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Isomorphic Finite Group Automata

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***Abstract**: Let B = (Q, *, Σ, δ, q_0, F) and B' = (Q', Δ , Σ, δ', q_0', F') be two Finite Group Automata. Then a mapping Ψ : B → B' is said to be a Finite Group Automata isomorphism or simply FGA isomorphism if 1. Ψ is a FGA homomorphism, 2.Ψ is 1-1 and 3.Ψ is onto. Examples of isomorphic Finite Group Automata are given. If there is a FGA isomorphism from B onto B' , there will be a FGA isomorphism from B' onto B. More generally, FGA Isomorphism is an equivalence relation among finite group automata.*

***Keywords:** Finite Group Automata, Finite Group Automata Homomorphism and Finite Group Automata isomorphism*

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I. Introduction

Finite Group Automata, and Finite Subgroup Automata were defined and many results were obtained. Commutative Finite Binary Automata, Associative Finite Binary Automata were defined. AC Finite Binary Automata was also defined. Many useful results were obtained. Now we define an isomorphism on Finite Group Automata. We see that FGA Isomorphism is an equivalence relation among finite group automata.

II. Preliminaries

Definition : Relation : Let A and B be non-empty sets. A subset ρ of A×B is called a relation.from A to B.
A subset of A×A is called a relation on A.
If an ordered pair (a,b) ϵ ρ, then we say a is related to b and we write it as aρb
A relation ρ defined on a set A is said to be reflexive if aρa, for all aϵA
A relation ρ defined on a set is said to be symmetric if aρa, then bρa
A relation ρ defined on a set is said to be transitive if aρb and bρc, then aρc
A relation is said to be an equivalence relation if it is reflexive, symmetric and transitive.

Definition : Finite Automaton: A finite automaton is a 5–tuple (Q, Σ, δ, q_0, F), where Q is a finite set of states, Σ is a finite input alphabet, q_0 in Q is the initial state, F $\subseteq$ Q is the set of final states, and δ is the transition function mapping Q x Σ to Q.
That is δ(q, a) is a state for each state and input symbol a.

Finite Group Automaton: A Finite Group Automaton B is a 6-tuple (Q, *, Σ, δ, q_0, F), where Q is a finite set of elements called states, Σ is a subset of non-negative integers, q_0 ϵ Q, q_0 is a state in Q called the initial state, F⊆Q and the set states (element) of F is said to be the set of final states, $\delta : Q \times \sum \rightarrow Q$ is the transition function defined by δ (q, n) = q^n = q * q *q *.......*q (n times) and * is a mapping from Q×Q to Q satisfying the following conditions.
(i) p * (q * r) = (p * q) * r , for all p,q,r in Q.
(ii) there exists a state denoted by 0 in Q such that p * 0 = p = 0 * p , for all p in Q
(iii) for each state p in Q there exists a state q in Q such that p * q = 0 = q * p.
Note : For n = 0, δ (q, n) = q^n => δ (q, 0) = q^0 , it is taken as 0

Definition : If for a state p in Q there exists a state q in Q such that p * q = 0 = q * p, then the state q is called the inverse state and the state p is called a invertible state in Q.
If a state p is invertible in Q and p * q = 0 = q * p, then the state q is also invertible.
If Σ* is the set of strings of inputs, then the transition function δ is extended as follows :
For m ϵ Σ* and n ϵ Σ, δ': Q×Σ* → Q is defined by δ'(q,mn) = δ(δ'(q,m),n).

If no confusion arises δ' can be replaced by δ.

Definition : Let B = (Q, *, Σ, δ, q_0, F) and B = (Q', Δ , Σ, δ', q_0', F') be two Finite Group Automata. Then a mapping Ψ : B → B' is said to be a **Finite Group Automata Homomorphism** or simply **FGA**

