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Numerical Solution of a Multilane Fluid Dynamic Traffic Flow Model with Three Lanes

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Abstract: *In this paper, we consider a multilane fluid dynamic traffic flow model for three lanes based on a linear velocity-density relationship which yields a non-linear first order system of hyperbolic partial differential equation as an IBVP. Due to the complexity of findings the analytical solution of the model, we investigate numerical solution by finite difference method. For numerical solution, we present the finite difference discretization of the model analogous to the second order Lax-Wendroff difference scheme and report on the stability and efficiency of the scheme by performing numerical experiments. The computed result satisfies some well known qualitative behaviors of the numerical solution based on artificial initial and boundary data.*

Keywords: *Multilane traffic flow model, Non-linear PDE and Numerical Simulation.*

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I. Introduction

Traffic congestion is one of the greatest problems in Bangladesh like some other countries of the world. In this respect, countries managing traffic in congested networks requires a clear understanding of traffic flow operations. Many research groups are involved in dealing with the different kinds of traffic models (like fluid dynamic models, kinetic models, microscopic models etc.) for several decades. Fluid dynamic traffic flow model which characterized by representations of traffic flow in terms of aggregate measures such as flux, space mean speed, and density was focused. A macroscopic traffic model first developed by Lighthill and Whitham (1955) and Richard (1956) shortly called LWR model ([2], [3]). Continuum models for multilane traffic are based on a system of conservation laws with source terms. A derivation of macroscopic multilane traffic flow model based on [8], [9], [10] and [16]. A hierarchy of a variety of traffic flow models is also presented here as in [8]. For elucidation of the derivation of multilane traffic model describe the balance equation of traffic flow and as well as closure relations on the balance equations follows from[8] and [9] etc. In this paper, we study a multilane fluid dynamic traffic flow model for three lanes based on a linear velocity-density relationship which yields a non-linear first order hyperbolic system of partial differential equations as an IBVP. Finding analytical solution of such system of PDEs complicated and we implement the second order Lax-Wendroff difference scheme for system of non-linear first order PDEs. We perform the finite difference descretization of the model analogous to the second order Lax-Wendroff difference scheme. The same stability conditions of the numerical schemes for single lane model are applied for three lanes of multilane model. In order to implement the numerical scheme we develop a computer programming code for the second order Lax-Wendroff difference scheme and perform numerical simulations of some qualitative behavior traffic flow with respect to various traffic parameters for artificially initial and boundary data and also present numerical simulation results of traffic flow for varying data of different situations of multilane traffic flow for a three lane highway.

II. Mathematical Model of Multilane Fluid Dynamic Traffic Flow

We assume a highway with N lanes. They are numbered by $\alpha = 1, 2, 3, ..., N.$ In our study, we focus on the macroscopic multilane traffic flow model which can be written in generalized form as flows from [8] and [10]:

$$\begin{aligned}
&\frac{\partial \rho_1}{\partial t}+\frac{\partial q_1}{\partial x}=\frac{\rho_2}{T_2^1}-\frac{\rho_1}{T_1^2}\\
&\frac{\partial \rho_j}{\partial t}+\frac{\partial q_j}{\partial x}=\frac{\rho_{j-1}}{T_{j-1}^j}-\frac{\rho_j}{T_j^{j-1}}+\frac{\rho_{j+1}}{T_{j+1}^j}-\frac{\rho_j}{T_j^{j+1}}\\
&\frac{\partial \rho_N}{\partial t}+\frac{\partial q_N}{\partial x}=\frac{\rho_{N-1}}{T_{N-1}^N}-\frac{\rho_N}{T_N^{N-1}}
\end{aligned} \qquad (1)$$

Here, the subscripts 1, $j=2,....,\mathrm{N}-1$ and N refer to the lane numbers. The quantities ρ_j and $q_j=\rho_j\, v_j$ are the vehicle density and the vehicle flux in the j-th lane respectively where as v_j is the vehicle velocity at the j -th lane for $j=1,2,....,\mathrm{N}$; at last $T_j^k=T_j^k\left(\rho_j,\,\rho_k\right)$ is the vehicle transition rate from lane j to lane k, with $\left|j-k\right|=1.$ In particular, we choose macroscopic multilane traffic flow model (1) for three lanes that is for $j=1,\,2,\,3\,(\mathrm{N}=3)$:

$$\begin{aligned}
&\frac{\partial \rho_1(t,x)}{\partial t}+\frac{\partial q_1\left(\rho_1(t,x)\right)}{\partial x}=\frac{\rho_2(t,x)}{T_2^1}-\frac{\rho_1(t,x)}{T_1^2}\\
&\frac{\partial \rho_2(t,x)}{\partial t}+\frac{\partial q_2\left(\rho_2(t,x)\right)}{\partial x}=\frac{\rho_1(t,x)}{T_1^2}-\frac{\rho_2(t,x)}{T_2^1}\\
&\frac{\partial \rho_3(t,x)}{\partial t}+\frac{\partial q_3\left(\rho_3(t,x)\right)}{\partial x}=\frac{\rho_2(t,x)}{T_2^3}-\frac{\rho_3(t,x)}{T_3^2}
\end{aligned} \qquad (2)$$

