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On Commutativity Property Of $Q_{k,m,n}$, $P_{k,m,n}$, $P_{k,m,\infty}$ and $Q_{k.m,\infty}$ Rings

[1] G.Gopalakrishnamoorthy, H.Habeeb Rani[2] and [3] V Thiripurasundari
[1]Advisor , Sri Krishnasamy arts and science college, Mettamalai, Sattur.
[2]Assistant Professor of Mathematics, H.K.R.H.College, Uthamapalayam, Theni district, Tamil nadu.
[3]Assistant Professor of Mathematics, Sri S.R.N.M.College, Sattur,Virudhunagar district,Tamil nadu
Corresponding Author: G.Gopalakrishnamoorthy

ABSTRACT: *We study commutativity in Rings R with the property that for fixed positive integers k,m,n, $x^kS^m = S^mx^k$ for all $x\in R$ and for all n-subsets S of R.*

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I. Introduction

Recently G.Gopalakrishnamoorthy and S.Anitha have defined $Q_{k,n}$ -rings by the property that $x^kS=Sx^k$ for all $x\in R$ and for all n-subsets S of R. They also have defined $Q_{k,,\infty}$- rings by the property that $x^k S = Sx^k$ for all $x\in R$ and for all infinite subsets S of R,and defined $P_{k,n}$- ring to be a ring R with the property that XY=YX for all k-subsets X of R and n-subsets Y of R.. Also they have defined $P_{k,\infty}$ -ring by the property that XY = YX for all k-subsets X of R and all infinite subsets Y of R. obviously every $Q_{k,n}$ -ring is a $P_{k,n}$- ring and every $P_{k,n}$-ring is a $P_{k,\infty}$ -ring .It is proved that any $Q_{k,n}$ –ring with identity such that $|R| > n$,is commutative. If $n \leq 4, Q_{k,n}$ –rings are commutative .If $n \leq 8$,every $Q_{k,n}$ –ring with 1 is commutative.
In this paper we define $Q_{k,m,n}$ -rings and $P_{k,m,n}$ –rings, thus generalizing the above concepts and discuss their commutativity.

II. Preliminaries

Let R be an arbitrary ring not necessarily with identity.Let D,N,Z and C(R) denote the set of zero divisors, the set of nilpotents, the center and the commutator ideal of R respectively. Let $|R|$ denote the cardinality of R.For any subset Y of R,let CR(Y),A1(Y), Ar(Y) and A(Y) denote the centralizer of Y,the left,right and two sided annihilators of Y respectively.For $x,y\in R$ the set Lx,y,k is defined to be $\{w \in R | x^k y = wx^k\}$where k≥1 is a fixed integer.

2.1 Definition
Let k,m,n be three fixed positive integers.A ring R is said to be $Q_{k,m,n}$ ring if $x^kS^m=S^mx^k$ for all $x\in R$ and for all n-subsets S of R.
where $|R|> n$ and $S^m=\{s^m/s\in S\}$

2.2 Definition
Let k,m,n be three fixed positive integers.A ring R is said to be $P_{k,m,n}$ ring
$X^mY^m = Y^mX^m$ for all k-subsets X of R and n-subsets Y of R.

2.3 Definition
Let k,m be two fixed positive integers.A ring R is said to be $Q_{k,m,\infty,}$ ring if $x^kS^m=S^mx^k$ for all $x\in R$ and for all infinite subsets S of R.
where $|R|> n$ and $S^m=\{s^m/s\in S\}$

2.4 Definition
Let k, m, be two fixed positive integers.A ring R is said to be $P_{k,m,\infty,}$ ring
$X^mY^m = Y^mX^m$ for all k-subsets X of R and for all infinite subsets Y of R.
Taking m=1 we note that XY=YX for all k-subsets X of R and for all infinite subsets Y of R.
We simply call $P_{k,1,\infty}$ ring as a $P_{k,,\infty}$ ring.

2.5 Note

- **i.** Every $Q_{k,m,n}$ ring is a $Q_{k,m,\infty}$ ring
- **ii.** Every $Q_{k,m,n}$ ring is a $P_{k,m,n}$ ring
- **iii.** Every $P_{k,m,n}$ ring is a $P_{k,m,\infty}$ ring
- **iv.** Every $Q_{k,m,\infty}$ ring is a $P_{k,m,\infty}$ ring
- **v.** Every $Q_{k,,\infty}$ ring is a $P_{k,\infty}$ ring

2.6 Definition

Let R be a ring and I be a subset of R.Let (k, m) be fixed positive integers.I is said to be a left (k,m)- ideal of R if

i. x^m , $y^m \in I \Rightarrow x^m - y^m \in I$ and

ii. $x^m \in I, r \in R \Rightarrow r^k x^m \in I$

Similarly the right (k,m)-Ideal and two sided (k,m) ideal can be defined.

