IOSR Journal of Mathematics (IOSR-JM)
e-ISSN: 2278-5728, p-ISSN: 2319-765X. Volume 15, Issue 1 Ser. I (Jan – Feb 2019), PP 57-70
www.iosrjournals.org

Fluctuation around the Gamma function and a Conjecture

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Abstract: *Using the expansion of the log of the ξ and the ζ functions in terms of the Pochammer's Polynomials, we obtain a fast convergence sequence for the first two Li-Keiper coefficients. The sequences are of oscillatory type. Then we study the oscillating part of the Li-Keiper coefficient (the "tiny" oscillations) and following some analytical calculations we pose a new conjecture in the form of a kind of "stability bound" for the maximum strength of the fluctuations around the mean staircase.*

Key words: *Pochammer's Polynomials, ξ function (Xi), Li-Keiper coefficients, Baez-Duarte and Maslanka expansions, trend and "tiny" fluctuations.*

Date of Submission: 25-01-2019 Date of acceptance: 07-02-2019

I. Introduction

Recently, Matiyasevich has given a new interesting Formula for the first Li coefficient, i.e. a representation of it with positive summands by means of the binary sequence for the Euler constant γ and for log(π), which reads [1]:

$$\lambda_1 = \sum_{j=1}^{\infty}\left(\frac{1}{\rho_j}+\frac{1}{1-\rho_j}\right) = \frac{1}{2}\cdot(\gamma-\log(4\pi)+2) = \sum_{n=3}^{\infty}\frac{\frac{1}{2}\cdot(2N_1(n)+3)}{[(2n)\cdot(2n+1)\cdot(2n+2)]}$$

Where: $\frac{1}{\rho_j}$ is the reciprocal of any nontrivial zero of the Zeta function and $N_1(n)$ is the number of units i.e. of 1 in the binary expansion of n. See also previous related works [2, 3]. In the first part of this work we will find another representation of λ_1 and of λ_2 which is not in the form of positive summands but that it is given by an alternating sequence of rational numbers and Zeta value of half integer arguments emerging from the representation of the ξ function. The sequence is nevertheless "fast" converging to the true values of the two constants. We use a Pochammer's representation of ξ at special values of the parameters (α, β) given in a systematic analysis of some representation of the Zeta function [4, 5, 6, 7, 8], that is here α =1/2, β =1. We do not comment on a point of view present in the literature concerning the utility or not of a representation of the Zeta function involving values of the Zeta function at integer arguments [9, 10, 14].

Some serious treatments have appeared in these years and various numerical experiments have been extensively pursued in addition to some rigorous partial results obtained in some pioneering works [4, 5,12]. Here, since dealing with representations of functions at the border of the domain of absolute convergence (i.e. at s ~ 1) no additional proofs seems to be necessary and our strategy is motivated by the results of our analytical and numerical experiments.

II. The expansion of the log of ξ (z) and others functions by means of Pochammer's Polynomials

We start with the expression of ξ (s), i.e. the Xi function where s = σ + i·t is the usual complex variable. Introducing the new variable z given by z = 1-1/s i.e. s = 1/(1-z), so that the critical line s = 1/2 + i·t is mapped onto the unit circle abs(z) = |z| = 1 we then have [13, 14]

$$\xi(z) = (½)\cdot z\cdot[1/(1-z)^2]\cdot\pi^{-(1/(2\cdot(1-z))}\cdot\Gamma(1/(2\cdot(1-z)))\cdot\zeta(1/(1-z)) \qquad (1)$$

where Γ(s/2) is the Gamma function of argument s/2 and ζ(s) is the Zeta function of argument s. Notice also that $\pi^{-(s/2)} = \pi^{-1/(2\cdot(1-z))}$.

We then consider the Pochammer's Polynomials in the complex variable s of degree k , given by [4 ,5]:

$P_K(s) = \prod_{r=1}^{k}\left(1-\frac{s}{r}\right)$ for k ϵ N and $P_0(s) = 1$ for all s. (2)

It is then interesting to consider the log of the ξ function where

$$\xi(s) = \xi(0)\cdot\prod_{\rho}\left(1-\frac{s}{\rho}\right)$$

[13, p.52] (ρ is also any nontrivial zero of ζ(s))and also the function

$$\log[(s-1)\cdot\zeta(s)] \qquad (3)$$

For Re(s) >Re(s_o) where s_o is a complex number we have:

$$\frac{1}{(s-s_{0)}} = \int_0^\infty e^{(-\lambda\cdot(s-s_0))}\cdot d\lambda = e^{(-\lambda\cdot(s-s_0))}\cdot\left[1-(1-e^{(-\beta)}\right]^{(s-\alpha)/\beta}\cdot d\lambda$$

$$= \sum_{k=0}^{\infty} P_k\left(\frac{s-\alpha}{\beta}+1\right)\cdot\left(\sum_{j=0}^{k}(-1)^j\cdot\binom{k}{j}\cdot\left(\frac{1}{\alpha+\beta\cdot j-s_0}\right)\right) \qquad (4)$$

in terms of a two-parameter family of Pochammer's Polynomials [8].
Notice that for a constant C we have:

$$= \sum_{k=0}^{\infty} P_k\left(\frac{s-\alpha}{\beta}+1\right)\cdot\left(\sum_{j=0}^{k}(-1)^j\cdot\binom{k}{j}\cdot(C)\right) = P_0\cdot C = C. \quad (5)$$

For Eq. (3) we then have:

log(ζ(s))·(s-1)) =

$$= \sum_{k=0}^{\infty} P_k\left(\frac{s-\alpha}{\beta}+1\right)\cdot\left(\sum_{j=0}^{k}(-1)^j\cdot\binom{k}{j}\cdot[\log((\alpha+\beta\cdot j-1)\cdot\zeta(\alpha+\beta\cdot j)]\right)$$

$$= \sum_{k=0}^{\infty} P_k\left(\frac{s-\alpha}{\beta}+1\right)\cdot d_k \qquad (6)$$

where $d_k = \left(\sum_{j=0}^{k}(-1)^j\cdot\binom{k}{j}\cdot[\log((\alpha+\beta\cdot j-1)\cdot\zeta(\alpha+\beta\cdot j)]\right)$ (7)

Notice that in d_k, in the last factor, it appears α+β·j instead of s.
Similarly for log(ξ(s)) we also have :

$$\log(\xi(s)) = \sum_{k=0}^{\infty} P_k\left(\frac{s-\alpha}{\beta}+1\right)\cdot\left(\sum_{j=0}^{k}(-1)^j\cdot\binom{k}{j}\cdot\log(\xi\cdot(\alpha+\beta\cdot j)\right)$$

$$= \sum_{k=0}^{\infty} P_k\left(\frac{s-\alpha}{\beta}+1\right)\cdot d_k \qquad (8)$$

where $d_k = \left(\sum_{j=0}^{k}(-1)^j\cdot\binom{k}{j}\cdot[\log(\xi(\alpha+\beta\cdot j)]\right)$ (9)

We analyse these Formulas without further comments.
In the next Section we will consider the case α =1/2 and β =1 and check the validity of the expressions given above by means of the Pochammer's Polynomials (a fast converging sequence of approximations) for ξ(s = 0) , λ_1 and λ_2,that is the first two Li-Keiper coefficients with the particular pair (α , β) given above.
In general we have [15] that

$$\log(\xi(z)) = \log(\xi(s=0)) + \sum_{n=1}^{\infty}\left(\frac{\lambda_n}{n}\right)\cdot z^n$$

i.e. the series around z = 0where λ_n is the n-then Li-Keiper coefficient.

