

## Yu mathematical Model of the Transmission Dynamics of HIV/Tb Co-Infection with Controls in a HIV Endemic Area.

<sup>1</sup>C.C.Asogwa, <sup>2</sup>G.C.E.Mbah, <sup>3</sup>S.E. Aniaku, <sup>4</sup>E.C.Mbah

<sup>1</sup>Department of Mathematics, University of Nigeria, Nsukka Nigeria

<sup>2</sup>Department of Mathematics, University of Nigeria, Nsukka Nigeria

<sup>3</sup>Department of Mathematics, University of Nigeria, Nsukka Nigeria

<sup>4</sup>Department of Mathematics, University of Nigeria, Nsukka Nigeria

---

**Abstract:** A model for the transmission dynamics of HIV/TB co-infection with controls in a HIV endemic area was formulated using differential equations. The effect of post-exposure prophylaxis was recognized and considered. The parameters responsible for the diseases spread were analyzed in order to find the most sensitive of them all. The effective basic reproduction number,  $R_0$  of the systems was obtained and shown that the disease will spread only if  $R_0 > 1$  and would die off with time if  $R_0 < 1$ . Numerical Simulations carried out using MATLAB on the model showed the dynamics of the infection in each class and also the impact of the variations of the control parameter values in the various classes.

**Key words:** HIV endemic area, post-exposure prophylaxis, basic reproduction number.

---

Date of Submission: 27-03-2019

Date of acceptance: 12-04-2019

---

### I. Introduction

Since its detection in the 1980s, the Human Immune-deficiency virus (HIV), the causative agent of the acquired immune deficiency syndrome (AIDS), has continued to pose major public health and socio-economic challenges globally. As at 2009, AIDS has caused 30 million deaths worldwide and as at 2010, approximately 34 million were living with HIV globally, UNAIDS (2011). Also UNAIDS (2013), global report on AIDS epidemic shows that an estimation of 35.5 million people were living with AIDS in 2012. Thus, AIDS is considered as a pandemic disease outbreak which is present over a large area and is actively spreading.

There is no known cure for AIDS. However antiretroviral (ART) treatment improves the health of the infected individuals, pro-logs the life, and subsequently reduces the risk of the infectivity on the victim. In both high-income and low-income countries, the life expectancy of patients infected with HIV who have access to ART is now measured in decades, and might approach that of uninfected population in patients who receive an optimum treatment, Deeks, *et al* (2013). According to Silva; *et al* (2015), with the increased use of ART, the number of AIDS death has declined with around 1.6 million AIDS death in 2012 down from 2.3 million in 2005.

Avert (2017) reported that Nigeria has the second largest HIV epidemic in the world. According to them an estimated 60% of the new HIV infections in western and central Africa in 2015 occurred in Nigeria. Together with South Africa and Uganda, the country accounts for almost half of all new HIV infections in sub-Saharan Africa every year.

*Mycobacterial tuberculosis* is the cause of most occurrence of tuberculosis (TB) and is usually acquired via airborne from someone who has active TB. TB is the most common opportunistic disease affecting people that have had their immune system compromised. The World Health Organization (WHO) in her October, 2016 fact sheet, estimated that about one third of the world's population has been infected with latent tuberculosis. All over the World, approximately 15% of TB patients have HIV co-infection, UNAIDS (2013). Tuberculosis is the most HIV-related opportunistic infection in the world and caring for patients with both diseases is a major public health challenge. In the recent decades, the dramatic spread of the HIV epidemic in sub-Saharan Africa has resulted in the sporadic rate of TB infection. The incidence of TB is also increasing in other higher HIV prevalence countries where the population with HIV infection and TB overlap, Shah *et al* (2014). This is the underlying factor that suggests that TB control will not make much headway in HIV prevalent setting unless HIV control is also archived.

Though HIV infective individuals cannot be cured, health experts have come out with therapeutic drug that can prevent individuals that are newly exposed to HIV infection from contacting the disease. The HIV