In this paper, the macroscopic multilane traffic flow model (2) is approximated by the Greenschield's linear density-velocity relationship as follows: $v(\rho)=v_{\max}\left(1-\frac{\rho}{\rho_{\max}}\right)$, where $v_{\max}$ is the maximum velocity and $\rho_{\max}$ is the maximum density which is based on bumper to bumper traffic.

III. Numerical Solution of Multilane Traffic Flow Model

In order to develop numerical solution of the model we have to make the model as an IBVP by inserting initial and boundary conditions. We present the descretization of our considered model which leads to second order Lax-Wendroff difference scheme. In the 2nd order P. Lax and B. Wendroff scheme, the model as an IBVP with two sided boundary conditions which takes the form:

$$\left.\begin{aligned}
&\frac{\partial \rho_1(t,x)}{\partial t}+\frac{\partial q_1\left(\rho_1(t,x)\right)}{\partial x}=\frac{\rho_2(t,x)}{T_2^1}-\frac{\rho_1(t,x)}{T_1^2}\\
&\frac{\partial \rho_2(t,x)}{\partial t}+\frac{\partial q_2\left(\rho_2(t,x)\right)}{\partial x}=\frac{\rho_1(t,x)}{T_1^2}-\frac{\rho_2(t,x)}{T_2^1}\\
&\frac{\partial \rho_3(t,x)}{\partial t}+\frac{\partial q_3\left(\rho_3(t,x)\right)}{\partial x}=\frac{\rho_2(t,x)}{T_2^3}-\frac{\rho_3(t,x)}{T_3^2}\\
&\text{with i.c. } \rho_1(0,x)=(\rho_1)_{\mathrm{o}}(x),\ \rho_2(0,x)=(\rho_2)_{\mathrm{o}}(x),\rho_3(0,x)=(\rho_3)_{\mathrm{o}}(x);a\le x\le b\\
&\text{and b.c. } \rho_1(t,a)=(\rho_1)_a(t),\ \rho_2(t,a)=(\rho_2)_a(t),\ \rho_3(t,a)=(\rho_3)_a(t);t_{\mathrm{o}}\le t\le T\\
&\qquad\qquad \rho_1(t,b)=(\rho_1)_b(t),\rho_2(t,b)=(\rho_2)_b(t),\rho_3(t,b)=(\rho_3)_b(t);\ t_{\mathrm{o}}\le t\le T\\
&\text{where } q_1=q_1\left(\rho_1(t,x)\right)=v_{1\max}\left(\rho_1-\frac{\rho_1^{\ 2}}{\rho_{1\max}}\right),\ q_2=q_2\left(\rho_2(t,x)\right)=v_{2\max}\left(\rho_2-\frac{\rho_2^{\ 2}}{\rho_{2\max}}\right)\\
&\qquad q_3=q_3\left(\rho_3(t,x)\right)=v_{3\max}\left(\rho_3-\frac{\rho_3^{\ 2}}{\rho_{3\max}}\right)
\end{aligned}\right\} \qquad (3)$$

In the study 2[nd] order Lax-Wendroff method, we discretize the time derivatives $\frac{\partial \rho_1(t,x)}{\partial t}$, $\frac{\partial \rho_2(t,x)}{\partial t}$ and $\frac{\partial \rho_3(t,x)}{\partial t}$ by first order forward difference in time and the space derivatives $\frac{\partial q_1(\rho_1(t,x))}{\partial x}$, $\frac{\partial q_2(\rho_2(t,x))}{\partial x}$ and $\frac{\partial q_3(\rho_3(t,x))}{\partial x}$ by second order centred difference in space.