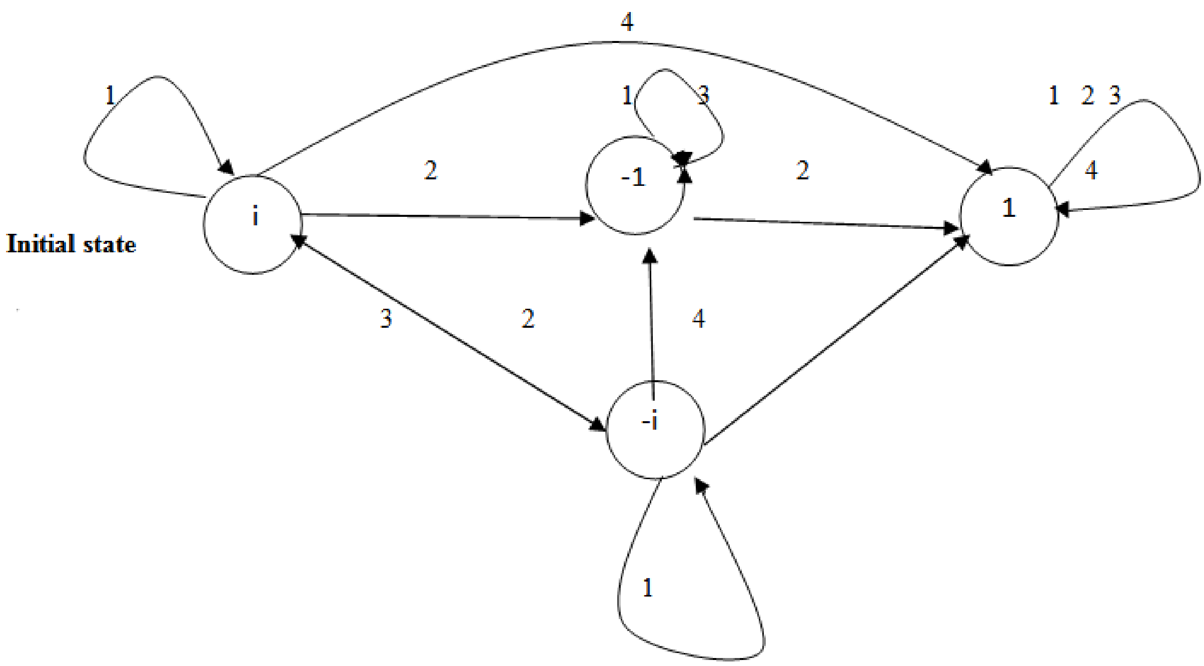
Homomorphism if
1. Ψ(a*b) = Ψ(a) Δ Ψ(b)
2. Ψ(δ(a,n)) = δ' (Ψ(a),n)
3. Ψ(q_0) = q_0'
4. a ϵ F if and only if Ψ(a) ϵ F'.

Definition : Let B = (Q, *, Σ, δ, q_0, F) and B' = (Q', Δ , Σ, δ', q_0', F') be two Finite Group Automata. Then a mapping Ψ : B → B' is said to be a **Finite Group Automata isomorphism** or simply **FGA isomorphism** if
1. Ψ is a FGA homomorphism
2. Ψ is 1-1 and
3. Ψ is onto.

If there is an isomorphism from B onto B', then we write it as B ≈ B'.
This isomorphism is called Finite Group Automata Isomorphism or simply FGA isomorphism.

Example : Consider the Finite Group Automaton B= (Q, *, Σ, δ, q_0, F), where Q = {1,-1,i,-i}, Σ = {1,2,3,4} q_0 = i is the initial state and F=Q, the set of final states , δ is the transition function mapping from Q×Σ to Q defined by δ(q,n) = q^n, and * is the mapping from Q×Q to Q defined by the following table.

*	1	-1	i	-i
1	1	-1	i	-i
-1	-1	1	-i	i
i	i	-i	-1	1
-i	-i	i	1	-1



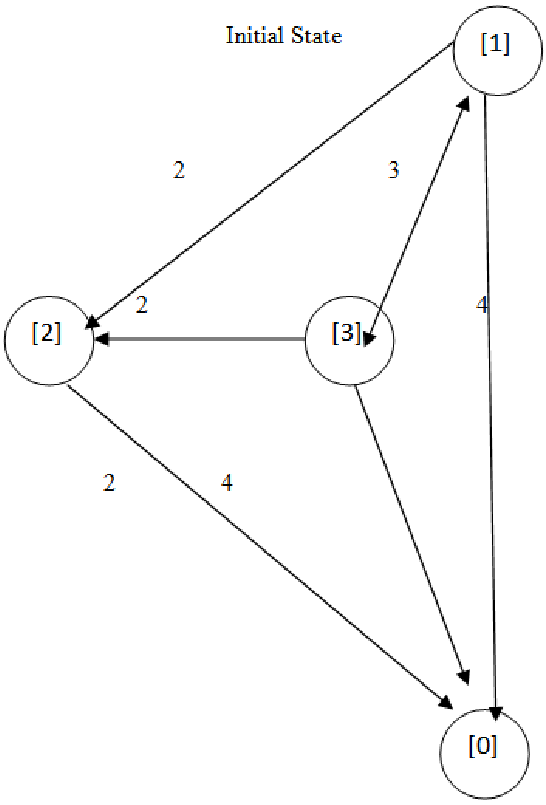
B= (Q, *, Σ, δ, q_0, F)