III. The case [α=1/2, β = 1]

We are now interested to check the Formulas following the approach with the Pochammer's polynomials (infinite series).To the best of our knowledge the following numerical experiments have not been considered before along these lines.

III.1 The case α = ½, β = 1

$$\log(\xi(s)) = \sum_{k=0}^{\infty} P_k\left(s+\frac{1}{2}\right)\cdot\left(\sum_{j=0}^{k}(-1)^j\cdot\binom{k}{j}\cdot\log\xi\cdot\left(\frac{1}{2}+j\right)\right) = \sum_{k=0}^{\infty} P_k\left(s+\frac{1}{2}\right)\cdot d_k \tag{10}$$

Where: $d_k = \sum_{k=0}^{\infty}(-1)^k\cdot\binom{k}{j}\cdot\log\xi\cdot\left(\frac{1}{2}+j\right)$

and Eq.(6),

$$\log(\zeta(s))\cdot(s-1) = \sum_{k=0}^{\infty} P_k\left(s+\frac{1}{2}\right)\cdot\left(\sum_{j=0}^{k}(-1)^j\cdot\binom{k}{j}\cdot\left[\log\left(\left(j-\frac{1}{2}\right)\cdot\zeta\left(\frac{1}{2}+j\right)\right)\right]\right) \tag{11}$$

Remark

About the convergence we just remark that we use the Baez-Duarte inequality given by $|P_k(s)| < (c/k^{Re(s)})$ [4,12], in our case we have

$|P_k(((s+1/2)| < c/k^{1+\delta}$ at large k for Re(s)> δ+1/2 and the boundedness would be "ensured" if $|d_k| < r$ where r is a constant.

Below we analyze Eq.(10) and also Eq.(11). For Eq.(11) we have that

$$[\log\xi(1/2+j)] = \log[(\pi)^{-(1/2+j)/2}\cdot\ \Gamma(5/4+j/2)\cdot(j-1/2)\cdot\zeta(1/2+j)] \tag{12}$$

III.2 The first two Li-Keiper coefficients λ_1 and λ_2.

We now check Eq. (11) and Eq. (10) first for the case α = 1/2, β = 1 and compute the first two Li-Keiper coefficients with these parameters.

From above we know that the Li-Keiper coefficients enters in the log of the ξ(s) function and of its log derivative as:

$\log(\xi(s)) = \log(\xi(z)) = \log\left(\frac{1}{2}\right) + \sum_{n=1}^{\infty}\lambda_n\cdot\frac{z^n}{n}$ i.e. the expansion in z = 0 (z = 1- 1/s: s = 1, z= 0), (13)

$\frac{\xi'(s)}{\xi(s)} = \sum_{n=1}^{\infty}\lambda_n\cdot z^{n-1}$ i.e. the expansion in z= 0 of the log derivative.

For Eq. (10) we have:

$\log(\xi(z)) =$

$$\sum_{k=0}^{\infty} P_k\left(\frac{1}{(1-z)}+\frac{1}{2}\right)\cdot d_k^* = \sum_{k=0}^{\infty}\left(\frac{1}{k!}\right)\cdot\left(\frac{1}{(1-z)}\right)^k\cdot\prod_{r=1}^{k}\left(\frac{1}{2}-r\right)\cdot(a_r+z)\cdot d_k \tag{14}$$

Where $a_r = (r-3/2)/(1/2-r)$, r =1..k. In general, we have that:

$$\prod_{r=1}^{k}(a_r+z) = \prod_{r=1}^{k}(a_r)\cdot\left(\sum_{r=1}^{k}\left(\frac{1}{a_r}\right)\cdot z + \left(\left(\frac{1}{2}\right)\cdot\left(\sum_{r=1}^{k}\left(\frac{1}{a_r}\right)\right)^2 - \frac{1}{2}\cdot\sum_{r=1}^{k}\left(\frac{1}{a_r^2}\right)\right)\cdot z^2 + O(z^2)\right) \tag{15}$$

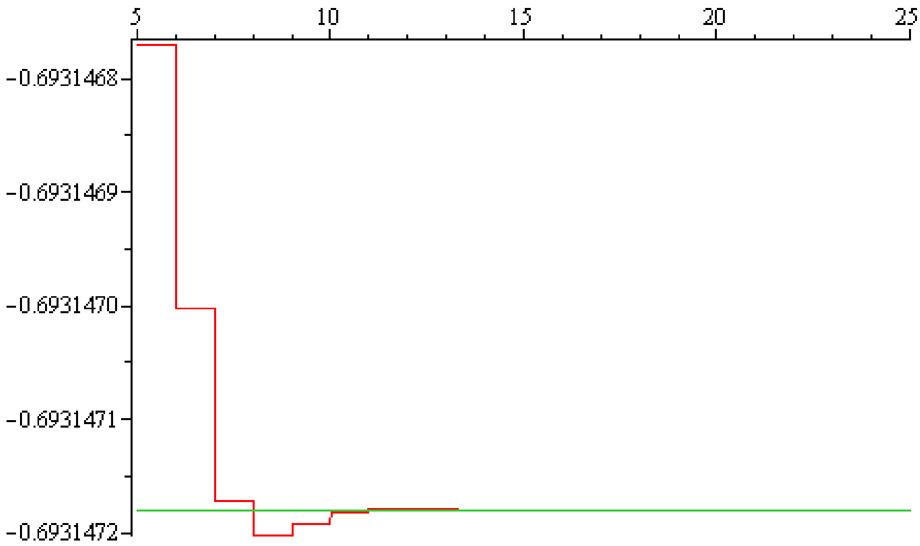

Fig.1. The constant log(ξ(1))= log(1/2) = - 0.69....

