IOSR Journal of Mathematics (IOSR-JM)
e-ISSN: 2278-5728, p-ISSN: 2319-765X. Volume 15, Issue 1 Ser. I (Jan – Feb 2019), PP 07-14
www.iosrjournals.org

βg^* – Separation Axioms

C. Dhanapakyam[1], K.Indirani[2].
Department of Mathematics Rathnavel subramaniam College of Arts & Science
Coimbatore-, India
Nirmala College for women Red fields,Coimbatore-,India
Corresponding Author: C. Dhanapakyam

Abstract: *In this paper, some new types of separation axioms in topological spaces by using βg^*-open sets are formulated. In particular the concept of βg^*-R_0 and βg^*-R_1 axioms are introduced. Several properties of these spaces are investigated using these axioms.*
Keywords: *βg^*-open set, βg^*-R_0, βg^*-R_1, βg^*-T_i(i=0,1,2)*

Date of Submission: 04-01-2019 Date of acceptance: 21-01-2019

I. Introduction

In 1970, Levine[4] introduced the concept of generalized closed set in topological spaces. In 2000, Veeerakumar [6] introduced several generalized closed sets namely g^* closed sets,$\hat{g}$ closed set. Andrijevic[1] introduced β-open set in general topology. The aim of this paper is to introduce the some new type of separation axioms via βg^*-open sets. Throughout this paper (X,τ) and (Y,σ)(or simply X and Y)represents the non-empty topological spaces on which no separation axioms are assumed,unless otherwise mentioned. For a subset A of X, cl(A) and int(A) represents the closure of A and interior of A respectively.

II. Preliminaries

Definition 2.1:A subset A of (X, τ) is called
1) Generalized closed[4] (briefly g-closed) if cl(A) ⊂ U whenever A⊂U and U is open.
2) βg^*-closed [3]if gcl(A)⊂U whenever A⊂U and U is β-open in X.

Definition 2.2: A map f: (X,τ) → (Y,σ) is called
1) Continuous [2] if f^{-1}(V) is closed subset in (X,τ) for every closed subset V in (Y,σ).
2). g continuous[5] if f^{-1}(V) is g closed subset in (X,τ) for every closed subset V in (Y,σ).
3) βg^*- continuous if f^{-1}(V) is βg^*- closed subset in (X,τ) for every closed subset V in (Y,σ).

Definition 2.3:A function f:(X,τ) →(Y,σ) from a topological space X into a topological space Y is called a βg* irresolute if f^{-1}(V) is βg* closed set in X for every βg*closed set V in Y.

III. βg^*-T_k (k = 0 , 1, 2) SPACES

In this section, a new type of separation axioms in topological spaces called βg^*-T_k spaces for k = 0 ,1, 2 are defined and their properties are studied.

Definition 3.1: A topological space (X, τ) is said to be
1. βg^*-T_0 if for each pair of distinct points x, y in X, there exists a βg^*-open set U such that either x ∈ U and y ∉ U or x ∉ U and y ∈ U.
2. βg^*-T_1 if for each pair of distinct points x, y in X, there exist two βg^*-open sets U and V such that x ∈ U and y ∉ U and y ∈ V but x ∉ V.
3. βg^*-T_2 if for each pair of distinct points x, y in X, there exist two disjoint βg^*-open sets U and V containing x and y respectively.

Example 3.2: (i) Let X = {a, b, c} with the topology τ ={X, ϕ, {a}}. Here βg^*-open sets are {X, ϕ, {a}, {b}, {c}, {a, b}, {b, c}, {a, c}}. Since for the distinct points a and b, there exist a βg^*-open set U= {a} such that a∈U and b∉ U or U={b} such that a∉ U and b∈U. In a similar manner other pairs of distinct points may also be discussed. Therefore X is βg^*-T_0 space.
(ii) Let X= {a, b, c } with the topology τ ={X, ϕ, {a}}. Here βg^*-open sets are {X, ϕ, {a}, {b}, {c}, {a, b}, {b, c}, {a, c}}. Since for the distinct points a and b, there exist βg^*-open sets U= {a} and V={b, c} such that a ∈U

but b $\notin$U and a $\notin$V but b $\in$V. In a similar manner other pairs of distinct points may also be discussed. Therefore X is βg^*-T_1 space.
(iii) Let X = {a, b, c} with the topology τ ={X, ϕ, {c}, {a, b}}. Here βg^*-open sets are {X, ϕ, {a}, {b}, {c}, {a, b}, {b, c}, {a, c}}. Since for the distinct points a and c, there exist two disjoint βg^*-open sets U= {a} and V={c} containing a and c. In a similar manner other pairs of distinct points may also be discussed. Therefore X is βg^*-T_2 space.

Remark 3.3: Let (X, τ) be a topological space, then the following statements are true:
1. Every βg^*-T_2 space is βg^*-T_1.
2. Every βg^*-T_1 space is βg^*-T_0.

Theorem 3.4: Every T_0 space is a βg^*-T_0 space.
Proof: Let X be a T_0 space. Let x, y be two distinct points in X. Since X is T_0 space, there exists an open set M in X such that x$\in$M, y $\notin$ M. Since every open set is a βg^*-open set , M is a βg^*-open set in X. Thus, for any two distinct points x, y in X, there exists a βg^*-open set M in X such that x $\in$ M, y $\notin$ M. Hence X is a βg^*-T_0 space.

Theorem 3.5: A topological space (X, τ) is βg^*-T_0 if and only if for each pair of distinct points x, y of X, βg^*-cl({x}) $\neq$ βg^*-cl({y}) .

Proof: Necessity: Let (X, τ) be a βg^*-T_0 space and x, y be any two distinct points of X. There exists βg^*-open set U containing x or y, say x but not y. Then X−U is a βg^*- closed set which does not contain x but contains y. Since βg^*-cl({y}) is the smallest βg^*-closed set containing y, βg^*-cl({y}) ⊆ X−U and therefore x $\notin$ βg^*-cl({y}). Consequently βg^*-cl({x}) $\neq$ βg^*-cl({y}).