2.7 Lemma

Let R be a $Q_{k,m,n}$ ring with $|R| > n$ .Then

i. for all $x \in R, x^k R^m = R^m x^k$

ii. If x^k is idempotent then x^k commutes with the m^{th} power of every $a \in R$

iii. N is a (k,m) ideal of R.

iv. $|A_r(x^k)^m| = |A_l(x^k)^m|$

v. If R is not commutative and $(x^k)^m \notin Z$ then $R \backslash A_l(x^k)^m \cup C_R(x^k)^m$ and $R \backslash A_r(x^k)^m \cup C_R(x^k)^m$ are non-empty.

Proof:

Let R be a $Q_{k,m,n}$ ring with $|R| > n$

i. $z \in R^m x^k$ iff $z = r^m x^k$ for some $r \in R$

iff $z \in S^m x^k$ for some n-subsets $S \subset R$

iff $z \in x^k S^m$ for some n-subsets $S \subset R$

iff $z = x^k (s^m)'$ for some $s \in S \subset R$

iff $z \in x^k R^m$

i.e, $R^m x^k = x^k R^m$ for all $x \in R$

ii. Let $x \in R$ be such that x^k is idempotent.Then for all $a \in R$

$x^k a^m = x^{2k} a^m$ (since x^k is idempotent)

$= x^k (x^k.a^m)$

$= x^k (a^m.x^k)$ (since $x^k R^m = R^m x^k$)

$= (x^k.a^m)\, x^k$

$= (a^m.x^k)\, x^k$ (since $x^k R^m = R^m x^k$)

$= a^m x^{2k}$

$= a^m x^k$ (since x^k is idempotent)

Hence x^k commutes with the m^{th} power of every $a \in R$

iii. Let $x^m, y^m \in N$.clearly $x^m + y^m \in N$ (adopt the standard proof that N is an ideal in commutative rings)

Since $x^m \in N$, $(x^m)^n = 0$ for some $n \geq 1$.

For all $r \in R$

$(r^k x^m)^n = (r^k x^m)(r^k x^m) \ldots\ldots (r^k x^m)$ n times

$= r^k (x^m r^k)(x^m r^k) \ldots\ldots (x^m r^k)\, x^m$.

$= r^k (r^k x^m)(r^k x^m) \ldots\ldots (r^k x^m)\, x^m$ (using (i))

$= r^{2k} (x^m r^k)(x^m r^k) \ldots\ldots (x^m r^k) x^{2m}$

$= r^{2k} (r^k x^m)(r^k x^m) \ldots\ldots (r^k x^m)\, x^{2m}$

$= r^{3k} (x^m r^k)(x^m r^k) \ldots\ldots (x^m r^k) x^{3m}$

$= \ldots\ldots\ldots\ldots\ldots\ldots\ldots$

$= r^{nk} x^{nm}$

$= 0$ (since $(x^m)^n = x^{mn} = 0$)

$(r^k x^m)^n = o$

Hence $r^k x^m \in N$

Thus $x^m \in N, r \in R \Rightarrow r^k x^m \in N$

So, N is a (k,m) ideal of R

iv. Also,

$z^m \in A_l(x^k)^m$ iff $z^m x^k = 0$
iff $x^k z^m = 0$ (using (i))
iff $z^m \in A_r(x^k)^m$

Hence $|A_l(x^k)^m| = |A_r(x^k)^m|$

v. Let R be a non-commutative ring and x^k does not belongs to Z.Then there exist y ∈R such that $x^k y^m \neq y^m x^k$ Consequently y does not belongs to $C_r(x^k)^m$.So $C_r(x^k)^m$ is a proper subgroups of (R,+).Then from (i) and (iv) imply that $A_l(x^k)^m$ and $A_r(x^k)^m$ are also proper subgroups of (R,+).Since a group cannot be the union of two proper subgroups,(v) is proved.

2.8 Note

This generalizes lemma 2.8[4].

2.9 Lemma

If R is an infinite $Q_{k,m,n}$ ring then R is commutative.