We then obtain the expression for logξ(1) and for the first two Li-Keiper coefficients using Eq.(10) as follows: log(ξ(1/(1-z)) at z = 0 is given by

$$\log(\xi(1)) = d_0 + \sum_{k=1}^{p} d_k \left(\frac{1}{k!}\right) \cdot \prod_{r=1}^{k} \left(r - \frac{3}{2}\right) = -0.69314718045 = log\left(\frac{1}{2}\right)$$

$$\lambda_1 = \sum_{k=1}^{p} d_k \left(\frac{1}{k!}\right) \cdot \prod_{r=1}^{k} \left(r - \frac{3}{2}\right) \cdot \left[k + \sum_{r=1}^{k} \frac{\left(\frac{1}{2} - r\right)}{\left(r - \frac{3}{2}\right)}\right] \quad (d_k\text{, from Eq.(10)}) \tag{16}$$

Up to some decimals we obtain: λ_1 = 0.02309570**90**. (0.02309570**89**...is the exact value up to 10 decimals). The exact value (Section1) is given by:

$$\lambda_1 = \frac{1}{2} \cdot (\gamma - \log(4\pi) + 2) = \frac{\gamma}{2} - \frac{1}{2}\, log 4\pi + 1 = 0.023 \ldots$$

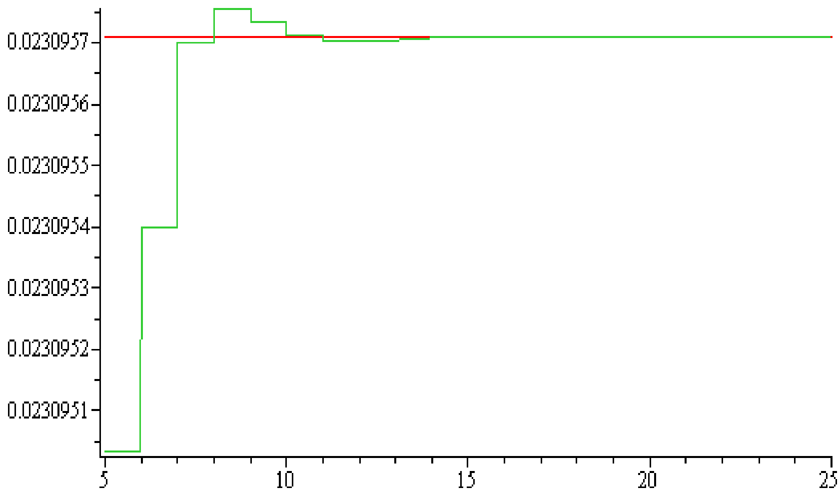

Fig.2. λ_1 (p) with Eq.(16) for p in the range 5-20.

At this point it is interesting to give a Table with our result given by Eq.(16) for λ_1 (See above Figure) and the formula of positive summands given by Matiyasevich. Notice that our Formula Eq.(16) is alternating but fast converging which may be of some interest for later more elaborated numerical experiments, also for $\lambda_k(p)$, k >1. We remember here that the formula of Matiyasevich is given by:

$$\lambda_1(p) = \sum_{n=3}^{\infty} \left(\frac{1}{2}\right) \cdot (2N_1(n) + 3) \cdot \left(\frac{1}{2n \cdot (2n+1) \cdot (2n+2)}\right)$$

where $N_1(n)$ is the number of units, i.e. of 1 in the binary expansion of the integer n (i.e. 1 for n=1, 1,10 for 2, 11 for 3 , 100 for 4 i.e. $N_1(4)=1$).

p	Matiyasevich	Eq.(16)	Exact
3	0.0104166667	0.0231132914	
6	0.0181429681	0.0230953998	
9	0.0205045706	0.0230957329	
12	0.0212906505	0.0230957042	
15	0.0216840359	0.0230957100	
18	0.0218474882	0.0230957089	
21	0.0219721434	0.0230957087	
24	0.0220600665	0.0230957090	0.02309570896

Table1 for λ_1 (p)

The formula of Matyiasevich for λ_1 is of course very interesting since it is also in connection with the binary system. We now continue with our Formulas and compute λ_2. From above:

$$\lambda_2 = \sum_{k=1}^{p} d_k \left(\frac{1}{k!}\right) \cdot \prod_{r=1}^{k} \left(r - \frac{3}{2}\right) \cdot \left[k \cdot \frac{(k+1)}{2} + \sum_{r=1}^{k} \frac{\left(\frac{1}{2} - r\right)}{\left(r - \frac{3}{2}\right)} + \frac{1}{2} \cdot \left(\sum_{r=1}^{k} \frac{\left(\frac{1}{2} - r\right)}{\left(r - \frac{3}{2}\right)}\right)^2 + \right.$$
$$\left. - \frac{1}{2} \cdot \left(\sum_{r=1}^{k} \frac{\left(\frac{1}{2} - r\right)}{\left(r - \frac{3}{2}\right)}\right)^2 \right] \qquad (17)$$

Up to some decimals we obtain: $\lambda_2 = 0.0923457352$.
The "exact" value is also known and given by $\lambda_2 = 0.0923457352$2

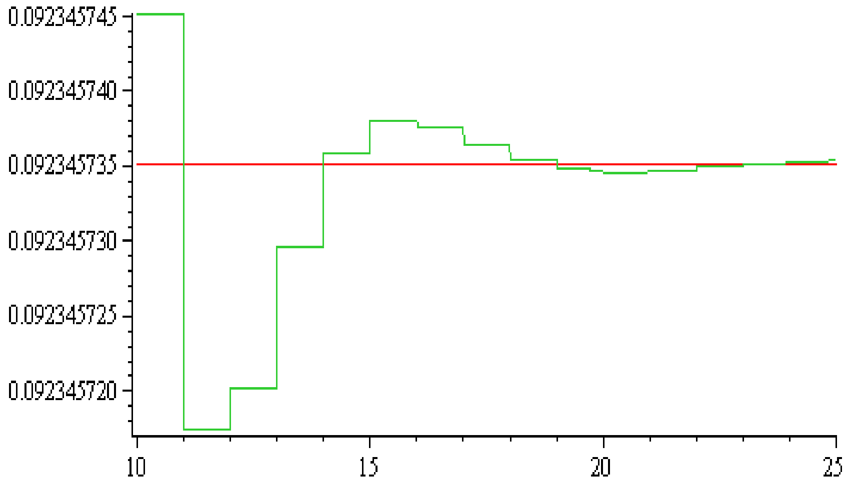

Fig. 3. λ_2 (k) from Eq.(17) for p in the range 10-25.

We conclude the analysis concerning the experiments using the Pochammer's polynomials for the first two Li-Keiper coefficients using the pair of parameters ($\alpha = 1/2$, $\beta = 1$) using directly the ξ function instead of its logarithm or its log derivative; we limit the calculation to the first coefficient λ_1. We have (See Eq.(1)):

$$\xi(z) = \frac{1}{2} \cdot z \cdot \left[\frac{1}{(1-z)^2}\right] \pi^{-\left(\frac{1}{2 \cdot (1-z)}\right)} \cdot \Gamma\left(\frac{1}{2 \cdot (1-z)}\right) \cdot \zeta\left(\frac{1}{1-z}\right)$$

$$= \pi^{-\left(\frac{1}{2\cdot(1-z)}\right)}\cdot\Gamma\left(1+\frac{1}{2\cdot(1-z)}\right)\cdot\left(\frac{z}{1-z}\right)\cdot\zeta\left(\frac{1}{1-z}\right)$$

(18)