Sufficiency: Suppose that x, y $\in$ X, x $\neq$ y and βg^*-cl({x}) $\neq$ βg^*-cl({y}). Let z be a point of X such that z $\in$ βg^*-cl({x}) but z $\notin$ βg^*-cl({y}). We claim that x$\notin$ βg^*-cl({y}). For if x $\in$ βg^*-cl({y}) then βg^*-cl({x}) ⊆ βg^*-cl({y}). This contradicts the fact that z $\notin$ βg^*-cl({y}). Consequently x belongs to the βg^*-open set X − βg^*-cl({y}) to which y does not belong to. Hence (X, τ) is a βg^*-T_0 space.

Theorem 3.6: In a topological space (X, τ), if the singletons are βg^*-closed then X is βg^*-T_1 space and the converse is true if βG^*O (X, τ) is closed under arbitrary union.

Proof: Let {z} is βg^*-closed for every z $\in$ X. Let x, y $\in$ X with x $\neq$ y. Now x $\neq$ y implies y $\in$ X –{x}. Hence X –{x} is a βg^*-open set that contains y but not x. Similarly X –{y} is a βg^*-open set containing x but not y. Therefore X is a βg^*-T_1 space.
Conversely, let (X, τ) be βg^*-T_1 and x be any point of X. Choose y $\in$X –{x}, then x $\neq$ y and so there exists a βg^*-open set U such that y $\in$U but x $\notin$ U. Consequently y $\in$ U⊆X –{x}, that is X –{x} = ∪{U_y: y$\in$X –{x}} which is βg^*-open. Hence {x} is βg^*-closed. That is every singleton set is βg^*-closed.

Theorem 3.7: The following statements are equivalent for a topological space (X, τ)
1. X is βg^*-T_2.
2. Let x $\in$ X. For each y $\neq$ x, there exists a βg^*-open set U containing x such that y $\notin$ βg^*-cl({U}).
3. For each x $\in$X, ∩{ βg^*-cl({U}): U$\in$ βG^*O(X, τ) and x$\in$U}={x}.

Proof: (1)⇒ (2): Let x$\in$X, and for any y$\in$X such that x $\neq$ y, there exist two disjoint βg^*-open sets U and V containing x and y respectively, since X is βg^*-T_2. So U⊆X−V. Therefore βg^*-cl({U})⊆X−V. So y $\notin$ βg^*-cl({U}).
(2)⇒(3) If possible for some y $\neq$ x, y$\in$∩{βg^*-cl({U}) : U$\in$ βG^*O(X, τ) and x$\in$U}. This implies y$\in$ βg^*-cl({U}) for every βg^*-open set U containing x, which contradicts (2). Hence ∩{ βg^*-cl({U}): U$\in$ βG^*O(X, τ) and x$\in$U}={x}.
(3)⇒(1) Let x, y$\in$X and x $\neq$ y. Then there exists at least one βg^*-open set U containing x such that y $\notin$ βg^*-cl({U}). Let V = X−βg^*-cl({U}), then y$\in$V and x $\in$ U and also U∩V=ϕ. Therefore X is βg^*-T_2.

Theorem 3.8: Let (X, τ) and (Y, σ) be two topological spaces and f: (X, τ)→ (Y, σ) be an one to one function. Then if f is
(1) βg^*-continuous and Y is a T_0 space then X is a βg^*-T_0 space.
(2) βg^*-irresolute and Y is a βg^*-T_0 space then X is a βg^*-T_0 space.

(3) Continuous and Y is a T_0 space then X is a βg^*-T_0 space.
(4) Onto, βg^*-irresolute and X is a βg^*-T_0 space then Y is a βg^*-T_0 space.

Proof: (1) Let x, y be two distinct points in X. Then f(x) and f(y) are distinct points in Y. Then there exists two open set U in Y such that f(x) $\in$U and f(y) $\notin$ U or f(y) $\in$ U and f(x) $\notin$ U. Then $f^{-1}(U)$ is a βg^*-open set in X such that x$\in f^{-1}(U)$ and y $\notin f^{-1}(U)$ or y$\in f^{-1}(U)$ and x $\notin f^{-1}(U)$. Therefore X is a βg^*-T_0 space.
Proof of (2) to (4) are similar.

Remark 3.9: The property of being a βg^*-T_0 space is preserved under one to one, onto and βg^*-irresolute mappings.

Theorem 3.10: Let (X, τ) and (Y, σ) be two topological spaces and f: (X, τ)→ (Y, σ) be an one to one function. Then if f is
(1) βg^*-continuous and Y is a T_1 space, then X is a βg^*-T_1 space.
(2) βg^*-irresolute and Y is a βg^*-T_1 space, then X is a βg^*-T_1 space.
(3) Continuous and Y is a T_1 space, then X is a βg^*-T_1 space.
(4) Onto and βg^*-irresolute and X is a βg^*-T_1 space then Y is a βg^*-T_1 space.

Proof: Let x, y be two distinct points in X. Then f(x) and f(y) are distinct points in Y. Then there exists two open sets U and V in Y such that f(x) $\in$U but f(y) $\notin$ U and f(y) $\in$ V and f(x) $\notin$ V. Then $f^{-1}(U)$ and $f^{-1}(V)$ are βg^*-open sets in X such that x$\in f^{-1}(U)$ and
y $\notin f^{-1}(U)$ and y$\in f^{-1}(V)$ and x $\notin f^{-1}(V)$. Therefore X is a βg^*-T_1 space.
Proof of (2) to (4) are similar.

Remark 3.11: The property of being a βg^*-T_1 space is preserved under one to one, onto and βg^*-irresolute mappings.