Forward difference in time: according to the Taylor's series expansion, we obtain

$$\left.\begin{aligned} \frac{\partial \rho_1}{\partial t} &\approx \frac{\rho_1(x,\, t+k)-\rho_1(x,t)}{k} \\ \frac{\partial \rho_2}{\partial t} &\approx \frac{\rho_2(x,\, t+k)-\rho_2(x,t)}{k} \\ \frac{\partial \rho_3}{\partial t} &\approx \frac{\rho_3(x,\, t+k)-\rho_3(x,t)}{k} \end{aligned}\right\}$$

Central difference in space: also we apply the Taylor's series and obtain

$$\left.\begin{aligned} \frac{\partial q_1}{\partial x} &\approx \frac{q_1(x+h,\, t)-q_1(x-h,t)}{2h} \\ \frac{\partial q_2}{\partial x} &\approx \frac{q_2(x+h,\, t)-q_2(x-h,t)}{2h} \\ \frac{\partial q_3}{\partial x} &\approx \frac{q_3(x+,\, t)-q_3(x-h,t)}{2h} \end{aligned}\right\}$$

Again, the Taylor's series expansion we write

$$\rho(x,\, t+\Delta t) = \rho(x,t) + \Delta t \frac{\partial \rho}{\partial t} + \frac{(\Delta t)^2}{2!}\frac{\partial^2 \rho}{\partial t^2} + \ \ldots\ldots\ldots\ldots \qquad (4)$$

In equation (4), where the time derivatives can be replaced space derivatives using $\frac{\partial \rho_1}{\partial t} + \frac{\partial q_1}{\partial x} = \frac{\rho_2}{T_2^1} - \frac{\rho_1}{T_1^2}$, this has been done by so called Cauchy-Kawalewski technique which implies

$$\frac{\partial \rho_1}{\partial t} = \left(\frac{\rho_2}{T_2^1} - \frac{\rho_1}{T_1^2}\right) - \frac{\partial q_1(\rho_1)}{\partial x}$$

$$\begin{aligned} \Rightarrow \frac{\partial^2 \rho_1}{\partial t^2} &= \frac{\partial}{\partial t}\left(\frac{\rho_2}{T_2^1} - \frac{\rho_1}{T_1^2}\right) - \frac{\partial}{\partial t}\left(\frac{\partial q_1(\rho_1)}{\partial x}\right) \\ &= \frac{\partial}{\partial t}\left(\frac{\rho_2}{T_2^1} - \frac{\rho_1}{T_1^2}\right) + \frac{\partial}{\partial x}\left(-\frac{\partial q_1(\rho_1)}{\partial t}\right) \\ &= \frac{\partial}{\partial t}\left(\frac{\rho_2}{T_2^1} - \frac{\rho_1}{T_1^2}\right) - \frac{\partial}{\partial x}\left\{q_1'(\rho_1)\right\}\left(\frac{\rho_2}{T_2^1} - \frac{\rho_1}{T_1^2}\right) - q_1'(\rho_1)\frac{\partial}{\partial x}\left(\frac{\rho_2}{T_2^1} - \frac{\rho_1}{T_1^2}\right) + \\ &\quad \left\{\frac{q_1'\left(\rho^n_{1_{(i+\frac{1}{2})}}\right)\left(q_1\left(\rho_1{}^n_{(i+1)}\right) - q_1\left(\rho_1{}^n_i\right)\right) - q_1'\left(\rho^n_{1_{(i-\frac{1}{2})}}\right)\left(q_1\left(\rho_1{}^n_i\right) - q_1\left(\rho_1{}^n_{(i-1)}\right)\right)}{(\Delta x)^2}\right\} \end{aligned}$$

Since we have Taylor's series, $\frac{\rho_1(x, t+k)-\rho_1(x,t)}{k} = \frac{\partial \rho_1}{\partial t} + \frac{\Delta t}{2!}\frac{\partial^2 \rho_1}{\partial t^2} + \dots\dots\dots\dots$

$$\Rightarrow \frac{\partial \rho_1}{\partial t} = \left(\frac{\rho_2}{T_2^1} - \frac{\rho_1}{T_1^2}\right) - \frac{\partial q_1(\rho_1)}{\partial x} + \frac{\Delta t}{2!}\left[\begin{array}{l} \frac{\partial}{\partial t}\left(\frac{\rho_2}{T_2^1} - \frac{\rho_1}{T_1^2}\right) - \frac{\partial}{\partial x}\left\{q_1'(\rho_1)\right\}\left(\frac{\rho_2}{T_2^1} - \frac{\rho_1}{T_1^2}\right) - q_1'(\rho_1)\frac{\partial}{\partial x}\left(\frac{\rho_2}{T_2^1} - \frac{\rho_1}{T_1^2}\right) + \\ \left\{\frac{q_1'\left(\rho^n_{1(i+\frac{1}{2})}\right)\left(q_1\left(\rho^n_{1\,(i+1)}\right) - q_1\left(\rho^n_{1\,i}\right)\right) - q_1'\left(\rho^n_{1(i-\frac{1}{2})}\right)\left(q_1\left(\rho^n_{1i}\right) - q_1\left(\rho^n_{1(i-1)}\right)\right)}{(\Delta x)^2}\right\} \end{array}\right]$$

