 8P_{k+1}, \ \textit{for even } k\end{cases} \text{ and } \left(E'_{2k-1},E'_{2k}\right)=\begin{cases}12, \textit{if } \ k\equiv 1 \pmod 4\\ 4, \quad \ \textit{otherwise}\end{cases}.$$

Proof For value E'_{2k}, substitute $k+1$ for m and $k-1$ for n in (12) and for value E'_{2k+1}, $k+1$ for m and k for n in (13), then desired results are obtained.

Since $\left(E'_{2k-1},E'_{2k}\right)=\left(Q_{2k-1}-2,Q_{2k}+6\right)$, let $n=2k-1$ and $m=\frac{k}{2}$ for k even in (15);

$$\begin{aligned}\left(Q_{2k-1}-2,Q_{2k}+6\right)&=\left(-2Q_k-4P_k,2Q_k+4P_{k-1}\right)\\&=4\left(-P_{k+1},q_k+P_{k-1}\right)\\&=4\left(-P_{k+1},3q_k\right).\end{aligned}$$

Since $(P_{k+1},3)=3$ if and only if $k+1\equiv 0\,(\mathrm{mod}\,4)$ but for k even $(P_{k+1},3)=1$, and $(P_{k+1},q_k)=1$, it follows that $(E'_{2k-1},E'_{2k})=4$ for k even.

Let $n=2k-1$ and $m=\frac{k-1}{2}$ for k odd in (15);

$$\begin{aligned}(Q_{2k-1}-2,Q_{2k}+6)&=(-4P_{k-1},8P_k+4P_{k-2})\\&=4(-P_{k-1},2P_k+P_{k-2}+2P_{k-1})\\&=4(-P_{k-1},3P_k).\end{aligned}$$

Let's remember that $(P_{k-1},3)=3$ if and only if $k-1\equiv 0\,(\mathrm{mod}\,4)$, it completes proof.

Theorem 8. If E'_n, $n\geq 1$ is the altered Pell Lucas numbers given in (3), then

$$(E'_{4k},E'_{4k+1})=8P_{2k+1},\ (E'_{4k+2},E'_{4k+3})=2Q_{2k+2},\ (E'_{4k-1},E'_{4k})=4$$

$$(E'_{4k+1},E'_{4k+2})=\begin{cases}12, & if\ k\equiv 0,2\,(\mathrm{mod}\,4)\\ 4, & otherwise\end{cases}.$$

Proof Using the Lemma 6 and the definition of the number E'_n given in (3), we rewrite $(Q_{4k}+6,Q_{4k+1}-2)$ $=8P_{2k+1}$, and $(Q_{4k+2}+6,Q_{4k+3}-2)=2Q_{2k+2}$.

Since $(E'_{4k+1},E'_{4k+2})=(Q_{4k+1}-2,Q_{4k+2}+6)$, let $n=4k+1$, and $m=k$ in (15);

$$\begin{aligned}(Q_{4k+1}-2,Q_{4k+2}+6)&=4(-P_{2k},2P_{2k+1}+P_{2k-1})\\&=4(-P_{2k},2P_{2k+1}+2P_{2k}+P_{2k-1})\\&=4(-P_{2k},3P_{2k+1}).\end{aligned}$$

Let's remember that $(P_{2k},3)=3$ if and only if $2k\equiv 0\,(\mathrm{mod}\,4)$, it completes proof.

Since $(E'_{4k-1},E'_{4k})=(Q_{4k-1}-2,Q_{4k}+6)$, let $n=4k-1$, and $m=k$ in (15);

$$\begin{aligned}(Q_{4k-1}-2,Q_{4k}+6)&=(Q_{2k-1}-Q_{2k+1}-4P_{2k},2Q_{2k}+4P_{2k-1})\\&=2(Q_{2k}-2P_{2k},Q_{2k}+2P_{2k-1})\\&=4(P_{2k-1},P_{2k}).\end{aligned}$$

The altered Pell Lucas numbers K'_n are introduced in the following table;

n	1	2	3	4	5	6	7	8	9	10	11	12	13
(K'_n,K'_{n+1})	4	$8P_2$	4	$2Q_3$	12	$8P_4$	4	$2Q_5$	4	$8P_6$	4	$2Q_7$	12

Be aware of $(K'_{2k-1},K'_{2k})=12$ if $k\equiv 3\,(\mathrm{mod}\,4)$ and otherwise, $(K'_{2k-1},K'_{2k})=4$. It is seen that $(K'_{2k},K'_{2k+1})=8P_{k+1}$ for $k=1,3,5,...$ and $(K'_{2k},K'_{2k+1})=2Q_{k+1}$ for $k=2,4,6,...$. In addition they can be rewritten $(K'_{4k-2},K'_{4k-1})=8P_{2k}$ and $(K'_{4k},K'_{4k+1})=2Q_{2k+1}=4q_{2k+1}$, $k\geq 1$.