Let B' = (Q', ⊕ , Σ, δ', q_0', F'),
where Q' = Z_4 = { {[0],[1],[2],[3]}
$\oplus_4$ is the operation of addition modulo 4
[0] = the equivalence class determined by 0
[1] = the equivalence class determined by 1
[2] = the equivalence class determined by 2

[3] = the equivalence class determined by 3
$\Sigma = \{1,2,3,4\}$
$\oplus : Z_4 \times Z_4 \rightarrow Z_4$ is defined by the following Table

⊕	[0]	[1]	[2]	[3]
[0]	[0]	[1]	[2]	[3]
[1]	[1]	[2]	[3]	[0]
[2]	[2]	[3]	[0]	[1]
[3]	[3]	[0]	[1]	[2]

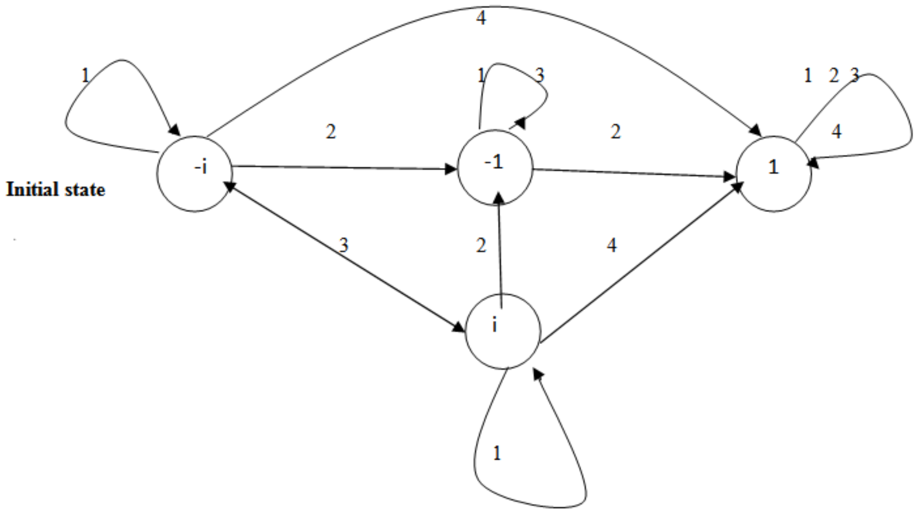
$\delta : Z_4 \times \Sigma \rightarrow Z_4$ is the transition mapping
$q_0' = [1]$
$F' = Q'$
Clearly B' = (Q', ⊕ , Σ, δ', q_0', F') is a Finite Group Automaton.



B' = (Q', ⊕ , Σ, δ', q_0', F'),

Define Ψ : B → B' by the following.
Ψ(1) = [0]
Ψ(i) = [1]
Ψ(-1) = [2]
Ψ(-i) = [3]
Ψ(δ(a,n)) = δ' (Ψ(a),n)
Then Ψ : B → B' is an FGA isomorphism.
Example : Consider the Finite Group Automaton B= (Q, *, Σ, δ, q_0, F), where Q = {1,-1,i,-i}, Σ = {1,2,3,4} q_0 = -i is the initial state and F=Q, the set of final states , δ is the transition function mapping from Q×Σ to Q defined by $\delta(q,n) = q^n$, and * is the mapping from Q×Q to Q defined by the following table.

