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Convergence of derivatives for certain mixed Baskakov-Szasz operators

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Abstract:*In this paper we discuss about the mixed summation-integral type operators having Baskakov basis functionin summation and Szasz-Mirakyan basis function in integration. We have central moments and some other basic results for these operators, and obtain the rate of point-wise convergence, a Voronovskaya type asymptotic formula and error estimate in simultaneous approximation.*

Keywords:*Baskakov operators; Szasz Mirakyan operators; Simultaneous approximation; Asymptotic formula; Error estimate;.Rate of convergence;*

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I. Introduction

In the year 2003, Srivastava-Gupta [5] proposed a family of summation-integral typeoperatorsincluding some well-known operators [4] as special cases. Some other operators have been proposed in[1],[2] and[3].

For$C_{\gamma[0,\infty)} = \{f \in C: f(t) \le M(1+t)^{\gamma},\ M > 0,\ \gamma > 0\}$, a new mixed type sequenceof operators is defined here as

$$P_n(f,x) = \int_0^{\infty} W_n(x,t) f(t)\,dt,$$

where

$$W_n(x,t) = n\sum_{v=0}^{\infty} p_{n,v}(x) q_{n,v}(t).$$

$$p_{n,v}(x) = \frac{(n+v-1)!}{v!\,(n-1)!}\frac{x^v}{(1+x)^{n+v}}, \qquad q_{n,v}(t) = e^{-nt}\frac{(nt)^v}{v!}$$

are Baskakov and Szasz basis functions respectively. Hencethe operators defined in this paper are

$$P_n(f,x) = n\sum_{v=0}^{\infty} p_{n,v}(x)\int_0^{\infty} q_{n,v}(t) f(t)\,dt. \qquad \ldots\ldots..(1.1)$$

It can easily be verified that these operators havingcombination of Baskakov basis function and Szasz basis function, are the linear positive operators. In the present paper we estimate some direct results for the operators $P_n$such as a point-wise rate of convergence, asymptotic formula and an error estimate in simultaneous approximation.

II. Auxiliary Results

We need the following lemmas to get appropriate results.

Lemma 1: For $m \in W \coloneqq wholenumbers$, if the m^{th}order moment is defined by

$$U_{n,m}(x) = \sum_{v=0}^{\infty} p_{n,v}\left(\frac{v}{n} - x\right)^m$$

Then $U_{n,0}(x) = 1;\ U_{n,1}(x) = 0$; and $U_{n,m}(x) = x(1+x)\left[U_{n,m}'(x) + mU_{n,m-1}(x)\right]$.
Consequently,

$$U_{n,m}(x) = O\left(n^{-\left[\frac{m+1}{2}\right]}\right).$$

Lemma 2: The central moment $T_{n,m}(x),\ m \in W$ is defined by

$$T_{n,m}(x) = n\sum_{v=0}^{\infty} p_{n,v}(x)\int_0^{\infty} q_{n,v}(t)\,(t-x)^m\,dt.$$

Taking $m = 0,1,2$ we have

$$T_{n,0}(x)=1,\qquad T_{n,m}(x)=\frac{1}{n};\qquad T_{n,2}(x)=\frac{nx^2+2nx+2}{n^2}.$$

and also we have the recurrence relation

$$nT_{n,m+1}(x)=x(1+x)\left[T'_{n,m}(x)+mT_{n,m-1}(x)\right]+(m+1)T_{n,m}(x)+mxT_{n,m-1}(x).$$

Consequently from this relation $T_{n,m}(x)=O\left(n^{-\left[\frac{m+1}{2}\right]}\right)$ for all $x\in[0,\infty)$.

Lemma 3: For our operators (1.1), we can easily find the identity given as below

$$P_n(t^i,x)=n^{-i}\frac{(n+i-1)!\ 2i!}{(n-1)!\,(i!)^2}x^i+n^{-i-1}\frac{(n+i-1)!\ (2i-1)!}{(n-1)!\left((i-1)!\right)^2}x^{i-1}+O(n^{-2}).$$

Lemma 4: There exists polynomial $Q_{i,j,r}(x)$ which is independent of $n$and v, such that

$$x^r(1+x)^rD^r\left[p_{n,v}(x)\right]=\sum_{\substack{2i+j\leq r,\\ i,j\geq 0}}n^i(v-nx)^jQ_{i,j,r}(x)p_{n,v}(x),\qquad D\equiv\frac{d}{dx}$$

III. Direct Results

In this section we prove direct results related to operators(1.1).