- $\psi$ : The rate of the recovery of exposed HIV individuals with the help of post-exposure prophylaxis back to susceptible class.
- $\tau$ : The rate at which individuals who are exposed to HIV become infectious of HIV.
- $\sigma$ : The rate at which infective HIV individuals are being treated of HIV to progress to treated HIV class ( $I_{TH}$ ).
- $1-\sigma$ : The rate at which untreated infective HIV individuals enter into infectious but not treated HIV group ( $I_{NH}$ ).
- $\omega_1$ : The proportion of the infected untreated HIV individuals or AIDS individuals that are being co-infected with active TB.
- $\omega_2 = 1 - \omega_1$ : The proportion of the infected untreated HIV individuals or AIDS individuals that are being co-infected with latent TB.
- $\lambda_2$ : The rate at which individuals in the untreated infectious HIV with latent TB progress to the untreated infective HIV with active TB.
- $\lambda_3$ : The rate at which untreated infectious HIV individuals with active TB are treated of TB only.
- $\lambda_4$ : The rate at which untreated infectious HIV individuals with latent TB are treated of TB only.
- $\lambda_5$ : The rate at which AIDS patients with active TB are treated of TB.
- $\lambda_6$ : The rate at which untreated AIDS individuals with latent TB are treated of TB only.
- $\delta_1$ : The rate at which the untreated infectious HIV individuals with active TB progress to full blown AIDS with active TB.
- $\delta_2$ : The rate at which the untreated infectious HIV individuals progress to the full blown AIDS class.  $\delta_4$ : is assumed to be the rate at which the untreated infectious HIV individuals with treated TB progress to full blown AIDS.
- $\delta_3$ : The rate at which the untreated infectious HIV individuals with latent TB progress to the full blown AIDS with latent TB.
- $\tau_1$ : The rate at which individuals with TB and HIV, treated of TB, fully recover from the TB disease.
- $\tau_2$ : The rate at which individuals in AIDS with latent TB class progress to AIDS with active TB class.
- $\tau_3$ : The rate at which individuals with TB and AIDS, are treated of TB and fully recover from the TB disease.
- $\mu$ : Natural mortality /death rate.
- $\delta_A$ : HIV-induced mortality/death rate.
- $\delta_T$ : TB-induced mortality/death rate.
- $\rho_1$ : Proportion of infected HIV individuals being treated of HIV that are also being co-infected with active TB.
- $\rho_2 = 1 - \rho_1$ : The proportion of individuals infected and being treated of HIV that are also being co-infected with latent TB.
- $\sigma_2$ : The rate of progression of TB from latent TB with infective HIV being treated to active TB with infective HIV being treated.
- $\sigma_3$ : The rate at which active TB with HIV being treated class also starts receiving TB treatment.
- $\sigma_4$ : The rate at which individual that is being treated of HIV with latent TB starts receiving TB treatment.

The following model was then built with the help of the diagram;

$$\frac{dS}{dt} = \Lambda + \psi E_H - (\beta_1 + \mu)S$$

$$\begin{aligned}
 \frac{dE_H}{dt} &= \beta_1 S - (\psi + \tau + \mu)E_H \\
 \frac{dI_{NH}}{dt} &= (1 - \sigma)\tau E_H + \tau_1 I_{NCT} - (\beta_2(\omega_1 + \omega_2) + \mu)I_{NH}, \quad \omega_1 + \omega_2 = 1 \\
 \frac{dI_{TH}}{dt} &= \sigma\tau E_H + \tau_1 I_{TCT} - ((\rho_1 + \rho_2)\beta_2 + \mu)I_{TH}, \quad \rho_1 + \rho_2 = 1 \\
 \frac{dI_{NC}}{dt} &= \beta_2\omega_1 I_{NH} + \lambda_2 I_{NE} - (\lambda_3 + \delta_1 + \mu + \delta_T)I_{NC} \\
 \frac{dI_{TC}}{dt} &= \beta_2\rho_1 I_{TH} + \sigma_2 I_{TE} - (\sigma_3 + \mu + \delta_T)I_{TC} \\
 \frac{dI_{TE}}{dt} &= \beta_2\rho_2 I_{TH} - (\sigma_2 + \sigma_4 + \mu)I_{TE} \\
 \frac{dI_{TCT}}{dt} &= \sigma_3 I_{TC} + \sigma_4 I_{TE} - (\tau_1 + \mu)I_{TCT} \\
 \frac{dI_{NE}}{dt} &= \beta_2\omega_2 I_{NH} - (\sigma_3 + \mu + \lambda_2 + \lambda_4)I_{NE} \\
 \frac{dI_{NCT}}{dt} &= \lambda_3 I_{NC} + \lambda_4 I_{NE} - (\tau_1 + \mu + \delta_4)I_{NCT} \\
 \frac{dA_{NE}}{dt} &= \delta_3 I_{NE} + \beta_2\omega_2 A - (\lambda_6 + \tau_2 + \mu + \delta_A)A_{NE} \\
 \frac{dA_C}{dt} &= \tau_2 A_{NE} + \beta_2\omega_1 A + \delta_1 I_{NC} - (\lambda_5 + \mu + \delta_A + \delta_T)A_C \\
 \frac{dA_{CT}}{dt} &= \lambda_5 A_C + \delta_4 I_{NCT} + \lambda_6 A_{NE} - (\tau_3 + \mu + \delta_A)A_{CT} \\
 \frac{dA}{dt} &= \delta_2 I_{NH} + \tau_3 A_{CT} - ((\omega_2 + \omega_1)\beta_2 + \mu + \delta_A)A \quad (1.10)
 \end{aligned}$$

The above fourteen equations sum up to give the population as N, that is

$$N = S + E_H + I_{NH} + I_{TH} + I_{NC} + I_{TC} + I_{TE} + I_{TCT} + I_{NE} + I_{NCT} + A_{NE} + A_C + A_{CT} + A$$

The force of infection of HIV is  $\beta_1$ , where

$$\beta_1 = \frac{K_H (n_1 I_{NH} + n_2 I_{TH} + n_3 I_{NC} + n_4 I_{TC} + n_5 I_{TE} + n_6 I_{TCT} + n_7 I_{NE} + n_8 I_{NCT} + n_9 A_{NE} + n_{10} A_C + n_{11} A_{CT} + n_{12} A)}{N}$$

infecting the susceptible class only, with  $K_H$  as the rate of transmission of HIV to the susceptible class and

$$n_1, n_2, n_3, \dots, n_{12} \text{ representing different infectivity rates of HIV classes. } \beta_2 = \frac{K_T (m_1 I_{NC} + m_2 I_{TC} + m_3 A_C)}{N}$$

is the force of infection of TB, infecting the already HIV infected classes, with  $K_T$  as the rate of transmission of

TB to a none TB patient while  $m_1, m_2, m_3$  are the different infectivity rates of active TB classes.