The Taylor expansion of $\pi^{-(1/(2\cdot(1-z))}\cdot\Gamma(1+1/(2\cdot(1-z))$ to first order is $(½)\cdot\exp(1-\gamma/2-\log(4\cdot\pi))$ and now we compute to first order the Taylor expansion of $(z/(1-z)\cdot\zeta(1/(1-z)) = (s-1)\cdot\zeta(s)$. By means of the Pochammer's expansion given by:

$$\xi(s) = \sum_{k=0}^{\infty} P_k\left(\frac{s-\alpha}{\beta}+1\right)\cdot\left(\sum_{j=0}^{k}(-1)^j\cdot\binom{k}{j}\cdot(\alpha+\beta\cdot j-1)\cdot\left(\xi(\alpha+\beta\cdot j)\right)\right)$$

(19)

so that still for the case $\alpha = 1/2$ and $\beta=1$ we have:

$$(s-1)\cdot\zeta(s) = \sum_{k=0}^{\infty} P_k\left(s+\frac{1}{2}\right)\cdot\left(\sum_{j=0}^{k}(-1)^j\cdot\binom{k}{j}\cdot\left(j-\frac{1}{2}\right)\cdot\left(\zeta\left(\frac{1}{2}+j\right)\right)\right)$$

(20)

where the second factor is the d_k for this expansion. With $s=(1/(1-z))$, we now check that we obtain $1+\gamma\cdot z = 1+e^{\gamma\cdot z}$, at $z\sim 0$.
For the constant we have as above

$$f(p) = d_0 + \sum_{k=1}^{p} d_k\left(\frac{1}{k!}\right)\cdot\prod_{r=1}^{k}\left(r-\frac{3}{2}\right) = \left(-\frac{1}{2}\right)\cdot\zeta\left(\frac{1}{2}\right) + \sum_{k=1}^{p} d_k\left(\frac{1}{k!}\right)\cdot\prod_{r=1}^{k}\left(r-\frac{3}{2}\right)$$

while for the linear term in z we obtain:

$$g(p) = \sum_{k=1}^{p} d_k\left(\frac{1}{k!}\right)\cdot\prod_{r=1}^{k}\left(r-\frac{3}{2}\right)\cdot\left(k+\sum_{r=1}^{k}\left(\frac{\frac{1}{2}-r}{r-\frac{3}{2}}\right)\right)$$

the plots of the functions f and g are given below.

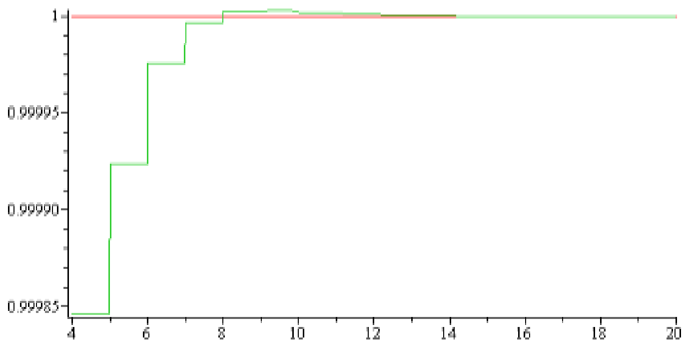

Fig. 4. Plot of f(p) in the range 1<p <30 and the constant 1 .

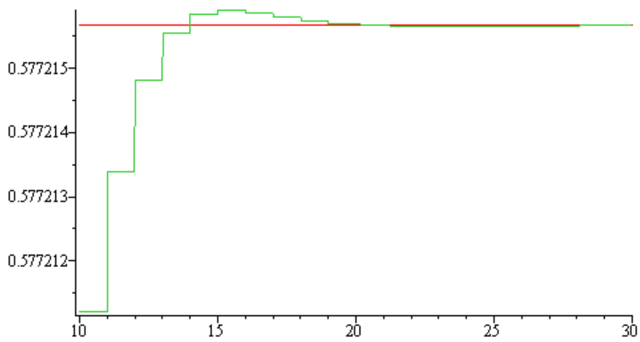

Fig. 5. Plot of g(p) and in the range 10<p<30 and γ = 0.577... , (γ is the Euler constant).

We thus have:

$\xi\,(z\rightarrow o)= \left(\frac{1}{2}\right)\cdot e^{\left(1-\frac{\gamma}{2}-\left(\frac{1}{2}\right)log(4\pi)+\gamma\right)\cdot z} = \left(\frac{1}{2}\right)\cdot e^{\lambda_1\cdot z}$

with

$\lambda 1 = 1+\frac{\gamma}{2}-\left(\frac{1}{2}\right)log(4\pi) = 0.0230957.$

It is interesting to write the formula in the Riesz case (where α = β =2) for the first Li-Keiper coefficient λ_1, following the expansion of (s-1)·ζ(s).We have that:

$$(s-1)\cdot\zeta(s) = \sum_{k=0}^{\infty} P_k\left(\frac{s-\alpha}{\beta}+1\right)\cdot\left(\sum_{j=0}^{k}(-1)^j\cdot\binom{k}{j}\cdot(\alpha+\beta\cdot j-1)\cdot\left(\zeta(\alpha+\beta\cdot j)\right)\right)$$

$$=\frac{1}{2}\cdot e^{\left(1-\frac{\gamma}{2}-\frac{1}{2}\log(4\pi)\right)\cdot z}\cdot\left(\left(\zeta(2)+\sum_{k=1}^{p}d_k\left(\frac{1}{k!}\right)\cdot\prod_{r=1}^{k}\left(r-\frac{1}{2}\right)\right)+\right.$$

$$\left.+\sum_{k=1}^{p}d_k\left(\frac{1}{k!}\right)\cdot\prod_{r=1}^{k}\left(r-\frac{1}{2}\right)\cdot\left(k+\sum_{r=1}^{k}\left(\frac{-r}{\left(r-\frac{1}{2}\right)}\right)\right)\cdot z\right) \tag{21}$$

where in $d_k = \sum_{j=0}^{k}(-1)^j\cdot\binom{k}{j}\cdot(1+2\cdot j)\cdot\left(\zeta\,(2+2\cdot j)\right)$, it appears, in addition, the factorial powers of π. Since

$\zeta\,(2+2\cdot j) = (-1)^{(1+2\cdot j)}\cdot 2^{(3+4\cdot j)}\cdot B_{4+4\cdot j}\cdot\pi^{4+4\cdot j}\cdot(1/(4+4\cdot j)!)$(22)

We should obtain from above: $(1/2)\cdot e^{(1-\gamma/2\ -(1/2)\cdot\log(4\pi))\cdot z}\cdot(1+\gamma\cdot z)$

The numerical computation gives in fact for p = 45:

$$\left(\zeta(2)+\sum_{k=1}^{p}d_k\left(\frac{1}{k!}\right)\cdot\prod_{r=1}^{k}\left(r-\frac{1}{2}\right)\right) = 0.9999974863435160$$

And

$$\sum_{k=1}^{p}d_k\left(\frac{1}{k!}\right)\cdot\prod_{r=1}^{k}\left(r-\frac{1}{2}\right)\cdot\left(k+\sum_{r=1}^{k}\left(\frac{-r}{r-\frac{1}{2}}\right)\right)\cdot z = 0.5772247299616524$$

The plot of the two above functions of p, approaching 1 and γ are given below.