3.1 Simultaneous Approximation

Theorem: If $f\in C_\gamma[0,\infty),\gamma>0$ and $f^{(r)}$ exists at a point $x\in(0,\infty)$, then

$$\lim_{n\to\infty}P_n^{(r)}(f(t),x)=f^{(r)}(x).$$

Proof: By Taylor's expansion of$f$, we have

$$f(t)=\sum_{i=0}^{r}\frac{f^{(i)}(x)}{i!}(t-x)^i+\epsilon(t,x)(t-x)^r,$$

where $\epsilon(t,x)\to 0$ as $t\to x$.Therefore

$$P_n^{(r)}(f(t),x)=\sum_{i=0}^{r}\frac{f^{(i)}(x)}{i!}\int_0^\infty W_n^{(r)}(t,x)\,(t-x)^idt+\int_0^\infty W_n^{(r)}(t,x)\epsilon(t,x)\,(t-x)^rdt$$

$$:=E_1+E_2.$$

To estimate E_1, we use binomial expansion and Lemma 2 as

$$E_1=\sum_{i=0}^{r}\frac{f^{(i)}(x)}{i!}\sum_{v=0}^{i}\binom{i}{v}(-x)^{i-v}\int_0^\infty W_n^{(r)}(t,x)\,t^vdt$$

$$=\frac{f^{(r)}(x)}{r!}\int_0^\infty W_n^{(r)}(t,x)\,t^rdt$$

$$=f^{(r)}(x)+o(1),\qquad (n\to\infty)$$

Next, using Lemma 4, we obtain

$$|E_2|\leq\sum_{\substack{2i+j\leq r,\\ i,j\geq 0}}n^{i+1}\frac{\left|Q_{i,j,r}(x)\right|}{x^r}\sum_{v=0}^{\infty}|v-nx|^j\,p_{n,v}(x)\int_0^\infty q_{n,v}(t)|\epsilon(t,x)|\,|t-x|^rdt$$

Since $\epsilon(t,x)\to 0$ as$t\to x$ for given$\epsilon>0$, there exists a $\delta>0$ such that $|\epsilon(t,x)|<\epsilon$for $0<|t-x|<\delta$. Further if $\mu\geq\max\{\gamma,r\};\mu\in Z$ then we can find a constant K such that $|\epsilon(t,x)||t-x|^r<K|t-x|^\mu$ for $|t-x|\geq\delta$.Thus with$M=\max_{2i+j\leq r}\frac{|Q_{i,j,r}(x)|}{x^r}$, we have

$$|E_2|\leq M\sum_{\substack{2i+j\leq r,\\ i,j\geq 0}}n^{i+1}\sum_{v=0}^{\infty}|v-nx|^j\,p_{n,v}(x)\int_0^\infty q_{n,v}(t)\epsilon\,|t-x|^rdt$$

$$\leq M\sum_{\substack{2i+j\leq r,\\ i,j\geq 0}}n^{i+1}\sum_{v=0}^{\infty}|v-nx|^j\,p_{n,v}(x)\int_0^\infty q_{n,v}(t)\epsilon\,|t-x|^rdt$$

$$\leq M\sum_{\substack{2i+j\leq r,\\ i,j\geq 0}}n^{i+1}\sum_{v=0}^{\infty}|v-nx|^j\,p_{n,v}(x)\left\{\epsilon\int_{|t-x|<\delta}q_{n,v}(t)\,|t-x|^rdt+\int_{|t-x|\geq\delta}q_{n,v}(t)K\,|t-x|^\mu dt\right\}$$

$$:=E_3+E_4.$$

For $E_{3,}$ we apply Schwaz inequality for integration and summation and then use lemmas1 and 2 as

$$|E_3| \leq \epsilon M \sum_{\substack{2i+j\leq r,\\ i,j\geq 0}} n^{i+1} \left\{\sum_{v=0}^{\infty} |v-nx|^{2j}\, p_{n,v}(x)\right\}^{1/2} \left\{\int_0^{\infty} q_{n,v}(t)\right\}^{\frac{1}{2}} \left\{\int_0^{\infty} q_{n,v}(t)\, |t-x|^{2r} dt\right\}^{\frac{1}{2}}$$

$$\leq \epsilon M \sum_{\substack{2i+j\leq r,\\ i,j\geq 0}} n^{i+1}\, O\left(n^{\frac{j}{2}}\right).\frac{1}{n}.O\left(n^{-\frac{r}{2}}\right)$$