$$\Rightarrow \frac{\rho_{1i}^{n+1} - \rho_{1i}^n}{\Delta t} = \left(\frac{\rho_{2i}^n}{T_2^1} - \frac{\rho_{1i}^n}{T_1^2}\right) - \frac{1}{2(\Delta x)}\left\{q_1\left(\rho^n_{1(i+1)}\right) - q_1\left(\rho^n_{1(i-1)}\right)\right\} + \frac{\Delta t}{2!}.\frac{1}{T_2^1}\left(\frac{\rho_{2i}^{n+1} - \rho_{2i}^n}{\Delta t}\right) - \frac{\Delta t}{2!}.\frac{1}{T_1^2}\left(\frac{\rho_{1i}^{n+1} - \rho_{1i}^n}{\Delta t}\right) -$$

$$\frac{\Delta t}{4(\Delta x)}\left\{q_1'\left(\rho^n_{1(i+1)}\right) - q_1'\left(\rho^n_{1(i-1)}\right)\right\}\left(\frac{\rho_{2i}^n}{T_2^1} - \frac{\rho_{1i}^n}{T_1^2}\right) - \frac{\Delta t}{2!}q_1'(\rho_{1i}^n)\left\{\frac{1}{T_2^1}\left(\frac{\rho^n_{2(i+1)} - \rho^n_{2(i-1)}}{2\Delta x}\right) - \frac{1}{T_1^2}\left(\frac{\rho^n_{1(i+1)} - \rho^n_{1(i-1)}}{2\Delta x}\right)\right\} +$$

$$\frac{\Delta t}{2!}\left\{\frac{q_1'\left(\rho^n_{1(i+\frac{1}{2})}\right)\left(q_1\left(\rho^n_{1\,(i+1)}\right) - q_1\left(\rho^n_{1\,i}\right)\right) - q_1'\left(\rho^n_{1(i-\frac{1}{2})}\right)\left(q_1\left(\rho^n_{1i}\right) - q_1\left(\rho^n_{1(i-1)}\right)\right)}{(\Delta x)^2}\right\}$$

$$\Rightarrow \rho_{1i}^{n+1} = \left(\frac{2T_1^2}{2T_1^2 + \Delta t}\right)\left[\begin{array}{l} \rho_{1i}^n + \Delta t\left(\frac{\rho_{2i}^n}{T_2^1} - \frac{\rho_{1i}^n}{T_1^2}\right) - \frac{\Delta t}{2(\Delta x)}\left\{q_1\left(\rho^n_{1(i+1)}\right) - q_1\left(\rho^n_{1(i-1)}\right)\right\} + \frac{\Delta t}{2!T_2^1}\left(\rho_{2i}^{n+1} - \rho_{2i}^n\right) \\ + \frac{\Delta t}{2!T_1^2}\rho_{1i}^n - \frac{(\Delta t)^2}{4(\Delta x)}\left\{q_1'\left(\rho^n_{1(i+1)}\right) - q_1'\left(\rho^n_{1(i-1)}\right)\right\}\left(\frac{\rho_{2i}^n}{T_2^1} - \frac{\rho_{1i}^n}{T_1^2}\right) - \frac{(\Delta t)^2}{2!}q_1'(\rho_{1i}^n) \\ \left\{\left(\frac{\rho^n_{2(i+1)} - \rho^n_{2(i-1)}}{2T_2^1\Delta x}\right) - \left(\frac{\rho^n_{1(i+1)} - \rho^n_{1(i-1)}}{2T_1^2\Delta x}\right)\right\} + \frac{1}{2!}\left(\frac{\Delta t}{\Delta x}\right)^2 \\ \left\{q_1'\left(\rho^n_{1(i+\frac{1}{2})}\right)\left(q_1\left(\rho^n_{1\,(i+1)}\right) - q_1\left(\rho^n_{1\,i}\right)\right) - q_1'\left(\rho^n_{1(i-\frac{1}{2})}\right)\left(q_1\left(\rho^n_{1i}\right) - q_1\left(\rho^n_{1(i-1)}\right)\right)\right\} \end{array}\right] \quad (5)$$