Proof

Let R be an infinite $Q_{k,m,n}$ ring.
If R is commutative then there is nothing to prove.
Suppose R is non-commutative.Since all $Q_{1,1,1}$ rings are commutative,k>1 ,m>1 and n>1 .
Assume that R is not a $Q_{k,m,s}$ ring for any s<n Then there exist x∈R and an (n-1) subset H of R such that $x^k H^m \neq H^m x^k$.Since R is infinite $R \backslash H \neq \emptyset$
For any $a \in R|H, x^k(H \cup \{a\})^m = (H \cup \{a\})^m x^k$
So if we take $h \in H$ for which $x^k h^m$ does not belongs to $H^m x^k$
We have $x^k h^m = a^m x^k$ (1)
Since (1) holds for all $a \in R \backslash H$ it follows that for fixed
$b \in R \backslash H$,$R \backslash H^m \subseteq b^m + A_l(x^k)^m$ (2)
Moreover
if $c \in A_l(x^k)$, $(b^m+c^m)x^k = b^m x^k + c^m x^k$
$= b^m x^k$
$= x^k h^m \notin H^m x^k$
So $b^m + c^m \notin H^m$
That is $b^m + c^m \notin b^m + A_l(x^k)^m$
This implies $b+c \notin H$
Hence $b^m + A_l(x^k)^m \subseteq R-H^m$ (3)
Hence by (2) and (3) we have
$R-H^m = b^m + A_l(x^k)^m$
Hence $|R-H^m| = |A_l(x^k)^m|$ and
$|R \backslash A_l(x^k)^m| = |H^m|$
Since $A_l(x^k)$ is a proper subgroup of (R,+)
we have
$|R-A_l(x^k)^m| \geq |A_l(x^k)^m|$
That is $|H^m| > |R-H^m|$
The finiteness of H^m yields finiteness of R,contradicting
R is infinite.
Hence R is commutative

2.10 Note :

This generalizes lemma 2.10[4]

2.11 Lemma: (See[6])

If R is a finite ring with $N \subseteq Z$,then R is commutative.

2.12 Remark

Inview of lemma 2.11,we assume henceforth that R is finite.

III. Commutativity of $Q_{k,m,n}$ Rings

3.1Theorem If R is any $Q_{k,m,n}$ ring with identity such that |R|>n, then R is commutative.
Proof : If R is infinite,Commutativity follows from Lemma 2.9.So, assume R is finite.
By Lemma 2.11, we need only to show that $N \subseteq Z$.
Since u ε N implies 1+u is invertible,it sufficies to prove that invertible elements are central.
Let x ε R be an invertible element. If x ε Z, there is nothing to prove. Assume $x \notin Z$,
then $x^m \notin Z^m$. Hence $C_R(x^m)$ is a proper subset of R. Choose y ε R such that $y^m \notin C_R(x^m)$.
Then $y^m x^m \neq x^m y^m$. If H is any (n-1) subset R, which does not contain y, the condition
$(x^k)^m(\{y^m\} \cup H^m) = (\{y^m\} \cup H^m)(x^k)^m$ yields an z ε H such that

$$(x^k)^m y^m = z^m (x^k)^m \qquad (1)$$

Since x is invertible, there is unique z ε R satisfying (1). Thus we have proved that every(n-1) subsets of R contains either y^m or z^m.
But S = $|R-\{y^m,z^m\}|$ does not contain y^m and $z^m$and $|S| \geq n-1$, a contradiction. This contradiction proves that non – central invertible elements cannot exist. This proves the theorem.

3.2 Remark
This theorem generalizes theorem 3.1[4]

3.3 Theorem
Let n≥4 and let R be a $Q_{k,m,n}$ ring |R|> 2n -2 or if n is even and |R|> 2n-4.
Then $(x^k)^m$ εZ for all x ε R.
Proof : le n≥4 and R be a $Q_{k,m,n}$ ring.
We shall prove that if there exists x ε R such that $(x^k)^m \notin Z$, then |R|≤ 2n -2 or |R|≤ 2n -4.
Since (n-1)< 2n-4,we may suppose that |R|≥ n. Suppose there exists x ε R such that$(x^k)^m \notin Z$, by Lemma 2.8(v), there exists