$$= \epsilon . o(1).$$

For $E_{4,}$ also applying Schwaz inequality, lemmas 1 and 2, and taking $C = MK$ we get

$$E_4 \leq C \sum_{\substack{2i+j\leq r,\\ i,j\geq 0}} n^{i+1} \left\{\sum_{v=0}^{\infty} |v-nx|^{2j}\, p_{n,v}(x)\right\}^{\frac{1}{2}} \left\{\sum_{v=0}^{\infty} p_{n,v}(x) \int_0^{\infty} q_{n,v}(t) \epsilon\, |t-x|^{2\mu} dt\right\}^{\frac{1}{2}}$$

$$\leq \epsilon M \sum_{\substack{2i+j\leq r,\\ i,j\geq 0}} n^{i+1}\, O\left(n^{\frac{j}{2}}\right) O\left(n^{-\frac{\mu}{2}-1}\right)$$

$$\leq O\left(n^{\frac{r-\mu}{2}}\right) = o(1).$$

Thus for arbitrary small $\epsilon > 0$, it follows that $E_2 = o(1)$.Hence collecting estimates of E_1andE_2, we get the objective.

3.2 Asymptotic Relation Formula

Theorem: If $f \in C_\gamma[0,\infty), \gamma > 0andf^{(r+2)}$ exists at a point $x \in (0,\infty)$, then

$$\lim_{n\to\infty} n\left\{P_n^{(r)}(f(t),x) - f^{(r)}(x)\right\} = \frac{(2r)! - (r!)^2}{(r!)^2} f^{(r)}(x) + \frac{(3r^2+4r+1)(2r)!}{\{(r+1)!\}^2} x f^{(r+1)}(x)$$
$$\frac{3(3r^4+14r^3+23r^2+16r+4)(2r)!}{2\{(r+2)!\}^2} x^2 f^{(r+2)}(x)$$

Proof: From Taylor expansion theorem of function

$$f(t) = \sum_{i=0}^{r+2} \frac{f^{(i)}(x)}{i!}(t-x)^i + \epsilon(t,x)(t-x)^{r+2},$$

where$\epsilon(t,x) \to 0$ as $t \to x$. Therefore applying Lemma 2

$$\lim_{n\to\infty}\left\{nP_n^{(r)}(f(t),x) - f^{(r)}(x)\right\} = \left\{n\sum_{i=0}^{r+2} \frac{f^{(i)}(x)}{i!} \int_0^{\infty} W_n^{(r)}(t,x)\,(t-x)^i dt - f^{(r)}(x)\right\}$$
$$+ n\int_0^{\infty} W_n^{(r)}(t,x)\,\epsilon(t,x)(t-x)^{r+2}, dt$$
$$:= J_1 + J_2.$$

From here

$$J_1 = n\sum_{i=0}^{r+2} \frac{f^{(i)}(x)}{i!} \sum_{j=0}^{i} \binom{i}{j} (-x)^{i-j} \int_0^{\infty} W_n^{(r)}(t,x)\, t^j\, dt - f^{(r)}(x)$$
$$= \frac{f^{(r)}(x)}{r!}\left\{nP_n^{(r)}(t^r,x) - r!\right\} + \frac{f^{(r+1)}(x)}{(r+1)!} n\left\{(r+1)(-x)P_n^{(r)}(t^r,x) + P_n^{(r)}(t^{r+1},x)\right\}$$
$$+ \frac{f^{(r+2)}(x)}{(r+2)!} n\left\{\frac{(r+1)(r+2)}{2} x^2 P_n^{(r)}(t^r,x) + (r+2)(-x)P_n^{(r)}(t^{r+1},x) + P_n^{(r)}(t^{r+2},x)\right\}.$$

Using Theorem 3, for each $x \in (0,\infty)$, we get

$$J_1 = \frac{f^{(r)}(x)}{r!} r! \left\{n\frac{n^{-r}(n+r-1)!\ (2r)!}{(n-1)!\,(r!)^2} - 1\right\} + \frac{nf^{(r+1)}(x)}{(r+1)!} \times$$
$$\left[(r+1)(-x)r!\left\{\frac{n^{-r}(n+r-1)!\ (2r)!}{(n-1)!\,(r!)^2}\right\} + \left\{\begin{array}{c}\frac{n^{-r-1}(n+r)!\ (2r+2)!}{(n-1)!\,\{(r+1)!\}^2}(r+1)!\,x + \\ \frac{n^{-r-2}(n+r)!\ (2r+1)!}{(n-1)!\,(r!)^2} r!\end{array}\right\}\right]$$