*	1	-1	i	-i
1	1	-1	i	-i
-1	-1	1	-i	i
i	i	-i	-1	1
-i	-i	i	1	-1

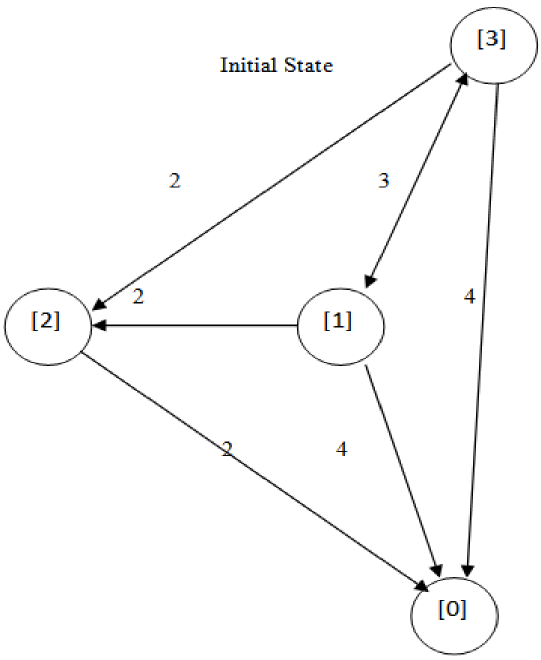


B= (Q, *, Σ, δ, q_0, F)

Let B' = (Q', ⊕ , Σ, δ', q_0', F'),
where Q' = Z_4 = { {[0],[1],[2],[3]}
$\oplus_4$ is the operation of addition modulo 4
[0] = the equivalence class determined by 0
[1] = the equivalence class determined by 1
[2] = the equivalence class determined by 2
[3] = the equivalence class determined by 3
Σ = {1,2,3,4}
⊕ : $Z_4 \times Z_4 \rightarrow Z_4$ is defined by the following Table

⊕	[0]	[1]	[2]	[3]
[0]	[0]	[1]	[2]	[3]
[1]	[1]	[2]	[3]	[0]
[2]	[2]	[3]	[0]	[1]
[3]	[3]	[0]	[1]	[2]

δ : $Z_4 \times \Sigma \rightarrow Z_4$ is the transition mapping
q_0' = [3], F' = Q'



B' = (Q', ⊕ , Σ, δ', q_0', F')

Clearly B' = (Q', $\oplus$, Σ, δ', q_0', F') is a Finite Group Automaton.
Define Ψ : B → B' by the following.
Ψ(1) = [0]
Ψ(i) = [1]
Ψ(-1) = [2]
Ψ(-i) = [3]
Ψ(δ(a,n)) = δ' (Ψ(a),n)
Then Ψ : B → B' is an FGA isomorphism.

Example : Let B= (Z_n, $\oplus$, Σ, δ, q_0, F),
where Z_n ={[0],[1],[2],[3],…………[n-1]}
$\oplus_n$ is the operation of addition modulo n
[0] = the equivalence class determined by 0
[1] = the equivalence class determined by 1
[2] = the equivalence class determined by 2
………………………………………………
[n-1] = the equivalence class determined by n-1,
Σ = {1,2,3,…….,n},
δ : Z_n× Σ → Z_n is the transition mapping,
q_0 = 0,
F = Z_n.
Then B= (Z_n, $\oplus_n$,Σ, δ, q_0, F) is a Finite Group Automaton.
Let B' = (Q', . , Σ, δ', q_0', F')
where Q'= the set of all n th roots of unity,
that is, Q' = { 1, ω,ω^2,ω^3,,……..,ω^{n-1}}, where ω^n = 1,
. is the multiplication,
δ' : Q'× Σ → Q' is the transition mapping,
q_0' = ω,
F' = Q'
B' = (Q', . , Σ, δ', q_0', F') is a Finite Group Automaton.
Define Ψ : B → B' by Ψ([k]) = ω^k.

1. Ψ is a homomorphism

(i) $\Psi(m \oplus_n k) = \omega^{m \oplus_n k}$
$= \omega^r$ ………………..(#)

$\Psi(m).\Psi(k) = \omega^m . \omega^k$
$= \omega^{m+k}$
$= \omega^{qn+r}$
$= \omega^{qn}\omega^r$
$= (\omega^n)^q\omega^r$
$= 1^q.\omega^r$
$= 1.\omega^r$
$= \omega^r$ ……………………(##)

From (#) and (##) we have $\Psi(m \oplus_n k) = \omega^r = \Psi(m).\Psi(k)$
Therefore $\Psi(m \oplus_n k) = \Psi(m).\Psi(k)$

(ii) $\Psi(\delta([m],k)) = \Psi([m]^k)$
$= \Psi([m] \oplus_n [m] \oplus_n [m] \oplus_n …….. \oplus_n [m])$
(k times)
$= \Psi([m])\Psi([m])\Psi([m]) …….. \Psi([m])$
(k times)
$= \omega^m\omega^m\omega^m …………… \omega^m$
$= \omega^{km}$ ………………($)