$$y \in R \setminus \{A_r(x^k)^m\} \cup C_R(x^k)^m\}.$$

If H is any (n-1) subset which doesnot contain y, we have
$(x^k)^m(\{y^m\} \} \cup H^m) = (\{y^m\} \} \cup H^m)(x^k)^m$.
Since $(x^k)^m y^m \neq z^m (x^k)^m$, there exists z^m ε H such that That $(x^k)^m y^m = z^m (x^k)^m$ is $z^m \in L_{x,y,k}$.
So $H^m \cap L_{x,y,k} \neq \phi$.
Thus we have proved that any (n-1) subset of R must either contain y or intersect $L_{x,y,k}$
This condition cannot hold if $|R - L_{x,y,k}| \geq n$.
So, $|R- L_{x,y}| \leq (n-1)$
That is $|R| \leq |L_{x,y,k}| + (n-1)$ (1)
Now, if w ε $L_{x,y,k}$ then $L_{x,y,k} = w + A_1(x^k)^m$
Hence $|L_{x,y,k}| = |A_1(x^k)^m|$.
Again by Lemma 2.8 (v), $A_1(x^k)^m \neq R$.
So $|L_{x,y,k}| = \frac{|R|}{P}$ for some p ≥ 2.
Substituting in(1), we get $|R| \leq \frac{|R|}{P} + (n-1)$
i.e, $|R|(1- 1/p) \leq (n-1)$
i.e, $|R| \leq \frac{p}{P-1}(n-1) \leq 2n-2$ (2)
Suppose that n is even, If $(A_1(x^k)^m)$ has index atleast 3 in (R,+), the inequality (2) yields

$$|R| \leq \frac{3(n-1)}{2} \leq 2n-4$$

Thus we may assume that $|A_1(x^k)^m| = \frac{|R|}{2}$ We shall show that |R|≠ 2n-2.
Suppose |R| = 2n – 2, then $|A_r(x^k)^m| = \frac{2n-2}{2} = (n-1)$. So $|A_l(x^k)^m| = (n-1)$
We note that $A_l(x^k)^m$ is an (n-1) subset not intersecting $L_{x,y,k}$.
Hence y ε $A_l(x^k)^m$.
Since $y \in R \setminus \{A_r(x^k)^m \cup C_R(x^k)^m\}$, we see that $y \notin A_r(x^k)^m$.
So, $A_l(x^k)^m \neq A_r(x^k)^m$ and consequently $A_r(x^k)^m (x^k)^m \neq 0$.
Now, $x^{km}(\{y^m\} \cup A_r(x^k)^m) = (\{y^m\} \cup A_r(x^k)^m)(x^k)^m$and therefore $A_r(x^k)^m(x^k)^m \subseteq \{(x^k)^m, y^m, 0\}$.
Hence $A_r(x^k)^m(x^k)^m = \{0, x^k, y^m\}$ is an additive subgroup of order 2.
Hence the map $\phi: A_r(x^k)^m \rightarrow A_r(x^k)^m(x^k)^m$ given by $\phi(w) = w(x^k)^m$ has kernel of index 2 in $A_r(x^k)^m$.
But $|A_r(x^k)^m|$ is odd and so we have a contradiction.
Hence $|R| \leq 2n - 4$.

3.4 Lemma (Theorem 5|5|)
Suppose the ring R is such that $x^{n(x)}$ ε Z, the centre of R, for all x ε R. Then if R has no non – zero nilideals, it must be commutative.

3.5 Theorem
Let n ≥ 4 and R be a $Q_{k,m,n}$ ring. If |R|> $2n-2$ or if n is even and |R| > $2n-4$, then R is commutative, provided R has no non- zero nilideals.
Proof: Follows from Theorem 3.3 and Lemma 3.4.

3.6 Theorem
Let n ≥ 4 and let R be a $Q_{k,m,n}$ ring with |R|> $\frac{3}{2}$(n-1). Then R is commutative, if one the following is satisfied

i. |R| is odd.
ii. (R,+) is not the union of three proper subgroups.
iii. N is commutative.
iv. $R^3 \neq \{0\}$.

