IOSR Journal of Mathematics (IOSR-JM)
e-ISSN: 2278-5728, p-ISSN: 2319-765X. Volume 15, Issue 2 Ser. II (Mar – Apr 2019), PP 52-55
www.iosrjournals.org

On the Numerical Range of λ-Commuting Operators

Wafula,J.Khalagai
Corresponding Author: Wafula,J.Khalagai

Abstract:*AproofisgiventhatifAandBareoperatorsonacomplexHilbertspacethatcommutebyascalorfactorthenthatscalorfactormustberealwhenevereitherAorBisselfadjoint.*
AMS2000Mathematicsclassification47B47,47A30,47B20
Keywords; Numericalrange,λ-

Date of Submission: 29-03-2019 Date of acceptance: 13-04-2019

CommutingOperators,normalandselfadjointoperatorsUniversityOfNairobi SchoolofMathematicsP.O.Box30197Nairobi,KenyaEmail:¡awafula@uonbi.ac.ke¿Introduc-tion

LetHbeaninfinitedimensionalHilbertspaceandB(H)denotetheBanachAlgebraofboundedlinearoperatorsonH,thenwesaythatAandBareλ-CommutingOperatorsif$AB=\lambda BA$,Commutatingoperatorshavehardextensiveapplicationinquantummechanical observationandanalysisofthespectra.InthisarticleweconsiderresultsobtainedbyPutnamandobtaincondi-tionstobeimposedonλsothat$AB=\lambda BA$.WearriveatsimmilarconditionstothoseobtainedbyBrookwithoutspectralanalysis oftheoperator.Weshowthatwhenoperatorscommutetoascalefactorthenthenumericalrange$W(AB)$oftheproductis realorpureimaginery

Anoperatorissaidtobenormalif$AA^*=A^*A$,selfajointif$A=A^*$andanti-selfajointif

$A=-A^*$,

Theorem1.1(Putnam-Fuglede)

IfTisanormaloperatorandTcommuteswithanyoperatorSthen$T^*S=ST^*$

Corollary1.2

IfAandBarenormaloperatorssuchthat$TA=BT$then$TA^*=B^*T$foralloperatorsTonH

Proof

Let $A=\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$

and

$$L=\begin{bmatrix} 0 & 0 \\ T & 0 \end{bmatrix}$$

thenNisnormalandfurthermore$LN=NL$.Soby(i)$LN^*=N^*L$. -

Hence$TA^*=B^*T$ApplyingFugledestheoremRehder(1982)madesomeremarkableobservationsontheproductof selfadjointoperatorsasfollows.

Theorem1.3

LetAandBbeselfadjointoperatorsandeitherAorBispositive.ThenABisselfadjointifandonlyifABisnormal

Proof

.ObviouslyifABisselfajointthenABisnormal.FortheconverseweNotethat

$A(BA)=(AB)A$

andsoweinvokeFugledestheoremtoobtain

$A(BA)^*=(AB)^*A$ i.e. $A^2B=BA^2$.

SupposethatAispositive(otherwiseinterchangesroleswithB)thenAisthesquarerootofA^2

henceAcommuteswithBandso$(AB)^*=BA=AB$
LetTbeaselfadjointoperatorthen$T=AB$whereAispositiveandBunitary be thepolarde-compositionofT.Theprevioustheoremgivesaconverseintheclassofnormaloperators.ThusifTisnormaland$T=AB$ whereAispositiveandBselfadjointthenTisselfadjoint.

Corollary1.4

Let$T=A+iB$bethecanonicalformofanoperatorwherebyAandBareselfadjoint.IfABisnormalandeitherAorBispositivethen Tisnormal

Proof

Wehavethat

$TT^*=(A+iB)(A-IB)=A^2+i(BA-AB)+B^2=A^2+B^2$.

Ontheotherhand

$T^*T=(A-iB)(A+iB)=A^2+i(AB-BA)+B^2=A^2+B^2=TT^*$

Corollary1.5

LetAandBbeSelfadjoint,andeitherAorBbepositive.If$AB-BA\neq 0$thenAlso$AB+BA\neq 0$
.

Proof

If$AB+BA=0$then$(AB)(AB)^*=(AB)(BA)=-(AB)^2$.Simmilarly$(AB)^*(AB)=(BA)(AB)=-(AB)^2$.
ConsequentlyABisnormalandbytheorem1.5.7ABisselfadjointi.e$AB-AB=0$.