Definition 3.12: Let A be a subset of a topological space (X, τ). The βg^*-kernel of A is defined as the intersection of all βg^*-open sets of (X, τ) which contains A (briefly βg^*-ker(A)). That is βg^*-ker(A) = $\cap$ {U$\in$ $\beta G^*O(X, \tau)$: A$\subseteq$U}.

Definition 3.13: Let x be a point of a topological space X. Then βg^*-ker(x) =$\cap$ {M :M$\in$ $\beta G^*O(X, \tau)$ and x$\in$M}.

Theorem 3.14: Let (X, τ) be a topological space and x $\in$X. Then y$\in$ βg^*-ker({x}) if and only if x$\in$ βg^*-cl({y}).

Proof: Suppose that y$\notin$ βg^*-ker({x}). Then there exists a βg^*-open set U containing x such that y $\notin$U. Therefore , x$\notin$ βg^*-cl({y}). The proof of the converse case can be done similarly.

Theorem 3.15: Let (X, τ) be a topological space and A be a subset of X. Then βg^*-ker({A}) = {x $\in$X : βg^*-cl({x})$\cap$A$\neq$ ϕ}.

Proof: x $\in$ βg^*-ker({A}) and suppose βg^*-cl({x}) $\cap$A = ϕ. Hence x $\notin$ X$-$ βg^*-cl({x}) which is a βg^*-open set containing A. This is impossible, since x $\in$ βg^*-ker({A}). Consequently, βg^*-cl({x}) $\cap$A$\neq$ ϕ. Next, let x $\in$X such that βg^*-cl({x}) $\cap$A$\neq$ ϕ and suppose that x $\notin$ βg^*-ker({A}). Then there exists a βg^*-open set U containing A and x $\notin$ U. Let y $\in$ βg^*-cl({x}) $\cap$A . Hence U is a βg^*-neighbourhood of y which does not contain x. By this contradiction x $\in$ βg^*-ker({A}) and hence the claim.

Theorem 3.16: The following properties hold for any two subsets A, B of a topological space (X, τ)
1. A $\subseteq$ βg^*-ker({A}).
2. A $\subseteq$ B implies that βg^*-ker({A})$\subseteq$ βg^*-ker({B}).
3. If A is βg^*-open in (X, τ), then A= βg^*-ker({A}).
4. βg^*-ker(βg^*-ker({A})) = βg^*-ker({A}).

Proof: The proof of (1), (2) and (3) are immediate consequences of Definition 3.12.
(4) By (1) and (2), we have βg^*-ker({A})$\subseteq$ βg^*-ker(βg^*-ker({A})). If x$\notin$ βg^*-ker({A}), then there exists U$\in$ $\beta G^*O(X, \tau)$ such that A$\subseteq$U and x$\notin$ U. Hence βg^*-ker({A})$\subseteq$U, and so x$\notin$$\beta g^*$-ker($\beta g^*$-ker({A})). Thus βg^*-ker(βg^*-ker({A})) = βg^*-ker({A}).

Definition 3.17: A topological space (X, τ) is said to be βg^*-symmetric if for any pair of distinct points x and y in X, x ∈ βg^*-cl({y}) implies y ∈ βg^*-cl({x}).

Theorem 3.18: For a topological space (X, τ), the following are equivalent:

1. (X, τ) is a βg^*-symmetric space.
2. {x} is βg^*-closed, for each x ∈X.

Proof: (1)⇒(2): Let (X, τ) be a βg^*-symmetric space. Assume that {x} ⊆U ∈ βG^*O (X, τ), but βg^*-cl({x})⊄U. Then βg^*-cl({x})∩(X−U)≠ ϕ. Now, we take y ∈ βg^*-cl({x}) ∩(X−U), then by hypothesis x ∈ βg^*-cl({y}) ⊆ X−U that is, x∉U, which is a contradiction. Therefore {x} is βg^*-closed, for each x ∈X.
(2) ⇒(1): Assume that x ∈ βg^*-cl({y}), but y ∉ βg^*-cl({x}). Then {y}⊆X−βg^*-cl({x}) and hence βg^*-cl({y})⊆X−βg^*-cl({x}). Therefore x ∈X−βg^*-cl({x}), which is contradiction and hence y∈ βg^*-cl({x}).

Corollary 3.19: Let βG^*O(X, τ) be closed under arbitrary union. If the topological space (X, τ) is a βg^*-T_1 space, then it is βg^*-symmetric.

Proof: In a βg^*-T_1 space, every singleton set is βg^*-closed and therefore, by theorem 3.18, (X, τ) is βg^*-symmetric.

Corollary 3.20: If a topological space (X, τ) is βg^*-symmetric and βg^*-T_0, then (X, τ) is a βg^*-T_1 space.

Proof: Let x ≠ y and as (X, τ) is βg^*-T_0, we may assume that x ∈U ⊆X −{y} for some U ∈ βG^*O (X, τ). Then x ∉ βg^*-cl({y}) and hence y∉βg^*-cl({x}). There exists a βg^*-open set V such that y ∈V ⊆X −{x} and thus (X, τ) is a βg^*-T_1 space.

IV. βg^*-R_k (k=0, 1) SPACES

In this section, a new class of topological spaces called βg^*-R_0 and βg^*-R_1 spaces are introduced and some of their properties are studied.

Definition 4.1: A topological space (X, τ) is said to be βg^*-R_0 if U is βg^*-open set and x ∈U then βg^*-cl({x}) ⊆U.

Theorem 4.2: For a topological space (X, τ) the following properties are equivalent:

(1) (X, τ) is βg^*-R_0 space.
(2) For any F ∈ βG^*C (X, τ), x ∉ F implies F⊆U and x∉U for some U∈ βG^*O (X, τ).
(3) For any F ∈ βG^*C (X, τ), x ∉ F implies F ∩ βg^*-cl({x}) = ϕ.
(4) For any two distinct points x and y of X, either βg^*-cl({x}) = βg^*-cl({y}) or βg^*-cl({x})∩ βg^*-cl({y}) = ϕ.