Similarly,

$$\rho_{2i}^{n+1} = \left(\frac{2T_2^1}{2T_2^1 + \Delta t}\right)\left[\begin{array}{l} \rho_{2i}^n + \Delta t\left(\frac{\rho_{1i}^n}{T_1^2} - \frac{\rho_{2i}^n}{T_2^1}\right) - \frac{\Delta t}{2(\Delta x)}\left\{q_2\left(\rho^n_{2(i+1)}\right) - q_2\left(\rho^n_{2(i-1)}\right)\right\} + \frac{\Delta t}{2!T_1^2}\left(\rho_{1i}^{n+1} - \rho_{1i}^n\right) \\ + \frac{\Delta t}{2!T_2^1}\rho_{2i}^n - \frac{(\Delta t)^2}{4(\Delta x)}\left\{q_2'\left(\rho^n_{2(i+1)}\right) - q_2'\left(\rho^n_{2(i-1)}\right)\right\}\left(\frac{\rho_{1i}^n}{T_1^2} - \frac{\rho_{2i}^n}{T_2^1}\right) - \frac{(\Delta t)^2}{2!}q_2'(\rho_{2i}^n) \\ \left\{\left(\frac{\rho^n_{1(i+1)} - \rho^n_{1(i-1)}}{2T_1^2\Delta x}\right) - \left(\frac{\rho^n_{2(i+1)} - \rho^n_{2(i-1)}}{2T_2^1\Delta x}\right)\right\} + \frac{1}{2!}\left(\frac{\Delta t}{\Delta x}\right)^2 \\ \left\{q_2'\left(\rho^n_{2(i+\frac{1}{2})}\right)\left(q_2\left(\rho^n_{2\,(i+1)}\right) - q_2\left(\rho^n_{2\,i}\right)\right) - q_2'\left(\rho^n_{2(i-\frac{1}{2})}\right)\left(q_2\left(\rho^n_{2\,i}\right) - q_2\left(\rho^n_{2\,(i-1)}\right)\right)\right\} \end{array}\right] \quad (6)$$

and

$$\rho_{3i}^{n+1} = \left(\frac{2T_3^2}{2T_3^2+\Delta t}\right)\left[\begin{array}{l} \rho_{3i}^{\ n} + \Delta t\left(\frac{\rho_{2i}^{\ n}}{T_2^3} - \frac{\rho_{3i}^{\ n}}{T_3^2}\right) - \frac{\Delta t}{2(\Delta x)}\left\{q_3\left(\rho_{3(i+1)}^{\ n}\right) - q_3\left(\rho_{3(i-1)}^{\ n}\right)\right\} + \frac{\Delta t}{2!T_2^3}\left(\rho_{2i}^{\ n+1} - \rho_{2i}^{\ n}\right) \\ + \frac{\Delta t}{2!T_3^2}\rho_{3i}^{\ n} - \frac{(\Delta t)^2}{4(\Delta x)}\left\{q_3'\left(\rho_{3(i+1)}^{\ n}\right) - q_3'\left(\rho_{3(i-1)}^{\ n}\right)\right\}\left(\frac{\rho_{2i}^{\ n}}{T_2^3} - \frac{\rho_{3i}^{\ n}}{T_3^2}\right) - \frac{(\Delta t)^2}{2!}q_3'(\rho_{3i}^{\ n}) \\ \left\{\left(\frac{\rho_{2(i+1)}^{\ n} - \rho_{2(i-1)}^{\ n}}{2T_2^3\Delta x}\right) - \left(\frac{\rho_{3(i+1)}^{\ n} - \rho_{3(i-1)}^{\ n}}{2T_3^2\Delta x}\right)\right\} + \frac{1}{2!}\left(\frac{\Delta t}{\Delta x}\right)^2 \\ \left\{q_3'\left(\rho_{3(i+\frac{1}{2})}^{n}\right)\left(q_3\left(\rho_{3\ (i+1)}^{\ n}\right) - q_3\left(\rho_{3\ i}^{\ n}\right)\right) - q_3'\left(\rho_{3(i-\frac{1}{2})}^{n}\right)\left(q_3\left(\rho_{3i}^{\ n}\right) - q_3\left(\rho_{3(i-1)}^{\ n}\right)\right)\right\} \end{array}\right] \quad (7)$$

where

$$q_1'\left(\rho_{1(i\pm\frac{1}{2})}^{n}\right) = v_{1\max}\left(1 - \frac{2.\frac{1}{2}\left(\rho_{1(i\pm1)}^{n} + \rho_{1i}^{n}\right)}{\rho_{\max}}\right) = v_{1\max}\left(1 - \frac{1}{\rho_{1\max}}\left(\rho_{1(i\pm1)}^{n} + \rho_{1i}^{n}\right)\right),$$

$$q_2'\left(\rho_{2(i\pm\frac{1}{2})}^{n}\right) = v_{2\max}\left(1 - \frac{1}{\rho_{2\max}}\left(\rho_{2(i\pm1)}^{n} + \rho_{2i}^{n}\right)\right), q_3'\left(\rho_{3(i\pm\frac{1}{2})}^{n}\right) = v_{3\max}\left(1 - \frac{1}{\rho_{3\max}}\left(\rho_{3(i\pm1)}^{n} + \rho_{3i}^{n}\right)\right)$$