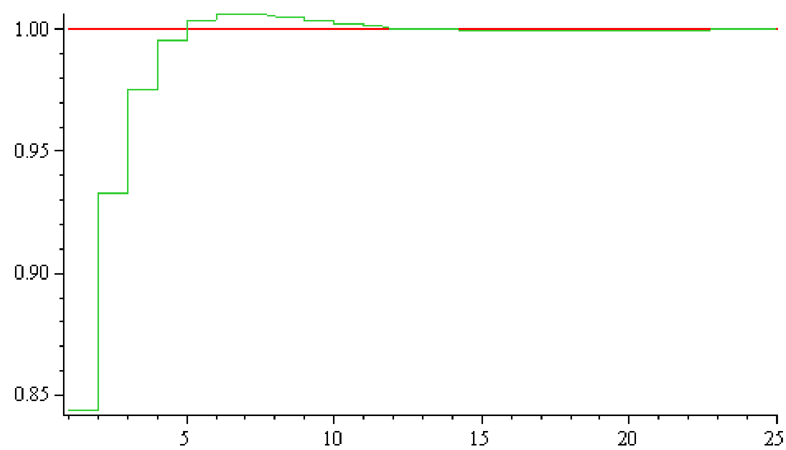

Fig. 6. The constant 1.

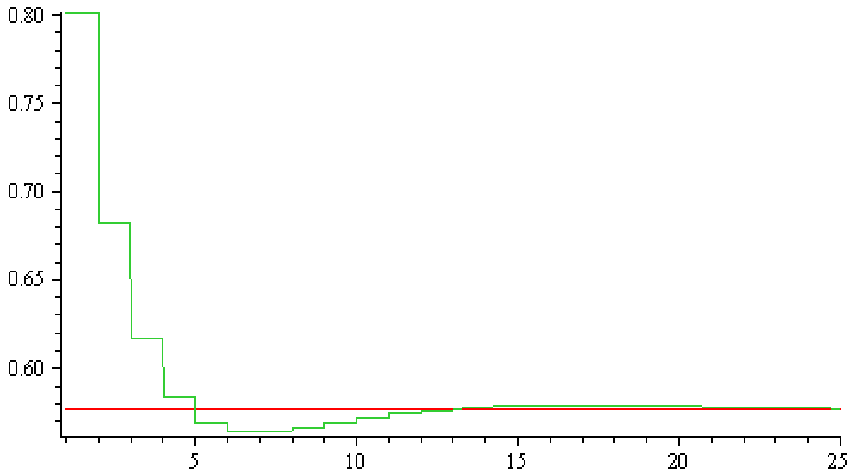

Fig.7. The "plot" of γ (p) in the range 1<p <25.

IV. Stability of the fluctuations: a conjecture

We now analyse more in details the fluctuations in the log of the ξ function that is the term log((s-1)·Zeta(s)). The term containing the Gamma function in ξ should give the major contribution to the n-then Li-Keiper coefficient of the order n , given by c.n.log(n), c> 0 for large n (called the trend).For some contributions in this direction, See the works in [15,16,17,18,19].

Here, concerning the tiny fluctuations [16] we make an analogy with the extensivity property of the free energy in statistical mechanics where the stability bound for the Hamiltonian take usually the form:

H > -c·N, c> o. Formula (10) for the fluctuations (still for the pairs α =1/2 , β=1) is given by:

log(ζ(s))·(s-1)) =

$$= \sum_{k=0}^{\infty} P_k\left(s+\frac{1}{2}\right)\cdot\left(\sum_{j=0}^{k}(-1)^j\cdot\binom{k}{j}\cdot\left[\log\left(\left(j-\frac{1}{2}\right)\cdot\zeta\left(j+\frac{1}{2}\right)\right)\right]\right)$$

where we call $d^{**}(k)$ the second factor containing the binomials. Then, for the contribution of the fluctuation to the first Li-Keiper coefficient we obtain, using Eq. (16) above:

$$\lambda_1^{**}(p) = \sum_{k=1}^{p} d_k^{**}\left(\frac{1}{k!}\right)\cdot\prod_{r=1}^{k}\left(r-\frac{3}{2}\right)\cdot\left[k+\sum_{r=1}^{k}\frac{\left(\frac{1}{2}-r\right)}{\left(r-\frac{3}{2}\right)}\right] = \sum_{r=1}^{p}\Delta_1(k)$$

Notice that $\Delta_1(1) = -d^{**}(1)$= -[log((-1/2·ζ (1/2)-log(1/2· ζ (3/2)]= 0.58158....

Moreover, we easily check that $\Delta_1(k)$ is a decreasing function of k. Thus - $\lambda_1^{**}(p)$is the sum of positive summands with an upper bound given by minus the Euler constant i.e. - γ = - 0.577.... and the lower bound given by – $\lambda_1^{**}(1)$ = - 0.58158... This bares some similarity with the property of positive summands given by Matiyasevich in [1] as reported above. Our function - $\lambda_1^{**}(p)$is represented on the Figure 8 with the constant - γ .

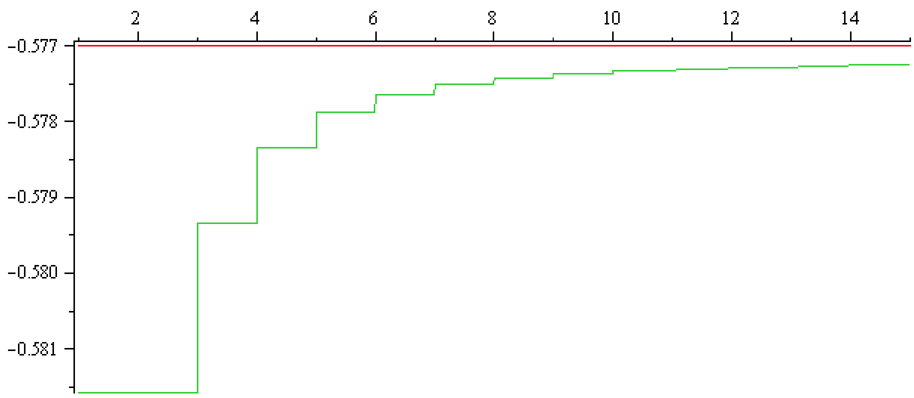

Fig.8. The function - $\lambda_1^{**}(p)$(in green) , - γ (in reed) , and
- $\lambda_1^{**}(p)$>> - $\lambda_1^{**}(1)$· 1 = - $\lambda_1^{***}(1)$= - 0.58158....