Theorem 9. If K'_n, $n\geq 1$ is the altered Pell Lucas number given in (3), then

$$(K'_{2k},K'_{2k+1})=\begin{cases}8P_{k+1}, & for\ odd\ k\\ 2Q_{k+1}, & for\ even\ k\end{cases}\ \text{and}\ (K'_{2k-1},K'_{2k})=\begin{cases}12, & if\ k\equiv 3\,(\mathrm{mod}\,4)\\ 4, & otherwise\end{cases}.$$

Proof For the value K'_{2k}, substitute $k+1$ for m and $k-1$ for n in (13) and for the value K'_{2k+1}, $k+1$ for m and k for n in (12), then desired results are obtained.

Since $(K'_{2k-1},K'_{2k})=(Q_{2k-1}+2,Q_{2k}-6)$, let $n=2k-1$ and $m=\frac{k}{2}$ for k even in (16);

$$\begin{aligned}(Q_{2k-1}+2,Q_{2k}-6)&=(Q_{k-1}+2P_{k+1}+6P_k,Q_k-2P_k-6P_{k-1})\\&=(12P_k,4P_{k-1})=4(3P_k,P_{k-1})\\&=4(3P_k,P_{k-1}).\end{aligned}$$

Since $(P_{k-1},3)=3$, if $k\equiv 1\,(\mathrm{mod}\,4)$, but for k even, $(3P_k,P_{k-1})=1$ is valid. Thus, $(K'_{2k-1},K'_{2k})=4$ is true for even k.

Now, let $n=2k-1$ and $m=\frac{k-1}{2}$ for k odd in (16);

$$\begin{aligned}(Q_{2k-1}+2,Q_{2k}-6)&=(Q_k+2P_k+6P_{k-1},Q_{k+1}-2P_{k-1}-6P_{k-2})\\&=(2P_{k+1}+6P_{k-1},4P_k+4P_{k-1}-4P_{k-2})\\&=4(P_k+2P_{k-1},3P_{k-1})\\&=4(q_{k-1},3P_{k-1}).\end{aligned}$$

$(q_{k-1},3)=3$ if and only if $k\equiv 3(\text{mod}\,4)$ and $(q_{k-1},P_{k-1})=1$ for every integer k, desired results is obtained.

Theorem 10. For any integer k,

$$(K'_{4k},K'_{4k+1})=2Q_{2k+1},\ (K'_{4k+2},K'_{4k+3})=8P_{2k+2},\ (K'_{4k-1},K'_{4k})=4,$$

$$(K'_{4k+1},K'_{4k+2})=\begin{cases}12, \textit{if } k\equiv 1,3\,(\text{mod}\,4)\\ 4, \quad \textit{otherwise}\end{cases}.$$

Proof By using $(K'_{4k},K'_{4k+1})=(Q_{4k}-6,Q_{4k+1}+2)$ and $(K'_{4k+2},K'_{4k+3})=(Q_{4k+2}-6,Q_{4k+3}+2)$ from the definition of the number K'_n given in (3), we obtain the first two equations in the Theorem 10.

Since $(K'_{4k-1},K'_{4k})=(Q_{4k-1}+2,Q_{4k}-6)$, let $n=4k-1$, and $m=k$ given in (16);

$$\begin{aligned}(Q_{4k-1}+2,Q_{4k}-6)&=(Q_{2k-1}+Q_{2k+1}+4P_{2k},2P_{2k-1}-6P_{2k-1})\\&=4(3P_{2k},-P_{2k-1}).\end{aligned}$$

Due to $2k\not\equiv 1(\text{mod}\,4)$, it follows $(3,-P_{2k-1})=1$, thus we have the result.

Since $(K'_{4k+1},K'_{4k+2})=(Q_{4k+1}+2,Q_{4k+2}-6)$, let $n=4k+1$, and $m=k$ in (16);

$$\begin{aligned}(Q_{4k+1}+2,Q_{4k+2}-6)&=(4P_{2k+1}+8P_{2k},4P_{2k+1}+4P_{2k}-4P_{2k-1})\\&=4(P_{2k+1}+2P_{2k},3P_{2k})=4(P_{2k+1}-P_{2k},3P_{2k})\\&=4(q_{2k},3P_{2k}).\end{aligned}$$

We know that $(q_{2k},3)=3$ if and only if $k\equiv 1,3\,(\text{mod}\,4)$. Thus desired results are obtained.

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