and

$$q_1\left(\rho_{1(i+1)}^{n}\right) = v_{1\max}\left(\rho_{1(i+1)}^{n} - \frac{\left(\rho_{1(i+1)}^{n}\right)^2}{\rho_{1\max}}\right), \quad q_1\left(\rho_{1i}^{n}\right) = v_{1\max}\left(\rho_{1i}^{n} - \frac{\left(\rho_{1i}^{n}\right)^2}{\rho_{1\max}}\right), q_1\left(\rho_{1(i-1)}^{n}\right) = v_{1\max}\left(\rho_{1(i-1)}^{n} - \frac{\left(\rho_{1(i-1)}^{n}\right)^2}{\rho_{1\max}}\right),$$

$$q_2\left(\rho_{2(i+1)}^{n}\right) = v_{2\max}\left(\rho_{2(i+1)}^{n} - \frac{\left(\rho_{2(i+1)}^{n}\right)^2}{\rho_{2\max}}\right), q_2\left(\rho_{2i}^{n}\right) = v_{2\max}\left(\rho_{2i}^{n} - \frac{\left(\rho_{2i}^{n}\right)^2}{\rho_{2\max}}\right), \quad q_2\left(\rho_{2(i-1)}^{n}\right) = v_{2\max}\left(\rho_{2(i-1)}^{n} - \frac{\left(\rho_{2(i-1)}^{n}\right)^2}{\rho_{2\max}}\right),$$

$$q_1\left(\rho_{3(i+1)}^{n}\right) = v_{3\max}\left(\rho_{3(i+1)}^{n} - \frac{\left(\rho_{3(i+1)}^{n}\right)^2}{\rho_{3\max}}\right), \quad q_3\left(\rho_{3i}^{n}\right) = v_{3\max}\left(\rho_{3i}^{n} - \frac{\left(\rho_{3i}^{n}\right)^2}{\rho_{3\max}}\right), q_3\left(\rho_{3(i-1)}^{n}\right) = v_{3\max}\left(\rho_{3(i-1)}^{n} - \frac{\left(\rho_{3(i-1)}^{n}\right)^2}{\rho_{3\max}}\right).$$

The equations (5), (6) and (7) are the Lax-Wendroff schemes for the IBVP (3).

IV. Numerical Simulation and Results Discussion

In this section, we present numerical results for some specific cases of traffic flow focusing on the traffic flow parameters of multilane traffic flow for a three lane highway. We choose maximum velocity $v_{1\max} = v_{2\max} = v_{3\max} = 60$ km/hour. It is notified that for satisfying the CFL condition we pick the unit of velocity as km/sec. We consider $\rho_{1\max} = \rho_{2\max} = \rho_{3\max} = 180/\text{km},$ and perform the numerical experiment for 6 minutes in 3600 time steps with $\Delta t = 0.1$ second for a three lanes highway of 10 km in 401 spatial grid points with step size $\Delta x = 100$ meters. We consider the initial density of multilane traffic flow model for three lanes are $\rho_1(0,x)$, $\rho_2(0,x)$ and $\rho_3(0,x)$. The transition rate from second lane to first lane 20%, first lane to second lane 10%, second lane to third lane 20% and third lane to second lane 10%. We run the program and attain the initial density profiles as shown in figure-I. The computational time is calculated as14.680696 seconds. In figure-II shows the traffic density behavior of multilane traffic flow for three lanes after six minutes.