The disease-free equilibrium (DFE) is the state of total absence of the disease in the entire population. Here it is the absence of both HIV and TB infection in the entire system.

Let  $\varepsilon_0 \left( S^*, E_H^*, I_{NH}^*, I_{TH}^*, I_{NC}^*, I_{TC}^*, I_{TE}^*, I_{TCT}^*, I_{NE}^*, I_{NCT}^*, A_{NE}^*, A_C^*, A_{CT}^*, A^* \right)$  be the equilibrium point of

the model. Since the recruitment term,  $\Lambda$ , can never vanish, that is  $N \neq 0$ , there exist no trivial equilibrium point like

$$\varepsilon_0 \left( S^*, E_H^*, I_{NH}^*, I_{TH}^*, I_{NC}^*, I_{TC}^*, I_{TE}^*, I_{TCT}^*, I_{NE}^*, I_{NCT}^*, A_{NE}^*, A_C^*, A_{CT}^*, A^* \right) = (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0),$$

so let

$$\varepsilon_0 \left( S^*, E_H^*, I_{NH}^*, I_{TH}^*, I_{NC}^*, I_{TC}^*, I_{TE}^*, I_{TCT}^*, I_{NE}^*, I_{NCT}^*, A_{NE}^*, A_C^*, A_{CT}^*, A^* \right) = (S, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)$$

Since, at equilibrium state,

$$\frac{dS}{dt} = \Lambda + \psi E_H^* - (\beta_1 + \mu) S^* = 0 \quad \text{since } \psi, \beta_1 = 0 \text{ (disease free)}$$

$$\Rightarrow S^* = \frac{\Lambda}{\mu}$$

Thus the disease free equilibrium state for the model exist at the point

$$\varepsilon_0 \left( S^*, E_H^*, I_{NH}^*, I_{TH}^*, I_{NC}^*, I_{TC}^*, I_{TE}^*, I_{TCT}^*, I_{NE}^*, I_{NCT}^*, A_{NE}^*, A_C^*, A_{CT}^*, A^* \right) = \left( \frac{\Lambda}{\mu}, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0 \right)$$

### III. Stability Analysis Of The HIV/TB Co-Infection

To determine the stability or otherwise of the disease free equilibrium state,  $\varepsilon_0$ , we shall examine the behavior of the model population near the equilibrium solution.

Note also that,

$$\beta_1 S = \frac{K_H (n_1 I_{NH} + n_2 I_{TH} + n_3 I_{NC} + n_4 I_{TC} + n_5 I_{TE} + n_6 I_{TCT} + n_7 I_{NE} + n_8 I_{NCT} + n_9 A_{NE} + n_{10} A_C + n_{11} A_{CT} + n_{12} A)}{N} \left( \frac{\Lambda}{\mu} \right),$$

since  $S = \frac{\Lambda}{\mu}$ ,  $\beta_2 I_{NH} = \beta_2 I_{NE} = \beta_2 I_{TH} = \beta_2 I_{TE} = \beta_2 I_{TC} = \beta_2 I_{NC} = 0$  and

$I_{NH} = I_{NE} = I_{TH} = I_{TE} = I_{TC} = I_{NC} = 0$  at the equilibrium point.

Recall that the disease free equilibrium is stable if all the eigenvalues of the Jacobian matrix of the system have negative real parts. Let the linearize Jacobian matrix of the system of equations (1.10) at the equilibrium point be  $J(\varepsilon_0)$  and the eigenvalues be  $\eta$ , then