Proof: (1)⇒(2) Let F ∈ βG^*C(X, τ) and x ∉ F. Then by (1), βg^*-cl({x}) ⊆ X−F. Set U = X−βg^*-cl({x}), then U is a βg^*-open set such that F⊆U and x ∉ U.
(2)⇒(3) Let F ∈ βG^*C(X, τ) and x ∉ F. There exists U∈ βG^*O(X, τ) such that F ⊆U and x ∉ U. Since U ∈ βG^*O(X, τ), U∩ βg^*-cl({x}) = ϕ and F∩ βg^*-cl({x}) = ϕ.
(3) ⇒(4) Suppose that βg^*-cl({x})≠ βg^*-cl({y}) for two distinct points x, y ∈X. There exists z ∈ βg^*-cl({x}) such that z ∉ βg^*-cl({y}) [or z∈ βg^*-cl({y}) such that z ∉ βg^*-cl({x})]. There exists V∈ βG^*O(X, τ) such that y ∉ V and z ∈ V, hence x∈V. Therefore, we have x ∉ βg^*-cl({y}). By (3), we obtain βg^*-cl({x})∩ βg^*-cl({y}) = ϕ.
(4) ⇒ (1) Let V∈ βG^*O(X, τ) and x∈V. For each y∉ V, x≠y and x ∉ βg^*-cl({y}). This shows that βg^*-cl({x}) ≠ βg^*-cl({y}). By (4), βg^*-cl({x})∩ βg^*-cl({y}) = ϕ for each y ∈ X−V and hence βg^*-cl({x})∩[∪ βg^*-cl({y}): y∈X−V}] = ϕ. On the other hand, since V∈ βG^*O(X, τ) and y∈X−V, we have βg^*-cl({y})⊆X−V and hence X−V = ∪{ βg^*-cl({y}) : y∈ X−V}. Therefore, we obtain (X−V) ∩ βg^*-cl({x}) = ϕ and βg^*-cl({x})⊆V. This shows that (X, τ) is a βg^*-R_0 space.

Theorem 4.3: If a topological space (X, τ) is βg^*-T_0 space and a βg^*-R_0 space then it is a βg^*-T_1 space.

Proof: Let x and y be any two distinct points of X. Since X is βg^*-T_0, there exists a βg^*-open set U such that x ∈U and y ∉ U. As x ∈U, βg^*-cl({x}) ⊆U. Since y ∉ U, y ∉ βg^*-cl({x}). Hence y ∈V= X −βg^*-cl({x}) and it is clear that x ∉ V. Hence it follows that there exist βg^*-open sets U and V containing x and y respectively, such that y ∉ U and x ∉ V respectively. This implies that X is a βg^*-T_1 space.

.Theorem 4.4: For a topological space (X, τ) the following properties are equivalent:
(1) (X, τ) is βg^*-R_0 space.
(2) x ∈ βg^*-cl({y}) if and only if y ∈ βg^*-cl({x}), for any two points x and y in X.
Proof: (1) ⇒ (2) Assume that X is βg^*-R_0. Let x ∈ βg^*-cl({y}) and V be any βg^*-open set such that y ∈V. Now by hypothesis, x ∈V. Therefore, every βg^*-open set which contain y contains x also. Hence y ∈ βg^*-cl({x}).
(2) ⇒(1) Let U be a βg^*-open set and x ∈U. If y ∉ U, then x ∉ βg^*-cl({y}) and hence y ∉ βg^*-cl({x}). This implies that βg^*-cl({x}) ⊆U. Hence (X, τ) is βg^*-R_0 space.

Remark 4.5: From Definition 3.17 and Theorem 4.4 the notion of βg^*-symmetric and βg^*-R_0 are equivalent.

Theorem 4.6: A topological space (X, τ) is βg^*-R_0 space if and only if for any two points x and y in X, βg^*-cl({x}) ≠ βg^*-cl({y}) implies βg^*-cl({x}) ∩ βg^*-cl({y}) = ϕ.

Proof: Necessity: Suppose that (X, τ) is βg^*-R_0 and x and y ∈ X such that βg^*-cl({x}) ≠ βg^*-cl({y}). Then, there exists z ∈ βg^*-cl({x}) such that z ∉ βg^*-cl({y}) [or z∈ βg^*-cl({y}) such that z ∉ βg^*-cl({x})]. There exists V ∈ βG^*O(X, τ) such that y ∉ V and z ∈V, hence x ∈ V. Therefore, we have x ∉ βg^*-cl({y}). Thus x ∈[X − βg^*-cl({y})] ∈ βG^*O(X, τ), which implies βg^*-cl({x}) ⊆[X − βg^*-cl({y})] and
βg^*-cl({x})∩ βg^*-cl({y}) = ϕ.

Sufficiency: Let V ∈ βG^*O(X, τ) and let x ∈V. To show that βg^*-cl({x}) ⊆V. Let y ∉ V, that is y ∈ X −V. Then x ≠ y and x ∉ βg^*-cl({y}). This shows that βg^*-cl({x}) ≠ βg^*-cl({y}). By assumption, βg^*-cl({x}) ∩ βg^*-cl({y}) = ϕ. Hence y ∉ βg^*-cl({x}) and therefore βg^*-cl({x}) ⊆V. Hence (X, τ) is βg^*-R_0 space.

Theorem 4.7: The following statements are equivalent for any two points x and y in a topological space (X, τ):
(1) βg^*-ker({x}) ≠ βg^*-ker({y}).
(2) βg^*-cl({x}) ≠ βg^*-cl({y}).