On the Table 2 we give explicitly the increments $\Delta_1(k)$.

k	$-\Delta_1(k)$
2	0
3	0.00223699
4	0.00099514
5	0.00046471
6	0.00024213
7	0.00013827
8	0.00008460
9	0.00005456
10	0.00003670
15	0.00000778
20	0.00000254
25	0.00000106
30	$5.191510298.10^{-7}$

Table 2

We now study the contribution of the fluctuations to the second Li-Keiper coefficient. We have similarly as above (Eq. (16)) that:

$$\lambda_2^{**}(\mathrm{p}) = \sum_{k=1}^{p} d^{**}(k)\cdot\left(\frac{1}{k!}\right)\cdot\prod_{r=1}^{k}\left(r-\frac{3}{2}\right)\cdot\left[\frac{k\cdot(k+1)}{2}+k\cdot\sum_{r=1}^{k}\frac{\left(\frac{1}{2}-r\right)}{\left(r-\frac{3}{2}\right)}+\right.$$

$$\left.+\frac{1}{2}\cdot\left(\sum_{r=1}^{k}\frac{\left(\frac{1}{2}-r\right)}{\left(r-\frac{3}{2}\right)}\right)^2-\frac{1}{2}\cdot\left(\sum_{r=1}^{k}\frac{\left(\frac{1}{2}-r\right)}{\left(r-\frac{3}{2}\right)}\right)^2 = \sum_{k=1}^{p}\Delta_2(k)\right]$$

Moreover, here too we easily check that $\Delta_2(k)$ is a decreasing function of k. Notice that $\lambda_2^{**}(1) = \lambda_1^{**}(1) = 0.58158...$

Thus – $\lambda_2^{**}(p)$ is here too the sum of positive summands with the lower bound equal to - 0.58158.The function – $\lambda_2^{**}(p)$ is represented on the Figure below.

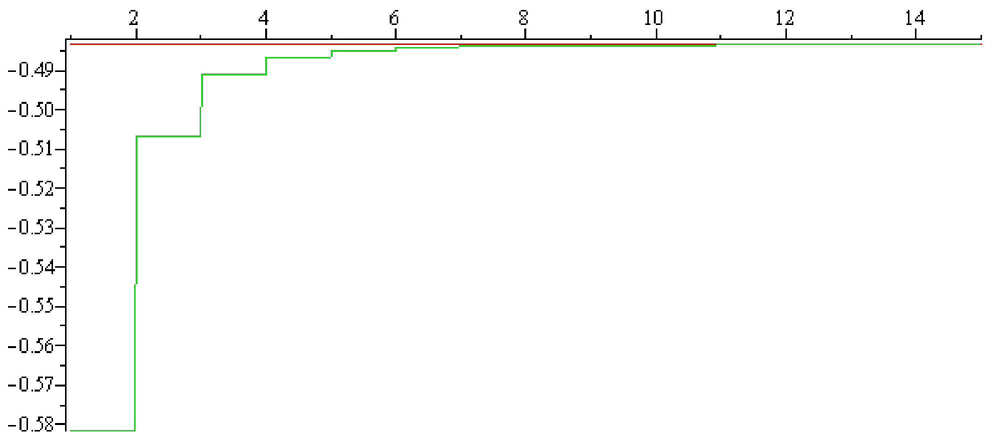

Fig. 9. The function – $\lambda_2^{**}(p)$ and his limit - 0.483448.. =-(1/2).0.9668850..(taken from Ref. [16] and $\lambda_2^{**}(1)$ = - 0.58158 . - $\lambda_2^{**}(p) > -\lambda_2^{**}(1) = -(1/2).\lambda_2^{***}(1).\lambda_2^{***}(1) = 2.\ \lambda_2^{**}(1)$ = 2.0.58158 = 1.16316.

Thus, - $\lambda_2^{***}(p)$ is the sum of positive summands and $\lambda_2^{***}(p) < 2.\ \lambda_2^{**}(1) = 2.\lambda_1^{***}(1)$= 2.0.58158...
The Table 3 gives explicitly the increments $\Delta_2(k)$.

k	$\Delta_1(k)$
2	0.07472470
3	0.01565897
4	0.00447817
5	0.00164285
6	0.00072314
7	0.00036198
8	0.00019849
9	0.00011642
10	0.00007197
15	0.00001091
20	0.00000273
25	$8.985216176\cdot 10^{-7}$
30	$3.481499094\cdot 10^{-7}$

Table 3

Here too we have the property of positive increments of $\Delta_2(k)$ as for $\Delta_1(k)$ bearing some analogy with the Matiyasevich pioneering result on λ_1, i.e. λ_1, as sum of positive summands.
For the contribution to the third coefficient λ_3, defining:

$$s(k) = \sum_{r=1}^{k} (a(r), q(k)) = \sum_{r=1}^{k} a(r)^2 \text{and } c(k) = \sum_{r=1}^{k} a(r)^3$$

Then (Eq. (15))

$$\lambda_3^{**}(p) = \sum_{k=1}^{p} d^{**}(k) \cdot \left(\frac{1}{k!}\right) \cdot \prod_{r=1}^{k} \left(r - \frac{3}{2}\right) \cdot$$

$$\cdot \left[\begin{array}{l} \left(\frac{1}{3}\right) \cdot c(k) - \left(\frac{1}{2}\right) \cdot s(k) \cdot q(k) + \left(\frac{1}{6}\right) \cdot s(k)^3 + \left(\frac{1}{6}\right) \cdot k \cdot (k-1) \cdot (k-2) - \left(\frac{1}{2}\right) \cdot k^2 \cdot (k-1) + \\ - \left(-\left(\frac{1}{2}\right) \cdot k^2 - \left(\frac{1}{2}\right) \cdot k\right) \cdot k + k \cdot \left(-\left(\frac{1}{2}\right) \cdot q(k) + \left(\frac{1}{2}\right) \cdot s(k)^2\right) + \left(-\left(\frac{1}{2}\right) \cdot k \cdot (k-1) + k^2\right) - s(k) \end{array} \right]$$

$= \sum_{k=1}^{p} \Delta_3\ (k).$

Here too $\Delta_3(k)$ is a decreasing function of k and $\lambda_3^{**}(1) = 0.58158..$ as above. Thus, $-\lambda_3^{**}(p)$ is of the kind above with a lower bound given by -0.58158..The function $-\lambda_3^{**}(p)$ is represented below.

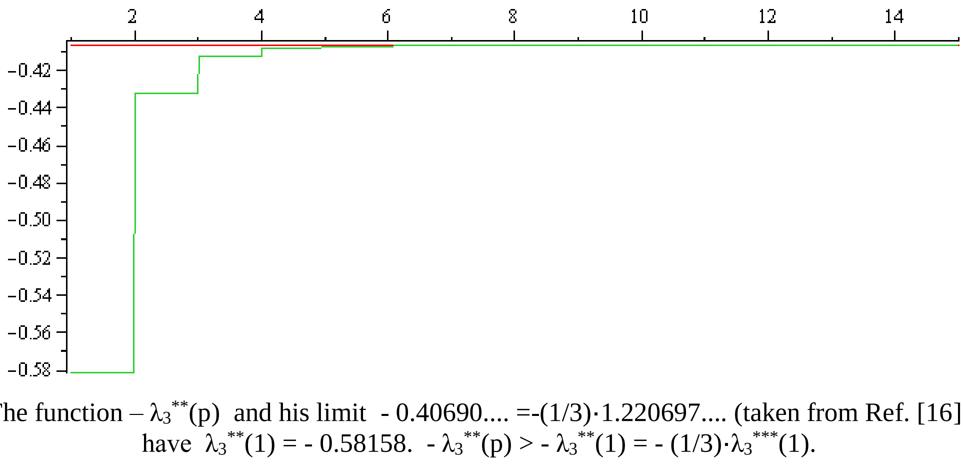

Fig. 10. The function $-\lambda_3^{**}(p)$ and his limit - 0.40690.... =-(1/3)·1.220697.... (taken from Ref. [16] . We also have $\lambda_3^{**}(1) = -0.58158$. $-\lambda_3^{**}(p) > -\lambda_3^{**}(1) = -(1/3)\cdot\lambda_3^{***}(1)$.