Now δ' (Ψ([m]),k) = δ'(ω^m , k)
$= (\omega^m)^k$
$= \omega^{mk}$ …………………….($$)

From ($) and ($$) we have Ψ(δ([m],k)) = ω^{mk} = δ' (Ψ([m]),k)
Therefore Ψ(δ([m],k)) = δ' (Ψ([m]),k)

(iii) Ψ(q_0) = Ψ([1]) = ω^1
= ω
= q_0'

(iv) [m] ϵ F if and only if Ψ([m]) = ω^m ϵ Z_n = F'

[m] ϵ F if and only if $\Psi([m])$ ϵ F'
Therefore $\Psi : B \to B'$ defined by $\Psi([k]) = \omega^k$ is a FGA homomorphism.
2. Ψ is 1-1
Let [m] , [k] ϵ Z_n

Suppose $\Psi([m]) = \Psi([k])$
$\omega^m = \omega^k$
[m] = [k]
3. Ψ is onto.
Let ω^k ϵ Q' = { 1, ω,ω^2,ω^3,,……..,ω^{n-1}} , for some k = 1,2,3,…..,n.
Clearly $\Psi(k) = \omega^k$
Therefore Ψ is onto.
Hence $\Psi : B \to B'$ defined by $\Psi([k]) = \omega^k$ is a FGA isomorphism.

Theorem : If there is a FGA isomorphism from B onto B' , there will be a FGA isomorphism from B' onto B. More generally, we have the following theorem.

Theorem : FGA Isomorphism is an equivalence relation among finite group automata.

Proof : Let B = (Q, *, Σ, δ, q_0, F) be a Finite Group Automaton.
Define I : B → B by I(a) = a for all a ϵ Q
Clearly 1) I(a*b) = a*b
=I(a)*I(b)
1) I(δ(a,n)) = δ(a,n)
=δ(I(a),n)
2) $I(q_0) = q_0$
3) For each a ϵ F, I(a) = a ϵ F
ie a ϵ F if and only if I(a) ϵ F
Therefore, I: B → B is a FGA isomorphism.
ie B ≈ B
Hence, ≈ is reflexive.
Let B = (Q, *, Σ, δ, q_0, F) and B' = (Q', Δ , Σ, δ', q_0', F') be two Finite Group Automata.
Assume B ≈ B'
Suppose Ψ : B → B' is a FGA isomorphism.
Then 1. Ψ is a homomorphism
2. Ψ is 1-1 and
3. Ψ is onto.
Since Ψ is 1-1 and onto, Ψ^{-1} exists.
Consider Ψ^{-1}: B' → B
Since Ψ is 1-1 and onto, Ψ^{-1} is also 1-1 and onto.
Since Ψ is a homomorphism, (i) $\Psi(a*b) = \Psi(a)\ \Delta\ \Psi(b)$
(ii) $\Psi(\delta(a,n)) = \delta'(\Psi(a),n)$
(iii) $\Psi(q_0) = q_0'$
(iv) a ϵ F if and only if $\Psi(a)$ ϵ F'
Now, let a', b' ϵ Q
Since Ψ is onto, there exist elements a and b in Q such that $\Psi(a)$ = a' and $\Psi(b)$ = b'
$\Rightarrow$ a = Ψ^{-1}(a') and b = Ψ^{-1}(b')
(i)' Ψ^{-1}(a' Δ b') = Ψ^{-1} ($\Psi(a)\ \Delta \Psi(b)$)
= Ψ^{-1} (Ψ(a*b)) (by (i))
= a*b
= Ψ^{-1}(a') * Ψ^{-1}(b')
(ii)' Let a' ϵ Q' and n ϵ Σ
There exists an element a in Q such that $\Psi(a)$ = a'.
$\Rightarrow$ a = Ψ^{-1}(a')
δ'(a', n) = δ'(Ψ(a) , n)
= Ψ(δ(a , n)) (by (ii))
$\Rightarrow$ Ψ^{-1}(δ'(a', n)) = δ(a , n)
= δ(Ψ^{-1}(a') , n) (since a = Ψ^{-1}(a'))
Now, clearly $\Psi(q_0) = q_0' \Rightarrow \Psi^{-1}(q_0') = q_0$