Proof: (1) ⇒ (2) Suppose that βg^*-ker({x}) ≠ βg^*-ker({y}), then there exists a point z in X such that z ∈ βg^*-ker({x}) and z ∉ βg^*-ker({y}). Theorem 3.14, implies that x ∈ βg^*-cl({z}), since z ∈ βg^*-ker({x}). By z ∉ βg^*-ker({y}), we have {y} ∩ βg^*-cl({z}) = ϕ. Since x ∈ βg^*-cl({z}), βg^*-cl({x}) ⊆ βg^*-cl({z}) and {y} ∩ βg^*-cl({x}) = ϕ. Therefore, it follows that βg^*-cl({x}) ≠ βg^*-cl({y}). Hence βg^*-ker({x}) ≠ βg^*-ker({y}) implies that βg^*-cl({x}) ≠ βg^*-cl({y}).
(2) ⇒ (1) Suppose that βg^*-cl({x}) ≠ βg^*-cl({y}). Then there exists a point z in X such that z ∈ βg^*-cl({x}) but z ∉ βg^*-cl({y}). We claim that x ∉ βg^*-cl({y}), for if x ∈ βg^*-cl({y}) then βg^*-cl({x}) ⊆ βg^*-cl({y}). This contradicts the fact that z ∉ βg^*-cl({y}). Hence x ∉ βg^*-cl({y}). Theorem 3.14, implies y ∉ βg^*-ker({x}). Therefore, βg^*-ker({x}) ≠ βg^*-ker({y}).

Theorem 4.8: Let (X, τ) be a topological space. Then ∩[βg^*-cl({x}) : x ∈ X} = ϕ if and only if βg^*-ker({x}) ≠ X for every x ∈ X.

Proof: Necessity: Suppose that ∩[βg^*-cl({x}) : x ∈ X} = ϕ. Assume that there is a point y in X such that βg^*-ker({y}) = X. Let x be any point of X. Then x∈U for every βg^*-open set U containing y and hence y∈ βg^*-cl({x}) for any x ∈ X. This implies that y∈∩{βg^*-cl({x}): x∈X}. But this is a contradiction. Hence βg^*-ker({x})≠X for every x∈X.

Sufficiency: Assume that βg^*-ker({x})≠X for every x∈X. If there exists a point y in X such that y∈∩{βg^*-cl({x}) : x∈X}, then every βg^*-open set containing y must contain every point of X. This implies that the space X is the only βg^*-open set containing y. Hence βg^*-ker({y}) = X which is a contradiction. Therefore ∩[βg^*-cl({x}) : x ∈ X} = ϕ.

Theorem 4.9: For a topological space (X, τ) the following properties are equivalent:
(1) (X, τ) is a βg^*-R_0 space.
(2) For any non-empty set A and G∈ βG^*O(X, τ) such that A∩G≠ ϕ, there exists F∈ βG^*C(X, τ) such that A∩F≠ ϕ and F⊆G.
(3) For any G∈ βG^*O(X, τ), we have G = ∪{F∈ βG^*C(X, τ) : F⊆G}.
(4) For any F∈ βG^*C(X, τ), we have F= ∩{G∈ βG^*O(X, τ) : F⊆G}.
(5) For every x ∈ X, βg^*-cl({x}) ⊆ βg^*-ker({x}).

Proof: (1) ⇒ (2) Let A be a non-empty subset of X and G∈ $\beta G^*O(X, \tau)$ such that A∩G≠ ϕ. Let x∈A∩G. Then x∈G⇒ βg^*-cl({x})⊆G, since (X, τ) is βg^*-R_0 space. Set F = βg^*-cl({x}), then F∈ $\beta G^*C(X, \tau)$, F⊆G and A∩F≠ ϕ.
(2) ⇒ (3) Let G∈ $\beta G^*O(X, \tau)$, choose x∈∪{ F ∈ $\beta G^*C(X, \tau)$: F⊆G}. Then x∈F for some F∈ $\beta G^*C(X, \tau)$ and F⊆G. Therefore, x∈ G. On the other hand, suppose x∈G. If we define A = {x}, then A∩G≠ ϕ. By our hypothesis, there exists F∈ $\beta G^*C(X, \tau)$ such that A∩F≠ ϕ, and F⊆G. Since A= {x}, x∈F⊆∪{F∈ $\beta G^*C(X, \tau)$: F⊆G}. Hence G = ∪{F ∈ $\beta G^*C(X, \tau)$: F⊆G}.
(3) ⇒ (4) Obvious.
(4) ⇒ (5) Let x be any point of X and y ∉ βg^*-ker({x}). There exists U ∈ $\beta G^*O(X, \tau)$ such that x ∈U and y ∉ U, hence βg^*-cl({y}) ∩U = ϕ. By (4) (∩{G ∈ $\beta G^*O(X, \tau)$: βg^*-cl({y}) ⊆G}) ∩U = ϕ and there exists G ∈ $\beta G^*O(X, \tau)$ such that x ∉G and βg^*-cl({y}) ⊆G. Therefore βg^*-cl({x}) ∩G = ϕ and y ∉ βg^*-cl({x}). Consequently, we obtain βg^*-cl({x}) ⊆ βg^*-ker({x}).
(5) ⇒ (1) Let G ∈ $\beta G^*O(X, \tau)$ and x ∈G. Let y ∈ βg^*-ker({x}), then x ∈ βg^*-cl({y}) and y ∈G. This implies that βg^*-ker({x}) ⊆G. Therefore x ∈ βg^*-cl({x}) ⊆ βg^*-ker({x}) ⊆G. Therefore (X, τ) is a βg^*-R_0 space.

Theorem 4.10: A topological space (X, τ) is βg^*-R_0 space if and only if βg^*-cl({x}) = βg^*-ker({x}), for each x ∈X.

Proof: Let (X, τ) be a βg^*-R_0 space. By theorem 4.9, βg^*-cl({x}) ⊆ βg^*-ker({x}) for each x ∈X. Let y ∈ βg^*-ker({x}), then x ∈ βg^*-cl({y}) and by theorem 3.14, y ∈ βg^*-cl({x}) and hence βg^*-ker({x}) ⊆ βg^*-cl({x}). Therefore βg^*-cl({x}) = βg^*-ker({x}). Converse part is true from theorem 4.9.