$$J(\varepsilon_0) = \begin{pmatrix} -\mu - \eta & \psi & \frac{-K_H n_1 \Lambda}{N\mu} & \frac{-K_H n_2 \Lambda}{N\mu} & \frac{-K_H n_3 \Lambda}{N\mu} & \frac{-K_H n_4 \Lambda}{N\mu} & \frac{-K_H n_5 \Lambda}{N\mu} & \frac{-K_H n_6 \Lambda}{N\mu} & \frac{-K_H n_7 \Lambda}{N\mu} & \frac{-K_H n_8 \Lambda}{N\mu} & \frac{-K_H n_9 \Lambda}{N\mu} & \frac{-K_H n_{10} \Lambda}{N\mu} & \frac{-K_H n_{11} \Lambda}{N\mu} & \frac{-K_H n_{12} \Lambda}{N\mu} \\ 0 & -(\psi + \tau) - \eta & \frac{K_H n_1 \Lambda}{N\mu} & \frac{K_H n_2 \Lambda}{N\mu} & \frac{K_H n_3 \Lambda}{N\mu} & \frac{K_H n_4 \Lambda}{N\mu} & \frac{K_H n_5 \Lambda}{N\mu} & \frac{K_H n_6 \Lambda}{N\mu} & \frac{K_H n_7 \Lambda}{N\mu} & \frac{K_H n_8 \Lambda}{N\mu} & \frac{K_H n_9 \Lambda}{N\mu} & \frac{K_H n_{10} \Lambda}{N\mu} & \frac{K_H n_{11} \Lambda}{N\mu} & \frac{K_H n_{12} \Lambda}{N\mu} \\ 0 & (1 - \sigma) \tau & -\mu - \eta & 0 & 0 & 0 & 0 & 0 & 0 & \tau_1 & 0 & 0 & 0 & 0 \\ 0 & \sigma & 0 & -\mu - \eta & 0 & 0 & 0 & \tau_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\left(\frac{\lambda_3 + \delta_1}{\mu + \delta_T}\right) - \eta & 0 & 0 & 0 & \lambda_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\left(\frac{\sigma_3 + \mu}{\mu + \delta_T}\right) - \eta & \sigma_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\left(\frac{\sigma_2 + \sigma_4}{\mu}\right) - \eta & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \sigma_3 & \sigma_4 & -(\tau_1 + \mu) - \eta & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\left(\frac{\sigma_3 + \mu}{\mu + \lambda_2 + \lambda_4}\right) - \eta & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda_3 & 0 & 0 & 0 & \lambda_4 & -\left(\frac{\tau_1 + \mu}{\mu + \delta_4}\right) - \eta & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \delta_3 & 0 & -\left(\frac{\lambda_6 + \tau_2}{\mu + \delta_A}\right) - \eta & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \delta_1 & 0 & 0 & 0 & 0 & 0 & \tau_2 & -\left(\frac{\lambda_5 + \mu}{\mu + \delta_A + \delta_T}\right) - \eta & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \delta_4 & \lambda_6 & -\left(\frac{\tau_3 + \mu}{\mu + \delta_A}\right) - \eta & 0 & 0 \\ 0 & 0 & \delta_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \tau_3 & -(\mu + \delta_A) - \eta \end{pmatrix} = 0$$

Solving we have the eigenvalues,  $\eta$ , of the system to be:

$$\eta = -\mu, -(\sigma_2 + \sigma_4 + \mu), -(\sigma_3 + \mu + \delta_T), -(\tau_1 + \mu), -(\sigma_3 + \mu + \lambda_2 + \lambda_4), -(\lambda_3 + \delta_1 + \mu + \delta_T),$$

$$-(\tau_1 + \mu + \delta_4), -(\lambda_6 + \tau_2 + \mu + \delta_A), -(\lambda_5 + \mu + \delta_A + \delta_T), -(\tau_3 + \mu + \delta A), -\mu,$$

$$\frac{-(2\mu + \delta_A) \pm \sqrt{(2\mu + \delta_A)^2 - 4\mu(\mu + \delta_A)}}{2}, -\left( \mu + (1 - \sigma)\tau \left( \frac{K_H n_1 \Lambda}{N\mu} \right) \right)$$

Since all the parameters are greater than zero, the real parts of the eigenvalues of the Jacobian matrix of the system are all negative, thus the disease free equilibrium state,  $\varepsilon_0$ , is asymptotically stable.

Using the next generation operator approach, we obtain the effective basic reproduction number,  $R_0$ , of the system (1.10) which is the spectra radius, ( $\rho$ ), of the next generation matrix,  $FV^{-1}$ , that is  $R_0 = \rho(FV^{-1})$ .

$F$  is the matrix of the new infection terms and  $V$ , the matrix of the transition terms obtained from the Jacobian matrix of linearized system of (1.10) about DFE.

$$\text{Let } X' = \varepsilon_0 (E_H^*, I_{NH}^*, I_{TH}^*, I_{NC}^*, I_{TC}^*, I_{TE}^*, I_{TCT}^*, I_{NE}^*, I_{NCT}^*, A_{NE}^*, A_C^*, A_{CT}^*, A^*, S)^T$$

$$\text{Therefore } X' = \frac{dX}{dt} = F(x) - V(x)$$

Where  $F(x)$  and  $V(x)$  are column matrix of (1.10) given as follows;