Let a’ ϵ F’
There exists an element a in Q such that Ψ(a) = a’.
Clearly Ψ(a) ϵ Q
By (iv) we have a ϵ F if and only if Ψ(a) ϵ F’
Therefore a ϵ F
But a = Ψ^{-1}(Ψ(a))
Therefore a = Ψ^{-1}(Ψ(a)) ϵ F
That is, a = Ψ^{-1}(a’) ϵ F
Therefore, a’ ϵ F’ if and only if Ψ^{-1}(a) ϵ F
Hence Ψ^{-1}: B’ → B is a homomorphism.
Ψ^{-1} is a bijective homomorphism of B onto B’ and hence is a FGA isomorphism .
Therefore, B’ ≈ B.
Hence ≈ is symmetric.
Suppose B ≈ B’ and B’ ≈ B’’.
Then there exist FGA isomorphisms f : B → B’ and g : B’ → B’’.
Since f is a FGA isomorphism, (i) f(a*b) = f(a) Δ f(b)
(ii) f(δ(a,n)) = δ’(f(a),n)
(iii) $f(q_0) = q_0$’
(iv) a ϵ F if and only if f(a) ϵ F’
and
Since g is a FGA isomorphism, (i) g(a’ Δ b’) = g(a’) θ g(b’)
(ii) g(δ’(a’,n)) = δ’’(g(a’),n)
(iii) $g(q_0$’) = q_0’’
(iv) a’ ϵ F’ if and only if g(a’) ϵ F’’
Consider g o f : B → B’’
Let Ψ = g o f
1. let a, b ϵ Q
Ψ(a*b) = (g o f) (a*b)
= g (f(a) Δ f(b))
= g(f(a)) θ g(f(b))
= (g o f)(a) θ (g o f)(b)
= Ψ(a) θ Ψ(b)
2. Ψ(δ(a,n)) = (g o f) (δ(a,n))
= g(f (δ(a,n)))
= g(δ’(f(a),n))
= δ’’((g(f(a)),n))
= δ’’((g o f) (a),n))
= δ’’(Ψ(a),n)
Let q_0, q_0’ and q_0’’ be the initial states of Q, Q’ and Q’’ respectively.
We have $f(q_0) = q_0$’ and $g(q_0$’) = q_0’’
$\Psi(q_0)$ = (g o f) (q_0)
= g(f (q_0))
= $g(q_0$’)
= q_0’’
We have a ϵ F if and only if f(a) ϵ F’ and a’ ϵ F’ if and only if g(a’) ϵ F’’
That is, a ϵ F if and only if f(a) = a’ ϵ F’ and f(a) = a’ ϵ F’ if and only if g(f(a)) ϵ F’’
That is, a ϵ F if and only if f(a) = a’ ϵ F’ and f(a) = a’ ϵ F’ if and only if (g o f) (a) ϵ F’’
That is, a ϵ F if and only if f(a) = a’ ϵ F’ and f(a) = a’ ϵ F’ if and only if Ψ (a) ϵ F’’
Therefore, a ϵ F if and only if Ψ (a) ϵ F’’
Hence Ψ = g o f : B → B’’ is a homomorphism.
Since f : B → B’ and g : B’ → B’’ are bijections, Ψ = g o f : B → B’’ is also a bijection.
Therefore, Ψ = g o f : B → B’’ is a FGA isomorphism.
Therefore, B ≈ B’’
That is, B ≈ B’ and B’ ≈ B’’ $\Longrightarrow$ B ≈ B’’
Therefore, ≈ is transitive.
Therefore, ≈ is an equivalence relation.
Hence, FGA Isomorphism is an equivalence relation among finite group automata.

Theorem : Let B = (Q, *, Σ, δ, q_0, F) and B' = (Q', Δ , Σ, δ', q_0', F') be two Finite Group Automata. Let f : B → B' be a FGA isomorphism. If S' is a Finite Subgroup Automaton of B', then f^{-1}(S') is a Finite Subgroup Automaton of B.

Proof : Let B = (Q, *, Σ, δ, q_0, F) and B' = (Q', Δ , Σ, δ', q_0', F') be two Finite Group Automata.
Let f : B → B' be a FGA isomorphism.
Let S' be a Finite Subgroup Automata of B'.

Therefore, f^{-1}(S') is a Finite Subgroup Automata of B.

III. Conclusion :

The theory of Finite Group Automata, isomorphism on Finite Group Automata will be useful in many areas. Further research can be done in this area.

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