Theorem 4.11: A topological space (X, τ) is βg^*-R_0 if and only if for any two points x and y in X, βg^*-ker({x})≠ βg^*-ker({y}) implies βg^*-ker({x}) ∩ βg^*-ker({y}) = ϕ.

Proof: Suppose that (X, τ) is a βg^*-R_0 space. Thus by theorem 4.7 for any two points x and y in X if βg^*-ker({x})≠ βg^*-ker({y}) then βg^*-cl({x})≠ βg^*-cl({y}). Now we prove that βg^*-ker({x}) ∩ βg^*-ker({y}) = ϕ. Assume that z ∈ βg^*-ker({x}) ∩ βg^*-ker({y}) . By z ∈ βg^*-ker({x}) and by theorem 3.14, we get x ∈ βg^*-cl({z}). Since x ∈ βg^*-cl({x}), by theorem 4.2, βg^*-cl({x})= βg^*-cl({z}). Similarly, we have βg^*-cl({y})= βg^*-cl({z})= βg^*-cl({x}). This is a contradiction. Therefore, we have βg^*-ker({x}) ∩ βg^*-ker({y}) = ϕ.
Conversely, let (X, τ) be a topological space such that for any points x and y in X, βg^*-ker({x}) ≠ βg^*-ker({y}) implies βg^*-ker({x}) ∩ βg^*-ker({y}) = ϕ.Theorem 4.7 states that, if βg^*-ker({x}) ≠ βg^*-ker({y}), then βg^*-cl({x})≠ βg^*-cl({y}). By theorem 4.6, it is enough to prove βg^*-cl({x}) ∩ βg^*-cl({y}) = ϕ. Suppose βg^*-cl({x}) ∩ βg^*-cl({y}) ≠ ϕ. Let z ∈ βg^*-cl({x})∩ βg^*-cl({y}) Then z∈ βg^*-cl({x}) and z∈ βg^*-cl({y}). Since z∈ βg^*-cl({x}), and by theorem 3.14, x∈ βg^*-ker({z}). Therefore, βg^*-ker({x}) ∩ βg^*-ker({y}) ≠ ϕ. Then by hypothesis, we get βg^*-ker({x}) = βg^*-ker({z}). Similarly from z ∈ βg^*-cl({y}), we can prove that βg^*-ker({y})= βg^*-ker({z}). Therefore βg^*-ker({x})= βg^*-ker({z})= βg^*-ker({y}). This is a contradiction to our assumption βg^*-cl({x})≠ βg^*-cl({y}). Therefore βg^*-cl({x})= βg^*-cl({y}). Hence (X, τ) is a βg^*-R_0 space.

Theorem 4.12: For a topological space (X, τ) the following properties are equivalent:
(1) (X, τ) is a βg^*-R_0 space.
(2) If F is βg^*-closed, then F= βg^*-ker(F).
(3) If F is βg^*-closed and x∈F, then βg^*-ker({x}) ⊆F.
(4) If x∈X, then βg^*-ker({x}) ⊆ βg^*-cl({x}).

Proof: (1)⇒(2) Let F be βg^*-closed and x∉F. Thus X−F is a βg^*-open set containing x. Since (X, τ) is βg^*-R_0, βg^*-cl({x}) ⊆X−F. Thus βg^*-cl({x}) ∩F = ϕ and by theorem 3.15, x∉ βg^*-ker(F). Therefore βg^*-ker(F) = F.
(2) ⇒ (3) In general, A⊆B implies βg^*-ker(A) ⊆ βg^*-ker(B). Therefore, it follows from (2), that βg^*-ker({x}) ⊆ βg^*-ker(F) =F.
(3) ⇒ (4) Since x∈ βg^*-cl({x}) and βg^*-cl({x}) is βg^*-closed, by (3), βg^*-ker({x}) ⊆ βg^*-cl({x}).
(4) ⇒ (1) Let x∈ βg^*-cl({y}). Then by theorem 3.14, y∈ βg^*-ker({x}). (4) ⇒ y∈ βg^*-ker({x})⊆ βg^*-cl({x}). Therefore x∈ βg^*-cl({y}) implies y∈ βg^*-cl({x}). Therefore (X, τ) is βg^*-R_0 space.

Definition 4.13: In a topological space (X, τ) is said to be βg^*-R_1 if for x, y, in X with βg^*-cl({x})≠ βg^*-cl({y}), there exist disjoint βg^*-open sets U and V such that βg^*-cl({x})⊆ U and βg^*-cl({y})⊆ V.

Theorem 4.14: A topological space (X, τ) is βg^*-R_1 space if it is βg^*-T_2 space.

Proof: Let x and y be any two points X such that βg^*-cl({x})≠ βg^*-cl({y}). By Remark 3.3 (1), every βg^*-T_2 space is βg^*-T_1 space. Therefore, by theorem 3.6, βg^*-cl({x})={x}, βg^*-cl({y})= {y} and hence {x}≠{y}. Since (X, τ) is βg^*-T_2 , there exist a disjoint βg^*-open sets U and V such that βg^*-cl({x}) = {x}⊆U and βg^*-cl({y}) = {y}⊆V. Therefore (X, τ) is βg^*-R_1 space.

Theorem 4.15: For a topological space (X, τ) is βg^*-symmetric, then the following are equivalent:

(1) (X, τ) is βg^*-T_2 space.
(2) (X, τ) is βg^*-R_1 space and βg^*-T_1 space.
(3) (X, τ) is βg^*-R_1 space and βg^*-T_0 space.