Proof: (i) Assume |R| is odd.
Suppose that R is not commutative.
Since, |R|> $\frac{3}{2}$ (n-1)> n
The arguments in the proof of throrem 3.3 gives
$| A_r(x^k)^m | = | A_r(x^k)^m | = |R|/2$
This is impossible. So, R must be commutative.
(ii) Assume (R,+) is not the union of three proper subgroups. Suppose that R is not commutative. Then by (i), |R| is even. By applying the first isomorphism throrem of groups:
$|(x^k)^m R| = |R(x^k)^m| = 2.$
Hence for any u ε R \ $A_r(x^k)^m$, $(x^k)^m R = \{0, (x^k)^m u\}$ and for any v ε R\ $A_r(x^k)^m$
$R(x^k)^m = \{0, v(x^k)^m\}$
By Lemma 2.8(i),
$(x^k)^m R = R(x^k)^m$.
If y ε R\ $A_l(x^k)^m \cup A_r(x^k)^m$ then
$\{0,(x^k)^m y\} = (x^k)^m R = R(x^k)^m = \{0, y(x^k)^m\}$
Hence y ε $C_R(x^k)^m$
Thus R= $A_l(x^k)^m \cup A_r(x^k)^m \cup C_R(x^k)^m$ which is a contradiction to our assumtion that (R,+) is not the union of three proper subgroups.
(iii) Assume N is commutative.
Suppose R is not commutative. Then by throrem 3.1, if R is any $Q_{k,m,n}$ ring with 1 such that
|R|> n, then R is commutative.
Now, R doesnot have 1. Hence, R = D fro R is finite. If $(x^k)^m \notin N$, some power of $(x^k)^m$ is an idempotent zero divisor e ≠ 0.
Since, $A_r(x^k)^m \subseteq A_1(e)$ and $A_1(e) \neq R$,
We must have $A_l(x^k)^m = A_1(e)$
And similarly $A_r(x^k)^m = A_r(e)$
By lemma 2.8 (ii),e is central.
Hence, $A_l(x^k)^m = A_r(x^k)^m = A(x^k)^m \subseteq C_R(x^k)^m$.
Thus if y $\notin A(x^k)^m$ then y $\notin A_l(x^k)^m$ and y $\notin A_r(x^k)^m$.
Y $\notin A_l(x^k)^m$ → y ε R\ $A_l(x^k)^m$.
→ $\{0,(x^k)^m y\} = (x^k)^m R = R(x^k)^m = \{0, y(x^k)^m\}$
Hence y ε $C_R(x^k)^m$ which is the contradiction to the assumption that $(x^k)^m \notin$ Z.
Hence $(x^k)^m$ is a non – central element.
If there exits two non – commutative elements, which is a contradicton to the assumption that N is commutative.
(iv) Assume $R^3 \neq 0$.
Suppose R is not commutative. Then there exists x ε R such that x $\notin$ Z. the fact that $(x^k)^m$ ε N yields
$A_r(x^k)^m \supseteq A_r(x^k)^m$
So, $A_r(x^k)^m = R$
Hence, $(x^{k+1})^m R = R(x^{k+1})^m = (0)$
Choose y ε R\ $A_r(x^k)^m \cup C_R(x^k)^m$) and w ε R\ $A_r(x^k)^m \cup C_R(x^k)^m$))
Then $y^{k+1} R = R y^{k+1}$
More over, $\{0, (x^k)^m y\} = (x^k)^m R = R(x^k)^m = \{0, w(x^k)^m\}$ so that $(x^k)^m y = w(x^k)^m$

Thus $(x^k)^m R^2 = (x^k)^m yR = w(x^k)^m R = \{w (x^k)^m y,0\} = \{xy^{k+1},0\} = (0)$
If z ε Z then $(x^k)^m + Z \notin Z$ so that $(x+Z)R^2 = \{0\}$
Hence $R^3 = \{0\}$, which is a contradiction to the fact that $R^3 \neq \{0\}$.
Hence R is commutative.

IV. Further Results for Small n

4.1 Theorem
If $n \leq 8$, then every $Q_{k,m,n}$ ring with 1 is commutative.
Proof: Let R be any $Q_{k,m,n}$ ring with identity. Suppose R is not commutative.
We any assume that n=8.
Then by theorem 3.1, $|R| \leq 8$.
Since all rings with 1 having fewer than 8 elements are commutative, |R| = 8 and R is indecomposable. Since idempotents are central, we must have no idempotents except 0 and 1. Hence every element is either nilpotent or invertible. Since u ε N,1+u is invertible. If follows from lemma 2.12, there exists a pair x^m,$y^m$of non- commuting invertible elements. The groups of units is not commutative and has almost 7 elements,hence is isomorphic to S_3. Thus, there exists a unique non – zero nilpotent element u which by Lemma 2.12 is not central. Hence there is an invertible element w such that $u(w^k)^m \neq (w^k)^m u$. By Lemma 2.8(iii), $(w^k)^m$ and $u(w^k)^m$ are non-zero nilpotents. This gives a contradiction.
So, R is ccommutative.

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