$$+\frac{nf^{(r+2)}(x)}{(r+2)!}\left[\begin{array}{c}\frac{(r+1)(r+2)}{2}x^2\left\{\frac{n^{-r}(n+r-1)!\ (2r)!}{(n-1)!\,(r!)^2}r!\right\}\\ +(r+2)(-x)\left\{\begin{array}{c}\frac{n^{-r-1}(n+r)!\ (2r+2)!}{(n-1)!\,\{(r+1)!\}^2}(r+1)!\,x+\\ \frac{n^{-r-2}(n+r)!\ (2r+1)!}{(n-1)!\,(r!)^2}r!\end{array}\right\}\\ +\left\{\begin{array}{c}\frac{n^{-r-2}(n+r+1)!\ (2r+4)!}{(n-1)!\,\{(r+2)!\}^2}\frac{(r+2)!}{2}x^2\\ +\frac{n^{-r-3}(n+r+1)!\ (2r+3)!}{(n-1)!\,\{(r+1)!\}^2}(r+1)!\,x\end{array}\right\}+O(n^{-2})\end{array}\right].$$

Taking limitsas $n\to\infty$, we get the coefficients of $f^{(r)}$, $f^{(r+1)}$ and $f^{(r+2)}$ as $\frac{(2r)!-(r!)^2}{(r!)^2}$, $\frac{(3r^2+4r+1)(2r)!}{\{(r+1)!\}^2}x$ and $\frac{3(3r^4+14r^3+23r^2+16r+4)(2r)!}{2\{(r+2)!\}^2}x^2$. In order to complete the theorem, it is sufficient to show that $J_2\to 0$ as $n\to\infty$, which can be shown easily along with the proof of previous theorem.

Remark: In particular if $r=0$, we conclude the following asymptotic formula in ordinary approximation-

$$\lim_{n\to\infty}\{nP_n(f(t),x)-f(x)\}=\frac{3}{2}x^2f^{(r+2)}.$$

.3.3 Error Estimate in Simultaneous Approximation

Theorem: If $f\in C_\gamma[0,\infty),\gamma>0$ and$r\le m\le(r+2)$. If $f^{(m)}$exists and is continuous for sufficiently large $n$on$(a-\eta,b+\eta)$,

$$\left\|P_n^{(r)}(f,x)-f^{(r)}(x)\right\|\le C_1n^{-1}\sum_{i=r}^{m}\left\|f^i\right\|+C_2n^{-1/2}\omega\left(f^{(r+1)},n^{-1/2}\right)+O(n^{-2}),$$

where the constants C_1 and C_2 are independent of f and n, $\omega(f,\delta)$ is the modulus of continuity of f on $(a-\eta,b+\eta)$ and $\|.\|$ denotes the sup-norm on $[a,b]$.

Proof: By Taylor's expansion of f, we have

$$f(t)=\sum_{i=0}^{m}(t-x)^i\frac{f^{(i)}(x)}{i!}+(t-x)^m\chi(t)\frac{f^{(m)}(\xi)-f^{(m)}(x)}{m!}+h(t,x)\big(1-\chi(t)\big),$$

where$x<\xi<t$and $\chi(t)$is the characteristics function on $(a-\eta,b+\eta)$. For $t\in(a-\eta,b+\eta)$ and $x\in[a,b]$, we have

$$f(t)=\sum_{i=0}^{m}(t-x)^i\frac{f^{(i)}(x)}{i!}+(t-x)^m\frac{f^{(m)}(\xi)-f^{(m)}(x)}{m!}\chi(t).$$

For $t\in[0,\infty)\backslash(a-\eta,b+\eta)$, we define

$$h(t,x)=f(t)-\sum_{i=0}^{m}(t-x)^i\frac{f^{(i)}(x)}{i!}.$$

Thus

$$P_n^{(r)}(f,x)-f^{(r)}(x)=\left\{\sum_{i=0}^{m}\frac{f^{(i)}(x)}{i!}\int_0^\infty W_n^{(r)}(t,x)\,(t-x)^i dt-f^{(r)}(x)\right\}+\int_0^\infty W_n^{(r)}(t,x)\times$$

$$\frac{f^{(m)}(\xi)-f^{(m)}(x)}{m!}(t-x)^m\chi(t)dt\ +\int_0^\infty W_n^{(r)}(t,x)\,h(t,x)\big(1-\chi(t)\big)dt$$