Proof: (1) ⇒(2) and (2) ⇒(3) obvious.
(3) ⇒(1) Let x, y ∈X such that x ≠ y . Since (X, τ) is βg^*-T_0 space. By theorem 3.5 βg^*-cl({x}) ≠ βg^*-cl({y}), since X is βg^*-R_1, there exist disjoint βg^*-open sets U and V such that βg^*-cl({x}) ⊆U and βg^*-cl({y})⊆V. Therefore, there exist disjoint βg^*-open set U and V such that x∈U and y∈V. Hence (X, τ) is βg^*-T_2 space.

Remark 4.16: For a topological space (X, τ) the following statements are equivalent:

(1) (X, τ) is βg^*-R_1 space.
(2) If x, y ∈X such that βg^*-cl({x}) ≠ βg^*-cl({y}), then there exist βg^*-closed sets F_1 and F_2 such that x∈F_1, y∉F_1, y∈F_2, x∉F_2 and X=F_1∪F_2.

Theorem 4.17: If a topological space (X, τ) is βg^*-R_1 space, then (X, τ) is βg^*-R_0 space.

Proof: Let U be a βg^*-open set such that x∈U. If y∉U, then x∉ βg^*-cl({y}), therefore βg^*-cl({x}) ≠ βg^*-cl({y}). So, there exists a βg^*-open set V such that βg^*-cl({y})⊆V and x∉V, which implies y∉ βg^*-cl({x}). Hence βg^*-cl({x}) ⊆U. Therefore, (X, τ) is βg^*-R_0 space.

Theorem 4.18: A topological space (X, τ) is βg^*-R_1 space if and only if x∈X − βg^*-cl({y}) implies that x and y have disjoint βg^*-open neighbourhoods.

Proof: Necessity: Let (X, τ) be a βg^*-R_1 space. Let x∈X − βg^*-cl({y}). Then βg^*-cl({x}) ≠ βg^*-cl({y}), so x and y have disjoint βg^*-open neighbourhoods.

Sufficiency: First to show that (X, τ) is βg^*-R_0 space. Let U be a βg^*-open set and x∈U. Suppose that y∉U. Then, βg^*-cl({y})∩U = ϕ and x ∉ βg^*-cl({y}). There exist a βg^*-open sets U_x and U_y such that x ∈ U_x, y ∈U_y and U_x ∩U_y = ϕ. Hence, βg^*-cl({x})⊆ βg^*-cl({U_x}) and βg^*-cl({x})∩U_y⊆ βg^*-cl({U_x})∩U_y = ϕ. [For since U_y is βg^*-open set, X−U_y is βg^*-closed set. So βg^*-cl({X−U_y}) = X−U_y. Also since U_x∩U_y = ϕ and U_x⊆U_y^c. So βg^*-cl({U_x})⊆ βg^*-cl({X−U_y}). Thus βg^*-cl({U_x})⊆X−U_y]. Therefore, y∉ βg^*-cl({x}). Consequently, βg^*-cl({x})⊆U and (X, τ) is βg^*-R_0 space. Next to show that (X, τ) is βg^*-R_1 space. Suppose that βg^*-cl({x})≠ βg^*-cl({y}). Then, assume that there exists z∈ βg^*-cl({x}) such that z∉ βg^*-cl({y}). There exist a βg^*-open sets V_z and V_y such that z∈V_z, y∈V_y and V_z∩V_y = ϕ**.** Since z∈ βg^*-cl({x}), x∈V_z. Since (X, τ) is βg^*-R_0 space, we obtain βg^*-cl({x})⊆V_z, βg^*-cl({y})⊆V_y and V_z∩V_y = ϕ. Therefore (X, τ) is βg^*-R_1 space.

Theorem 4.19: A topological space (X, τ) is βg^*-R_1 space if and only if for each x ≠y ∈X with βg^*-ker({x}) ≠ βg^*-ker({y}), then there exist βg^*-closed sets G_1, G_2 such that βg^*-ker({x}) ⊆G_1, βg^*-ker({x}) ∩G_2 = ϕ and βg^*-ker({y}) ⊆ G_2, βg^*-ker({y}) ∩ G_1 = ϕ and G_1∪G_2= X.

Proof: Let (X, τ) be a βg^*-R_1 space such that for each x ≠ y ∈X with βg^*-ker({x})≠ βg^*-ker({y}). Since every βg^*-R_1 space is βg^*-R_0 space. By theorem 4.7, βg^*-cl({x}) ≠ βg^*-cl({y}). As X is βg^*-R_1 space there exists βg^*-open sets U_1, U_2 such that βg^*-cl({x}) ⊆ U_1 and βg^*-cl({y}) ⊆ U_2 and U_1 ∩U_2= ϕ then X –U_1 and X –U_2 are βg^*-closed sets such that (X −U_1∪X – U_2)= X. Put G_1= X –U_2 and G_2 = X –U_1. Thus x ⊆G_1 and y ⊆G_2, so that βg^*-ker({x})⊆G_1, βg^*-ker({y}) ⊆G_2 and G_1∪G_2= X and βg^*-ker({x}) ∩G_2 = ϕ, βg^*-ker({y}) ∩ G_1 = ϕ.
Conversely, let for each x ≠ y∈X with βg^*-ker({x}) ≠ βg^*-ker({y}), there exists βg^*-closed sets G_1 and G_2 such that βg^*-ker({x}) ⊆G_1, βg^*-ker({x}) ∩G_2 = ϕ and βg^*-ker({y}) ⊆ G_2, βg^*-ker({y}) ∩ G_1 = ϕ and G_1∪G_2= X, then X –G_1 and X –G_2 are βg^*-open sets such that (X −G_1∩X – G_2)= ϕ. Put X−G_1= U_2 and X−G_2= U_1. Thus βg^*-ker({x})⊆U_1 and βg^*-ker({y})⊆U_2 and U_1∩U_2= ϕ, so that x∈U_1 and y∈U_2 implies x∉ βg^*-cl({y}) and y∉ βg^*-cl({x}), then βg^*-cl({x})⊆U_1 and βg^*-cl({y})⊆U_2. Thus (X, τ) is βg^*-R_1 space.