$$F(x) = \begin{bmatrix} K_H S (n_1 I_{NH} + n_2 I_{TH} + n_3 I_{NC} + n_4 I_{TC} + n_5 I_{TE} + n_6 I_{TCT} + n_7 I_{NE} + n_8 I_{NCT} + n_9 A_{NE} + n_{10} A_C + n_{11} A_{CT} + n_{12} A) \\ N \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$V(x) = \left[ \begin{aligned} & -(\psi + \tau + \mu)E_H \\ & - (1 - \sigma)\tau E_H - \tau_1 I_{NCT} + \frac{K_T (m_1 I_{NC} + m_2 I_{TC} + m_3 A_C)}{N} (\omega_1 + \omega_2) I_{NH} + \mu I_{NH} \\ & - \sigma E_H - \tau_1 I_{TCT} + (\rho_1 + \rho_2) \frac{K_T (m_1 I_{NC} + m_2 I_{TC} + m_3 A_C)}{N} I_{TH} + \mu I_{TH} \\ & - \frac{K_T (m_1 I_{NC} + m_2 I_{TC} + m_3 A_C)}{N} \omega_1 I_{NH} - \lambda_2 I_{NE} + (\lambda_3 + \delta_1 + \mu + \delta_T) I_{NC} \\ & - \frac{K_T (m_1 I_{NC} + m_2 I_{TC} + m_3 A_C)}{N} \rho_1 I_{TH} - \sigma_2 I_{TE} + (\sigma_3 + \mu + \delta_T) I_{TC} \\ & - \frac{K_T (m_1 I_{NC} + m_2 I_{TC} + m_3 A_C)}{N} \rho_2 I_{TH} + (\sigma_2 + \sigma_4 + \mu) I_{TE} \\ & - \sigma_3 I_{TC} - \sigma_4 I_{TE} + (\tau_1 + \mu) I_{TCT} \\ & - \frac{K_T (m_1 I_{NC} + m_2 I_{TC} + m_3 A_C)}{N} \omega_2 I_{NH} + (\sigma_3 + \mu + \lambda_2 + \lambda_4) I_{NE} \\ & - \lambda_3 I_{NC} - \lambda_4 I_{NE} + (\tau_1 + \mu + \delta_4) I_{NCT} \\ & - \delta_3 I_{NE} - \frac{K_T (m_1 I_{NC} + m_2 I_{TC} + m_3 A_C)}{N} \omega_2 A + (\lambda_6 + \tau_2 + \mu + \delta_A) A_{NE} \\ & - \tau_2 A_{NE} - \frac{K_T (m_1 I_{NC} + m_2 I_{TC} + m_3 A_C)}{N} \omega_1 A - \delta_1 I_{NC} + (\lambda_5 + \mu + \delta_A + \delta_T) A_C \\ & - \lambda_5 A_C - \delta_4 I_{NCT} + \lambda_6 A_{NE} + (\tau_3 + \mu + \delta_A) A_{CT} \\ & - \delta_2 I_{NH} - \tau_3 A_{CT} + ((\omega_2 + \omega_1) \frac{K_T (m_1 I_{NC} + m_2 I_{TC} + m_3 A_C)}{N} A + (\mu + \delta_A) A \\ & - \Lambda - \psi E_H + \beta_1 S + \mu S \end{aligned} \right] \dots 1.32$$

The derivatives  $DF(\varepsilon_0)$  and  $DV(\varepsilon_0)$  at disease free equilibrium, DFE, point,  $\varepsilon_0$  are partitioned as

$$DF(\varepsilon_0) = \begin{bmatrix} F & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad DV(\varepsilon_0) = \begin{bmatrix} V & 0 \\ J_1 & J_2 \end{bmatrix}$$

Where  $F$  and  $V$  are 13 by 13 matrices given by

$$F = \begin{bmatrix} 0 & n_1 K_H \Lambda & n_2 K_H \Lambda & n_3 K_H \Lambda & n_4 K_H \Lambda & n_5 K_H \Lambda & n_6 K_H \Lambda & n_7 K_H \Lambda & n_8 K_H \Lambda & n_9 K_H \Lambda & n_{10} K_H \Lambda & n_{11} K_H \Lambda & n_{12} K_H \Lambda \\ 0 & N\mu \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

and also  $V$ , the matrix of the transition terms as:

$$V = \begin{bmatrix} (\psi+\tau+\mu) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -(1-\sigma)\tau & \mu & 0 & 0 & 0 & 0 & 0 & 0 & -\tau_1 & 0 & 0 & 0 & 0 \\ -\sigma & 0 & \mu & 0 & 0 & 0 & -\tau_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & (\lambda_3+\delta_1+\mu+\delta_T) & 0 & 0 & 0 & -\lambda_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & (\sigma_3+\mu+\delta_T) & -\sigma_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & (\sigma_2+\sigma_4+\mu) & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\sigma_3 & -\sigma_4 & (\tau_1+\mu) & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & (\sigma_3+\mu+\lambda_2+\lambda_4) & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\lambda_3 & 0 & 0 & 0 & -\lambda_4 & (\tau_1+\mu+\delta_4) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\delta_3 & 0 & (\lambda_6+\tau_2+\mu+\delta_A) & 0 & 0 & 0 \\ 0 & 0 & 0 & -\delta_1 & 0 & 0 & 0 & 0 & 0 & -\tau_2 & (\lambda_5+\mu+\delta_A+\delta_T) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\delta_4 & -\lambda_6 & -\lambda_5 & (\tau_3+\mu+\delta_A) & 0 \\ 0 & -\delta_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\tau_3 & (\mu+\delta_A) \end{bmatrix}$$

Now the characteristics equation  $|FV^{-1} - I\xi| = 0$  is solved and the maximum value of  $\xi$ , the spectra radius of the matrix  $FV^{-1}$  gave the reproductive number as

$$R_0 = \frac{K_H \Lambda \tau [n_1(1-\sigma)(\mu + \delta_A) + n_2\sigma(\mu + \delta_A) + n_{12}\delta_2(1-\sigma)]}{N\mu^2(\psi + \tau + \mu)(\mu + \delta_A)}$$

#### IV. Sensitivity Analysis Results And Discussions

According to Elaine *et al.* (2012), sensitivity analysis is the study of the contributions of different parameters to the uncertainty present in the outcome of a system.