$\lambda_3^{***}(1) = 3\cdot \lambda_3^{**}(1) = 3\cdot 0.58158 = 1.74474$.
Thus, $-\lambda_3^{***}(p)$ is the sum of the kind above and $\lambda_3^{***}(p) < 3\cdot \lambda_2^{**}(1) = 3\cdot\lambda_1^{***}(1)$= 3.0.58158...Finally we also see that $-\lambda_4^{**}(p)$ is increasing and then stabilizes at the limit - 0.34389= - (1/4)·1.375588 in agreement with the value taken from Ref [161 and $\lambda_4^{***}(p) < 4\cdot\lambda_2^{**}(1) = 4\cdot 0.58158 = 2.32632$.
The function is represented below with his limit.

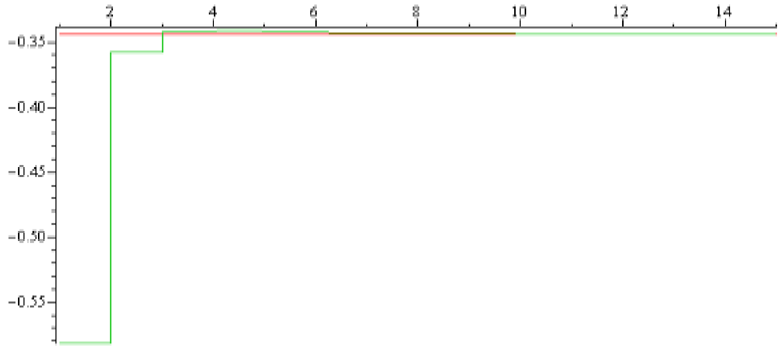

Fig.11 The function above and his limit -0.34389...

We have thus found that $-\Delta_i(k)$ is negative for i=1,2,3,4 and that $\lambda_i^{**}(1) = 0.58158$ indicating that $\lambda_i^{***}(p) < i\cdot\lambda_1^{***}(1) = i\cdot 0.58158$.
Moreover, from the Formula above, for any n> 0, the first term i.e. at p= k =1 is obtained as:

$$P_1\left(s+\frac{1}{2}\right)\cdot d_1^{**}(k=1) = P_1\left(\frac{1}{(1-z)}+\frac{1}{2}\right)\cdot 0.58158$$

since

$$d_1^{**}(k=1) = \left(\sum_{j=0}^{1}(-1)^j\cdot\binom{k}{j}\log\left(\left(j-\frac{1}{2}\right)\cdot\zeta\left(j+\frac{1}{2}\right)\right)\right) =$$

$$= \log\left(-\frac{1}{2}\right)\cdot\zeta\left(\frac{1}{2}\right) - \log\left(\frac{1}{2}\right)\cdot\zeta\left(\frac{3}{2}\right) = 0.58158\ldots$$

and for n>0, the series expansion of

$$P_1\left(\frac{1}{(1-z)}+\frac{1}{2}\right) = \prod_{r=1}\left(1-\frac{\frac{1}{(1-z)}+\frac{1}{2}}{r}\right) =\cdot \frac{\left(-\frac{1}{2}\right)(1+z)}{(1-z)} = \left(-\frac{1}{2}\right)\sum_{n=1}1\cdot z^n$$

and

$$P_1\left(s+\frac{1}{2}\right)\cdot d_1^{**}(k=1) = -P_1\left(\frac{1}{(1-z)}+\frac{1}{2}\right)\cdot d_1^{**}(k=1) = -0.58158\cdot\left(\sum_{n>0}z^n\right)$$

i.e. the stability bound for each n> 0 (k=1!) is expected to be given by:
$H(n) > -0.58158\cdot n$

We stop here and we introduce the following conjecture.

Conjecture
The fluctuations are bounded for every n by $n.\lambda_1^{***}(1) = n..0.58158$ in **absolute value** reflecting the extensivity (in n) of the "stability" bound $H(n) > -n.\ 0.58158$ where $H(n) = \text{limit}(-n.\ \lambda_n^{**}(p)\ \ p\rightarrow\infty)$ i.e. from Ref [16] the "tiny" fluctuations (> or <0) are very small .
The main contribution (the trend as reported in Ref. [16] , given in terms of the Gamma function , behaves like c. n. log(n) at large n and fit well with the explicit formula reported below: so there is no doubt that the trend behaves [16], as

$$\lambda_n(\text{trend}) = (3/4)+(½)\cdot(\gamma -1-\log(2\cdot\pi))\cdot n +(½)\cdot n\cdot\log(n) = (¾) + c\cdot n + (½)\cdot n\cdot\log(n) \tag{A}$$

where $c = (½)\cdot(\gamma -1-\log(2\cdot\pi) = -1.13....$
A numerical experiment to obtain c within 1% -2% of the true value above is reported in the Appendix, by means of a sequence of approximations, the first few values of which are given by

$c_1 = -1.07582,\ \ c_2 = -1.10465,\ c_3 = -1.12023$

Moreover the trend is given by the exact formula [17,18]

$\lambda_n(trend) = 1 - \frac{1}{2} \cdot (log(4\pi) + \gamma) \cdot n + \sum_{j=2}^{n} (-1)^j \cdot \binom{n}{j} \cdot (1 - 2^{-j}) \cdot \zeta(j)$ (B)

Notice that in the first formula (A) we take ¾ instead of ½ as in Ref [16].The excellent agreement between the Maslanka results and the above expression (B) for the trend is illustrated on the Figure below.

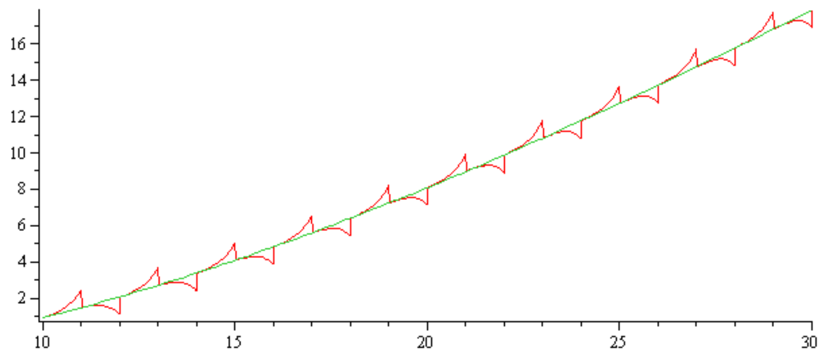

Fig.12. In red Eq. (B) and in green Eq. (A),10 <n <30 .