λ-commutingoperators

WeconsideroperatorsthatcommuteuptoscalarmultiplesandmakesomeapplicationofthePutnamFugledeproperty toobtainthefollowing

Theorem2.1

LetA,BbeboundedoperatorsonaHilbertspaceHsuchthat$AB=\lambda BA$and$AB\neq 0$where

λisacomplexnumber.Then

(i) IfAorBisSelfadjointwemusthavethatλisarealnumber(ii)IfAandBareSelfadjointthen$\lambda\in\{1,-1\}$
(iii) IfAandBareselfadjointandeitherAorBispositivethen$\lambda=1$.

Proof

(i) $AB=\lambda BA$ifandonlyif

$B^*A^* = \bar{\lambda}A^*B^*$.

Butifsay$A=A^*$then

$B^*A = \bar{\lambda}AB^*$.

BuyPutnamFuegledetheorem$B^*A^* = \lambda A^*B^*$.Hencewehavethat

$\bar{\lambda}A^*B^* = \lambda A^*B^*$

Consequentlywehavethat$(\bar{\lambda} - \lambda)A^*B^* = 0$Since$AB \neq 0$weobtaintheresult$(\bar{\lambda}-\lambda)=0$.So

$\bar{\lambda}=\lambda$

(ii) If$A=A^*, B=B^*$then$AB = \lambda BA$andso$(AB)^* =$

$\bar{\lambda}A^*B^*$.Hence$BA=$

$\bar{\lambda}AB=$

$\bar{\lambda}\lambda BA=|\lambda|^2 BA$.Consequently$(1-|\lambda|^2)BA=0$.Sofrom(i)$\lambda=\pm 1$.

(iii) Let$AB=-BA$thenConsiderthecommutatorofAandBnamely$AB - BA = AB + AB = 2AB \neq 0$.Howeverthisleadstotheanticommutaorbeingnonzero.i.e$AB + BA \neq 0$.Thisisacontradictionhencewemust havethat$\lambda=1$

AnoperatorAiscalledanti-selfadjointif$A+A^*=0$

Remark:IfAisananti-selfadjointoperatorthenAisnormaloperator.Toseethisnotethat

$AA^*=-A^2=A^*A$

Theorem2.2

IfanoperatorTisanti-selfadjointthenW(T)isapureimaginary.

Proof

Notethat$\lambda \in W(T) \Leftrightarrow \lambda=(Tx,x)=(x,T^*x)=-(x,Tx)=-\bar{\lambda}$.Thus$\lambda+\bar{\lambda}=0$andhence$\lambda$ isapureimaginarynumber.

Theorem2.3

Let$AB = \lambda BA, AB \neq 0$andBbeanormaloperator.IfAisanti-selfadjointthenλisareal number

Proof

If$AB = \lambda BA$thenwetaketheadjointonbothsidestofind$B^*A^* = \bar{\lambda}A^*B^*$.Henceapplying

thePutnam-Fuegledetheoremwehavethat

$B^*A=\lambda AB^* \Leftrightarrow -B^*A^*=-\lambda A^*B^* \Leftrightarrow (AB)^*=\lambda A^*B^* \Leftrightarrow (\lambda BA)^*=\lambda A^*B^* \Leftrightarrow$

$\lambda A^*B^* \Leftrightarrow (\bar{\lambda}-\lambda)A^*B^*=0$.Hence$\bar{\lambda}=\lambda$

Corollary2.4

$\bar{\lambda}A^*B^*=$

IfAandBareanti-selfadjointoperatorssuchthat$AB = \lambda BA and AB \neq 0$then$\lambda \in \{1,-1\}$

Proof
SinceAisnormalwehavethatλisarealnumber.Furthermore,fromAB=λBAweobtain

$B^{*}A^{*}=\lambda A^{*}B^{*}\Leftrightarrow BA=\lambda AB=\lambda^{2}BA\Leftrightarrow(1-(\lambda)^{2})BA=0$.Since$AB\neq 0$weobtain$\lambda$=$\pm 1$
Corollary2.5

IfAandBareanti-selfadjointoperatorssuchthatAB=$\lambda BA$and$AB\neq$
0theneither$W(AB)$isarealorpureimagineryset.

Proof
Ifλ=1thenABisselfadjointandso$W(AB)$isreal.Ifλ=-1thenABisanti-selfadjointandso$W(AB)$
ispureimaginery.

References

[1]. BrookJames,PaulBuschandDavidPearson,Commutativityuptoascalorfactorof
[2]. BoundedOperatorsinComplexHilbertSpace
[3]. CarlosSKubrusly,HilbertSpaceOperators;Aproblemsolvingappraoch,(2008)p84WulfRehder,OntheProductOfSelfadjointOperators,Internationaljournalofmathe-maticsandsciencevol5no4(1982)813-816

Wafula,J.Khalagai. " On the Numerical Range of λ-Commuting Operators." IOSR Journal of Mathematics (IOSR-JM) 15.2 (2019): 52-55.