Corollary 4.20: A topological space (X, τ) is βg^*-R_1 space if and only if for each x≠y∈X with βg^*-cl({x})≠ βg^*-cl({y}) there exist disjoint βg^*-open sets U and V such that βg^*-cl(βg^*-ker({x}))⊆U and βg^*-cl(βg^*-ker({y}))⊆V.

Proof: Let (X, τ) be a βg^*-R_1 space and let x≠y∈X with βg^*-cl({x})≠ βg^*-cl({y}), then there exist disjoint βg^*-open sets U and V such that βg^*-cl({x})⊆U and βg^*-cl({y})⊆V. Also (X, τ) is βg^*-R_0 space implies by theorem 4.10, for each x∈X, then βg^*-cl({x})=βg^*-ker({x}), but βg^*-cl({x})= βg^*-cl(βg^*-cl({x})) = βg^*-cl(βg^*-ker({x})). Thus βg^*-cl(βg^*-ker({x}))⊆U and βg^*-cl(βg^*-ker({y}))⊆V.
Conversely, let for each x≠y∈X with βg^*-cl({x})≠ βg^*-cl({y}), there exist disjoint βg^*-open sets U and V such that βg^*-cl(βg^*-ker({x}))⊆U and βg^*-cl(βg^*-ker({y}))⊆V. Since {x}∈ βg^*-ker({x}) then βg^*-cl({x})⊆ βg^*-cl(βg^*-ker({x})) for each x∈X, so we get βg^*-cl({x})⊆U and βg^*-cl({y})⊆V. Thus (X, τ) is βg^*-R_1 space.

Theorem 4.21: A topological space (X, τ) is βg^*-T_0 space if and only if either y∉ βg^*-ker({x}) or x∉ βg^*-ker({y}), for each x≠y∈X.

Proof: Let (X, τ) be a βg^*-T_0 space then for each x≠y∈X, there exist βg^*-open set U such that x∈U, y ∉ U or x ∉ U, y ∈U. Thus if x∈U and y ∉ U then y∉ βg^*-ker({x}) or else if x ∉ U and y∈U then x∉ βg^*-ker({y}).
Conversely, let either y∉ βg^*-ker({x}) or x∉ βg^*-ker({y}), for each x≠y∈X. Then there exists βg^*-open set U such that x∈U, y ∉ U or x ∉ U, y ∈U. Thus (X, τ) is βg^*-T_0 space.

Theorem 4.22: A topological space (X, τ) is βg^*-T_1 space if and only if for each x≠y∈X, y∉ βg^*-ker({x}) and x∉ βg^*-ker({y}).
Proof: Let (X, τ) be a βg^*-T_1 space then for each x≠y∈X, there exists βg^*-open sets U, V such that x∈U, y ∉ U and y ∈V, x ∉ V implies y∉ βg^*-ker({x}) and x∉ βg^*-ker({y}).
Conversely, let y∉ βg^*-ker({x}) and x∉ βg^*-ker({y}), for each x≠y∈X. Then there exists βg^*-open sets U, V such that x∈U, y ∉ U and y ∈V, x ∉ V. Thus (X, τ) is βg^*-T_1 space.

Theorem 4.23: A topological space (X, τ) is βg^*-T_1 space if and only if for each x≠y∈X, βg^*-ker({x})∩ βg^*-ker({y}) = ϕ.

Proof: Let (X, τ) be a βg^*-T_1 space. Then βg^*-ker({x}) = {x} and βg^*-ker({y}) = {y}. Thus βg^*-ker({x}) ∩ βg^*-ker({y}) = ϕ.
Conversely, let for each x≠y∈X implies βg^*-ker({x}) ∩ βg^*-ker({y}) = ϕ and suppose that (X, τ) be not βg^*-T_1 space then by theorem 4.21 we get for each x≠y∈X implies y∈ βg^*-ker({x}) or x∈ βg^*-ker({y}), then βg^*-ker({x}) ∩ βg^*-ker({y})≠ ϕ this is contradiction. Thus (X, τ) is βg^*-T_1 space.

Corollary 4.24: Let (X, τ) be a topological space. A βg^*-T_1 space is βg^*-T_2 space if and only if one of the following conditions holds:

1. For each x≠y∈X with βg^*-cl({x})≠ βg^*-cl({y}), then there exist βg^*-open sets U, V such that βg^*-cl(βg^*-ker({x}))⊆ U and βg^*-cl(βg^*-ker({y}))⊆ V.
2. For each x≠y∈X with βg^*-ker({x})≠ βg^*-ker({y}), then there exist βg^*-closed sets F_1, F_2 such that βg^*-ker({x})⊆F_1, βg^*-ker({x}) ∩F_2= ϕ and βg^*-ker({y})⊆ F_2, βg^*-ker({y}) ∩F_1 and $F_1 \cup F_2$ = X.

References

[1]. D.Andrijevic, semi preopen sets, Mat.Vesnik, 38(1) (1986), 24-32
[2]. K.Balachandran, P.Sundaram and H.Maki, On generalized continuous maps in topological spaces, Mem.Fac.sci.Kochi.Univ.Math.,12(1991),5-13.
[3]. C.Dhanapakyam ,K.ndirani,On βg*closed sets in topological spaces, Int. J. App. Research (2016),388-391
[4]. N.Levine, Generalized Closed sets in Topology, rend.Cir.Mat.palermo,2(1970),89-96. N.Levine, Semiopen sets and semi continuity in topological spaces.,Amer.Math.Monthly, 70(1963),36-41.
[5]. M.K.R.S Veerakumar, Between closed sets and g-closed sets, Mem. Fac. Sci.Kochi Univ.Ser.A,Math., 21 (2000) 1-19.

C. Dhanapakyam. "βg* – Separation Axioms." IOSR Journal of Mathematics (IOSR-JM) 15.1 (2019): 07-14