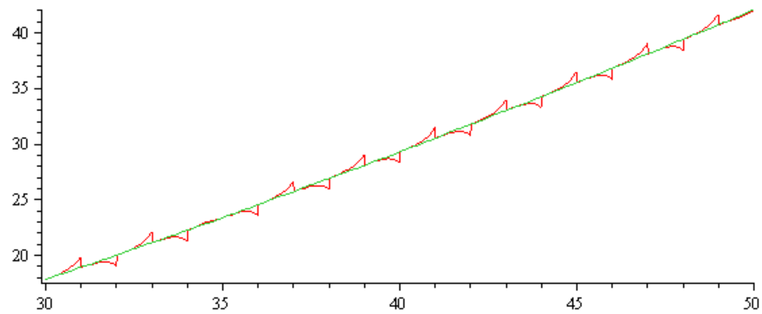

Fig. 13. In red Eq. (B) and in green Eq. (A), 30<n <50.

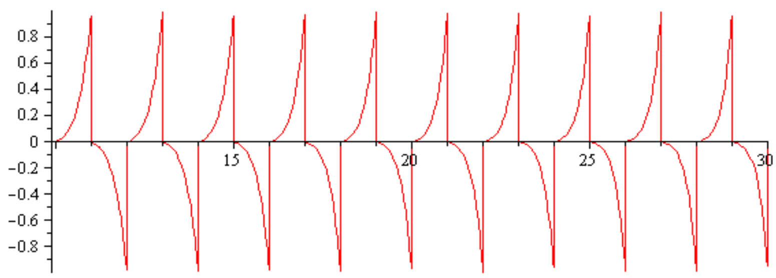

Fig.14. The difference Δ between the function in Eq. (A) and that in Eq.(B) , 10<n <30. Δ = 0.000...

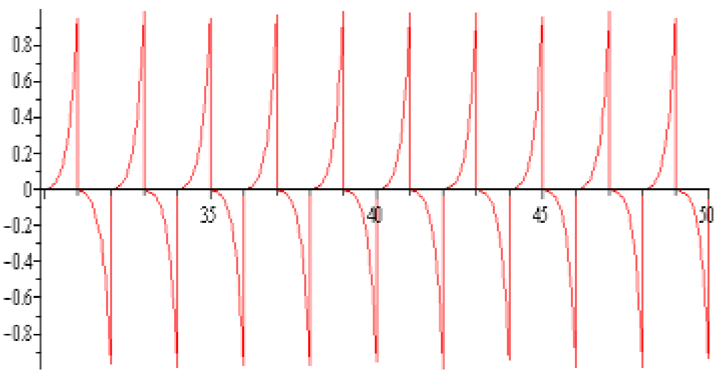

Fig.15. The difference Δ between the function in Eq. (A)

and that in Eq.(B), 30 <n <50. (Δ=0.000..)
If our conjecture is true, then the Gamma function is "responsible" for the positivity property of the Li-Keiper coefficients which ensure the truth of the Riemann Hypothesis .

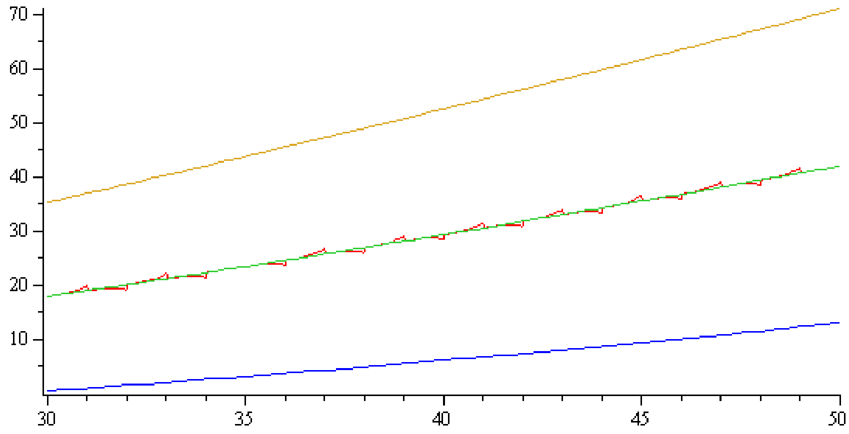

Fig. 16. The two function (trends) and the same increased or decreased by the maxima of the fluctuations (± 0.58158·n) following the above conjecture. In the plot, 30 <n <50.

Remark
Assuming our conjecture we have the inequality: λ_n> (3/4)+ (½)· (γ -1-log(2·π))·n +(½)·n ·log(n)-(log((1/2)·Zeta(3/2))·n= (½)·n· log(n)+ n.(γ-1-log(2· π)-(log((1/2)·Zeta(3/2))+(3/4).
Notice that (log[(1/2)·Zeta(3/2)] = 0.58158.. as given above.
The coefficient of n in our conjecture is given by the number - (½)· 3.42 near and a little above the lower bound conjectured in [19] and given by - (½))· 3.56.

V. Concluding remark

In the first part of this work we have found - using an expansion by means of the Pochammer's Polynomials - a new alternating series for the first two Li-Keiper coefficients. The series converge very fast and the values have been compared with those of the interesting sequence of positive summands of Matiyasevich for the first Li-Keiper coefficients λ_1. For λ_k, k > 2 further investigations are necessary to test more about the fast or not convergence of the associated sequences.
We then studied the fluctuations (the oscillatory part of the Li-Keiper coefficients) and then proposed a conjecture that these should be bounded in absolute value by c·n where c= 0.58158.. so that the trend -which behaves like c.n.log(n)with c>0 - should be the dominant part, which would ensure the truth of the RH ($\lambda_k \geq 0$ for all integers k).

Appendix

We want to obtain an approximation to the constant c appearing in the Eq. (A) above for the trend. For this, we start with the Formula

$$\log(\xi\,(z)) = log\left(\frac{1}{2}\right) + \sum_{n=1}^{\infty} \lambda_n \cdot \frac{z^n}{n} = log\left(\left(\frac{1}{2}\right) \cdot s \cdot (s-1) \cdot \pi^{-\frac{s}{2}} \cdot \Gamma\left(\frac{s}{2}\right) \cdot \zeta(s)\right)$$

where s=1/(1-z) .
We take the first 19 calculated values of lambda from Ref [16] and from n=20 to infinity we assume the Formula $\lambda_n \sim (n/2)\cdot\log(n) + c\cdot n$.
In the right hand side of the above Formula we insert s= 1/(1-z) and calculate with the sequence of the first few values $z_1 = 0.9$, $z_2 = 0.99$, $z_3 = 0.999$. With these three values of z we have solved the linear Equation for c and we have found the values c_n, n=1,2,3 given above exact up to few decimals of the exact value given by c = -1.13... [16].

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Danilo Merlini. " Fluctuation around the Gamma function and a Conjecture." IOSR Journal of Mathematics (IOSR-JM) 15.1 (2019): 57-70.