$$:=K_1+K_2+K_3.$$

Using Theorem 3, we find

$$K_1=\sum_{i=0}^{m}\frac{f^{(i)}(x)}{i!}\sum_{j=0}^{i}\binom{i}{j}(-x)^{i-j}\int_0^\infty W_n^{(r)}(t,x)\,t^j\,dt-f^{(r)}(x)$$

$$= \sum_{i=0}^{m} \frac{f^{(i)}(x)}{i!} \sum_{j=0}^{i} \binom{i}{j} (-x)^{i-j} \frac{\partial^r}{\partial x^r} \left\{ \begin{array}{c} n^{-r} \frac{(n+r-1)!\ 2r!}{(n-1)!\,(r!)^2} x^r + n^{-r-1} \\ \times \frac{(n+r-1)!\ (2r-1)!}{(n-1)!\left((i-1)!\right)^2} x^{r-1} + O(n^{-2}) \end{array} \right\} - f^{(r)}(x)$$

Therefore

$$\|K_1\| \leq C_1 n^{-1} \sum_{i=r}^{m} \|f^i\| + O(n^{-2}),$$

uniformly in $x \in [a,b]$. Next, using Lemma 4 and Schwaz inequality

$$|K_2| \leq \int_0^\infty W_n^{(r)}(t,x) \frac{\left|f^{(m)}(\xi) - f^{(m)}(x)\right|}{m!} |t-x|^m \chi(t) dt$$

$$\leq \frac{\omega\left(f^{(m)},\delta\right)}{m!} \int_0^\infty \left|W_n^{(r)}(t,x)\right| \left(1 + \frac{|t-x|}{\delta}\right) |t-x|^m dt.$$

$$\leq \frac{\omega\left(f^{(m)},\delta\right)}{m!} C_2 \sum_{\substack{2i+j\leq r,\\ i,j\geq 0}} n^{i+1} \left\{\sum_{v=0}^{\infty} |v-nx|^{2j} p_{n,v}(x)\right\}^{\frac{1}{2}} \times$$

$$\left[\left\{\int_0^\infty q_{n,v}(t)dt\right\}^{\frac{1}{2}} \left\{\int_0^\infty q_{n,v}(t)\,|t-x|^{2m} dt\right\}^{\frac{1}{2}} + \delta^{-1} \left\{\int_0^\infty q_{n,v}(t)dt\right\}^{\frac{1}{2}} \left\{\int_0^\infty q_{n,v}(t)\,|t-x|^{2(m+1)} dt\right\}^{\frac{1}{2}}\right]$$

uniformly in x. Choosing$\delta = n^{-1/2}$, we get

$$\|K_2\| \leq C_2 \frac{\omega\left(f^{(m)},n^{-1/2}\right)}{m!} \left[O\left(n^{\frac{r-m}{2}}\right) + n^{1/2} O\left(n^{\frac{r-m-1}{2}}\right)\right] \leq C_2 \omega\left(f^{(m)},n^{-1/2}\right) O\left(n^{-\left(\frac{m-r}{2}\right)}\right).$$

For K_3 , applying Lemma 4

$$\|K_3\| \leq \sum_{\substack{2i+j\leq r,\\ i,j\geq 0}} n^{i+1} \frac{\left|Q_{i,j,r}(x)\right|}{x^r} \sum_{v=0}^{\infty} |v-nx|^j p_{n,v}(x) \int_0^\infty q_{n,v}(t) |h(t,x)| \left(1-\chi(t)\right) dt$$

$$\leq M_1 \sum_{\substack{2i+j\leq r,\\ i,j\geq 0}} n^{i+1} \sum_{v=0}^{\infty} |v-nx|^j p_{n,v}(x) \int_{|t-x|\geq\delta} q_{n,v}(t) |h(t,x)|\, dt$$

where$M_1 = \max_{\substack{2i+j\leq r,\\ i,j\geq 0}} \frac{\left|Q_{i,j,r}(x)\right|}{x^r}$.

Further we take an integer $\mu = max\{\gamma, m\}$ so that there is a constant M_2such that $|h(t,x)| \leq M_2(t-x)^\mu$ for $|t-x| \geq \delta$.Nowapplying Lemmas 1 and 2, we can easily verify that $\|K_3\| = O(n^{-s})$for any $s > 0$ uniformly on $[a,b]$. The estimates of K_1, $K_2$and K_3 together give the required result and hence the proof is completed.

Note: Some readers can oppose the result in asymptotic formula for these operators, obtained by me. I request them for some improvement in this formula.

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