Thus, we carry out sensitivity analysis of the basic reproduction number, with respect to the model parameters in order to determine the relative importance of the different factors responsible for the transmission and the prevalence of the disease.

The normalized forward sensitivity index of the basic reproduction number  $R_0$ , with respect to the parameter

value, given as  $\frac{\partial R_0}{\partial p} \times \frac{p}{R_0}$ ; where  $p$  is the parameter of the model and  $R_0$ , the basic reproduction

number of the model, was used to calculate the sensitivity indices at the baseline parameter values.

Thus, calculating the sensitivity indices for those parameters on which the values of  $R_0$  depends on and substituting the rest of the values of the parameters from Appendix A gives the result in table below:

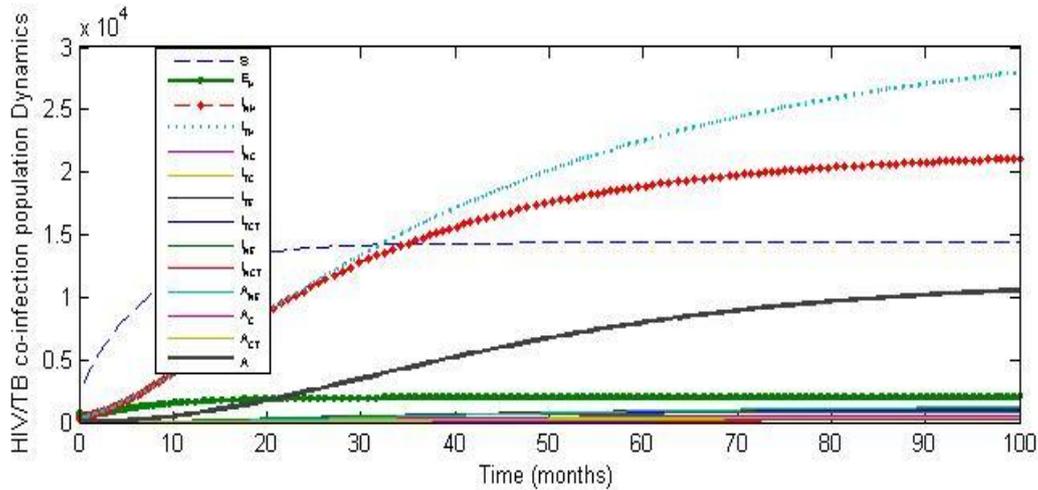
Signs of sensitivity index of  $R_0$

S/N	Parameters	Sensitivity index
	$K_H$	+1
	$\Lambda$	+1
	$\tau$	+0.1426
	$\mu$	-0.0172383
	$\sigma$	-0.5472
	$\delta_A$	-0.0006684
	$\delta_2$	+0.399674267
	$\psi$	-0.1226321
	$N$	-1
	$K_T$ , and others	0

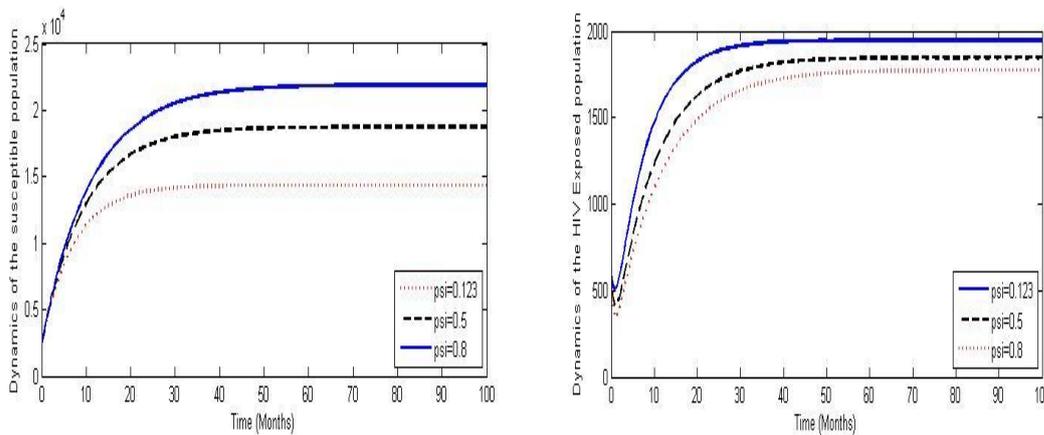
Thus if we allow any of the parameters,  $K_H, \Lambda, \delta_2$  and  $\tau$  to increase; then the disease will persist and will with time invade the population. Also if we increase any of  $\psi, \mu, \sigma, N$  and  $\delta_A$ ; then the disease will gradually reduce and with time phase out in the population. Sensitive index of other parameters that did not appear in  $R_0$ , like  $K_T$ , are zero, that is, the increase or decrease of such parameter has no influence on the reproductive number.