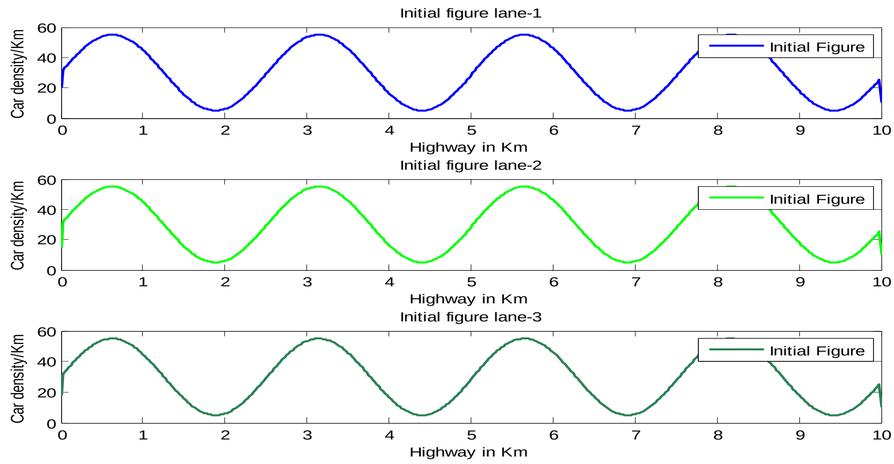

Figure-I: Initial Density Profile

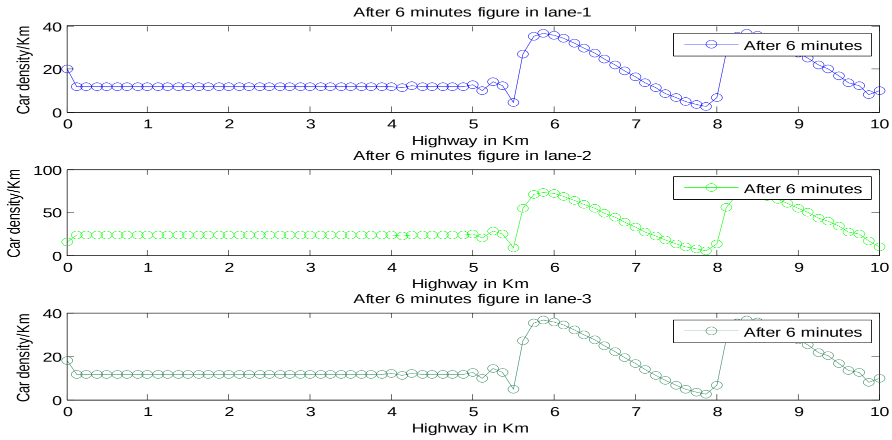

Figure-II: Density Profile after 6 minutes

In figure-III, we run the behavior of car density with respect to highway in km for 6 minutes Here, we observed that as time goes on the traffic wave is moving forward with reducing wave height. Using Lax-Wendroff scheme we get the density for three lanes of multilane traffic flow model. We can also compute the velocity from the density with the aid of our considered velocity- density relationship. The computed velocity is presented with respect to the distance in the following figure-IV.