**V. Numerical Results And Discussion**

In this section we used MATLAB to simulate the model for HIV/TB co-infection in a HIV endemic area. This will enable us see the population dynamics of each class of the HIV/TB co-infection. We also vary the values of the control parameters  $\psi$  and see the effect on both the Susceptible, the Exposed and infective HIV compartments, taking a sample population of 4000. The results are as shown below:



**Figure 1:** A graph showing the dynamics of HIV/TB co-infection in the population.



**Figure 2:** Graphs showing the impact of  $\psi$  (recovery rate of exposed HIV individuals with the help of Post Exposure Prophylaxis back to susceptible class) on the population of the susceptible and the Exposed classes.

As  $\psi$  increases, the number of the susceptible individuals in the population increases while the number of the exposed individuals in the population decreases. Reading from the 100<sup>th</sup> month, the population of the susceptible class rose from 14,344 to 18739 due to the increment rate of  $\psi$  from 0.123 to 0.5. Thus, the rate of increment of the population is  $\frac{18739 - 14344}{0.5 - 0.123} = 11,658$ . That is 11658 persons per a unit increase in the number of people in the exposed class that receive the post exposure prophylaxis. Also, the decrease in the Exposed class is from 1947 to 1847 giving a decrease rate of 265 persons per a unit increase in the number of people in the exposed class that receive the post exposure prophylaxis.

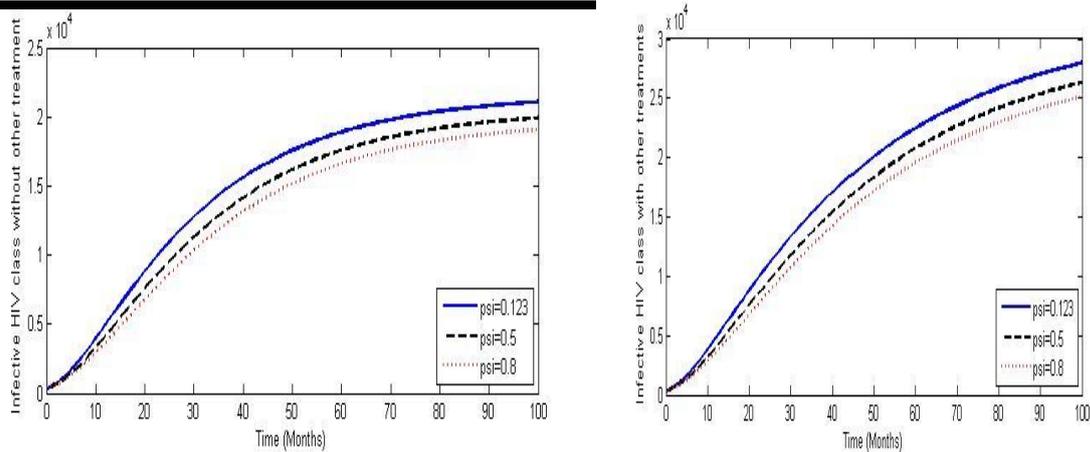


Figure 3: Graphs showing the impact of  $\psi$  on the HIV infective classes.

From the above graphs displayed, the number of both of the HIV infective classes are decreasing as  $\psi$  is increased. The data from the 100<sup>th</sup> month shows that the class of infective HIV without other treatment decreased from 19912 to 19066, while that with other HIV treatment decreased from 26315 to 25112 at the increased rate of  $\psi$  from 0.5 to 0.8. This implies that the population of infective HIV class without any other treatment and that of infective HIV with some other treatment decreased at the rate of 2820 and 4010 persons per unit increment in post exposure prophylaxis dispensation respectively.

### VI. Results and Conclusion

We divided the entire population into 14 compartments and formulated a mathematical model using differential equations for HIV/TB co-infection in a HIV endemic area. We introduced the use of post exposure prophylaxis (PEP) as a control measure during the Exposed period of HIV infection. The basic reproduction number was derived and found that the disease will persist if it exceeds one. From the sensitive analysis of the parameters we note that to control the infection, one should aim at minimizing the values of the HIV contact rate ( $K_H$ ), the rate of influx of people in the susceptible class ( $\Lambda$ ), the rate of progression of infective HIV individuals to AIDS class ( $\delta_2$ ) and rate of progression of HIV infected individuals from the exposed class to infective HIV class ( $\tau$ ) and maximize the dispensation rate of post exposure prophylaxis ( $\psi$ ), the rate of use of other HIV treatments ( $\sigma$ ), death rate of infective HIV individuals ( $\mu$ ), the death rate of AIDS class ( $\delta_A$ ).

People should have the knowledge of PEP and 72 hours HIV exposure duration, when it must be administered, for the HIV exposed patient to recover from the infection. They should be sensitive enough to detect or suspect when they are exposed and seek medical advice immediately. Anti-retroviral HIV drugs and TB drugs should be made available and at easy reach of the people. Aggressive campaign against the stigma towards the diseases should be pursued.

### APPENDIX A

Values of different parameters in the model.

Parameters	Nominal values	References
$\Lambda, K_H, \mu, \delta_T$	2000, 0.45, 0.02, 0.24	Sharef <i>et al</i> (2014)
$K_T, \tau$	0.76, 0.86	Tebeje, <i>et al</i> (2010)
$\psi$	0.123	Calculated (Appendix B)
$\sigma, 1-\sigma$	0.48, 0.52	Assumed
$\omega_1, \omega_2, \lambda_3, \lambda_4$	0.3, 0.7, 0.04, 0.08	Assumed
$\lambda_5, \delta_3, \delta_4, \tau_1, \tau_3, \sigma_2, \lambda_2$	0.5, 0.06, 0.03, 0.5, 0.5, 0.03, 0.03	Castilo-chavezC., <i>etal</i> (2004)
$\lambda_6, \delta_1$	0.08, 0.03	Assumed
$\tau_2, \delta_A$	0.05, 0.03	Silva <i>et al</i> (2015)
$n_1, n_2, \dots, n_{12}$	0.002, 0.001, 0.003, 0.002, 0.001, 0.001, 0.002, 0.002, 0.004, 0.005, 0.004, 0.003,	Assumed

$m_1, m_2, m_3$	0.002, 0.002, 0.003	Assumed
$\rho_1, \rho_2, \sigma_3, \sigma_4, \beta_1$	0.3, 0.7, 0.3, 0.35, 0.05	Assumed
N	35,000	Assumed
$\beta_2$	0.024	Crampin, <i>et al</i> (2011)

### Appendix B

#### NUMERICAL CALCULATION OF THE RATE OF RECOVERY OF HIV EXPOSED INDIVIDUALS DUE TO THE USE OF POST EXPOSURE PROPHYLAXIS

The rate of recovery of HIV exposed individuals due to the use of post-exposure prophylaxis is calculated based on the following findings:

According to Bosenatebeje, et al (2010), finding in the HIV endemic areas of Ethiopia shows that;

The rate of HIV exposure is approximated to be 76%

The rate of awareness of post-exposure prophylaxis (PEP) is 20%

Percentage rate of PEP success when fully administered is 81% and above.

With these data, we calculate that the rate of recovery of the HIV exposed individuals,  $\psi$  as;

$$\psi = \text{rate of HIV exposure (76\%)} \times \text{rate of PEP awareness (20\%)} \times \text{rate of PEP success (81\%)} \\ = 0.123$$

### References

- [1]. AIDS gov. (2015). <https://www.aids/hiv-aids-basics/hiv-aids-101/how-to-get-hiv-aids>.
- [2]. AIDS gov. (2015). Post-exposure prophylaxis. <https://www.aids.gov/hiv-aids-basics/prevention/reduce-your-risk/post-exposure>.
- [3]. Avert (2017). Information on AIDS: HIV and AIDS in Nigeria. <https://www.avert.org>sub-sahara-africa>.
- [4]. Bosenatebeje, ChevetHailu (2010): Assesment of HIV Post Prophylaxis use Among Health Workers of Government Health Institutions in Jimma zone, Oromiya Region Southwest Ethiopia.
- [5]. Crampin A.C., Kasimba S., Mwaungulu N.J, Dacombe R., Floyd S., Glynn J.R., Fine P.E.M. (2011). Married to M. Tuberculosis: Risk of infection and disease in spouses of smear positive TB Patients. *Trop. Med. Int. Health.*16 (7): 811-818.
- [6]. Cristina, J. Silva and Delfim, F.M.Torres, (2015). A TB/HIV co-infection model and optimal control treatment. Centre for Research and development in Mathematics and Applications (CIDMA). Department of Mathematics, University of Aveiro, 3810-193 Aveiro, Portugal.
- [7]. Deeks S.G., Lewin S.R., Havlir D.V., (2013). The end of AIDS: HIV infection as a chronic disease. *The lancet*, Vol. 382, issue 9903, 1525-1533.
- [8]. Eliane O. N.,Richard J. B., and Madhav V.,M. (2012), Sensitivity analysis of an individual-Based Model for simulation of epidemics; PLOS <https://doi-org/10.1371/journal>.
- [9]. Shah, N.H. and Gupta,J. (2014). Modelling of HIV-TB co-infection Transmission Dynamics, *American Journal of Epidemiology and infectious diseases*, vol. 2 No. 1 pg 1-7.
- [10]. UNAIDS (2013). UNAIDS report on Global AIDS epidemics, World Health Organization.
- [11]. UNAIDS (2016). Fact sheet <http://www.unaids.org./en/resources/fact-sheet>.
- [12]. WHO (2015). World Tuberculosis day report.
- [13]. WHO (2017). 'Tuberculosis'. [www.who.int/mediacentre/fact-sheets/fs\\_104/en/](http://www.who.int/mediacentre/fact-sheets/fs_104/en/)

C.C.Asogwa. "Mathematical Model of the Transmission Dynamics of HIV/TB Co-Infection with Controls in A HIV Endemic Area." *IOSR Journal of Mathematics (IOSR-JM)* 15.2 (2019): 01-11