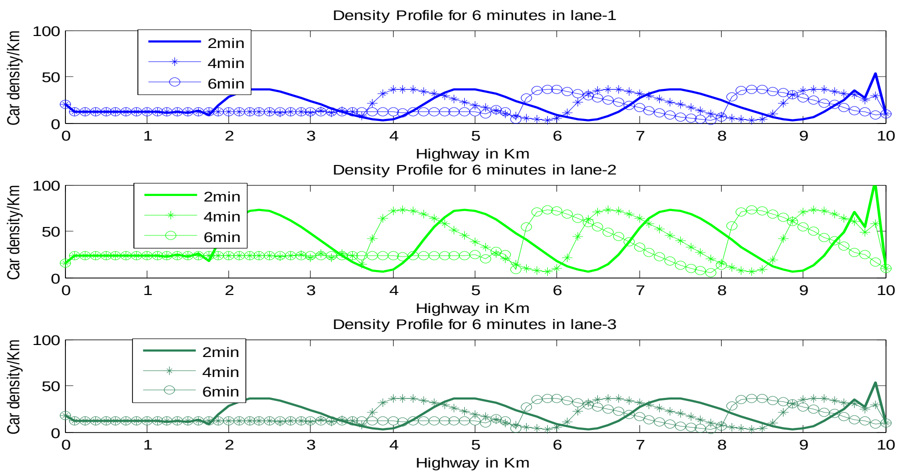

Figure-III: Computed Density Profile for 6 minutes

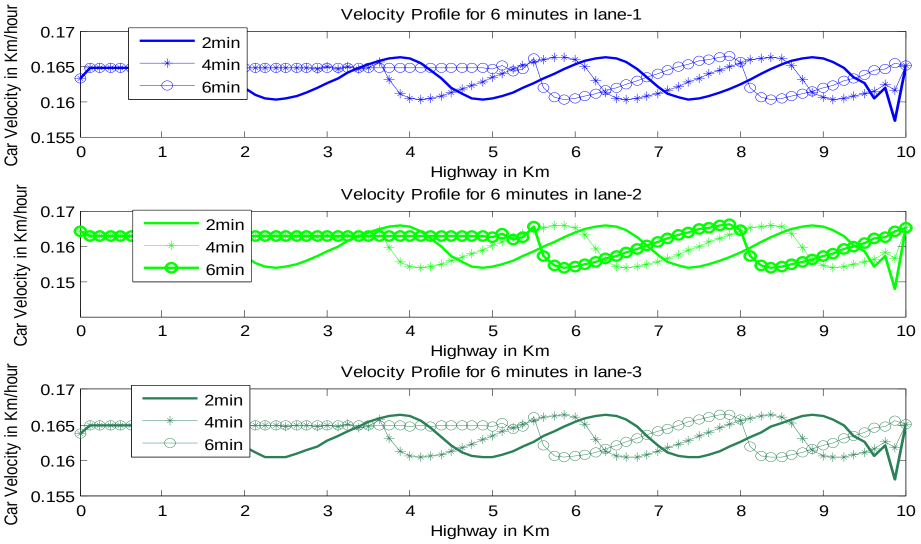

Figure-IV: Computed Velocity Profile for 6 minutes

With the help of our chosen flux-density relationship i.e. $q = q(\rho) = v_{\max}\left(\rho - \frac{\rho^2}{\rho_{\max}}\right)$, we can compute the flux at different points of the 10 km highway. The computed flux is presented with respect to the distance in the figure-V for three different time step as follows for three lanes. The velocity is depicted also with respect to the density using the Lax-Wendroff scheme in the figure-VI. The figure shows that the velocity and density relationship is linear which agrees accurately with our assumptions.

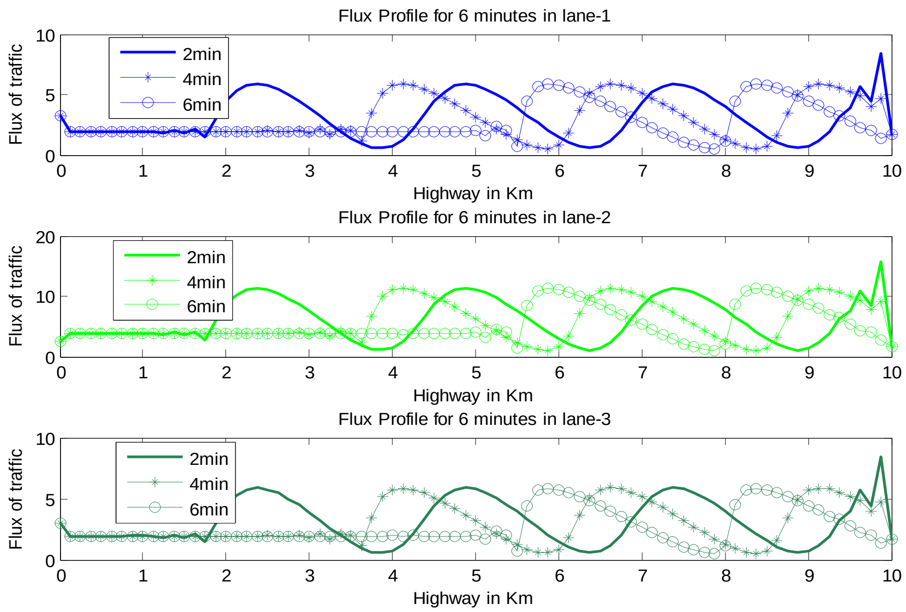

Figure-V: Computed Flux Profile for 6 minutes

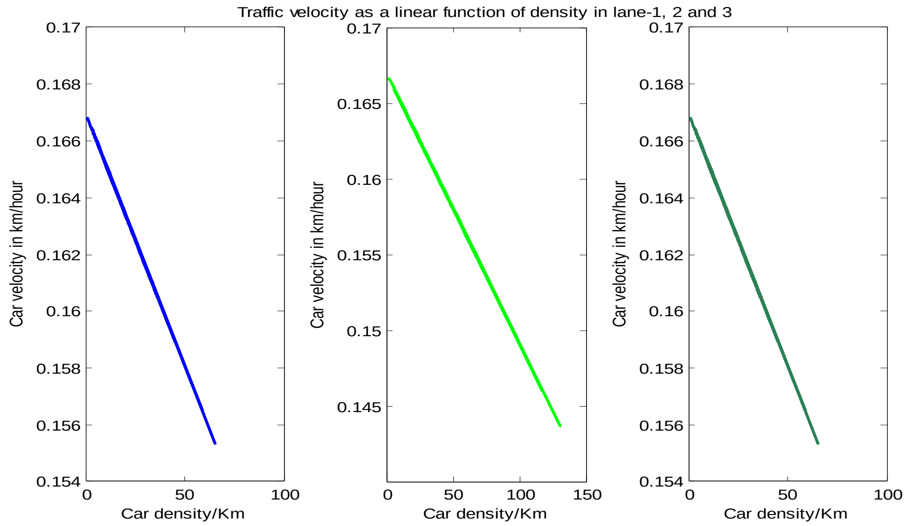

Figure-VI: Traffic velocity as a function of density

The numerical solution also converges with respect to the smaller discretization spatial grid size, Δx and temporal grid size, Δt which is a very good qualitative behavior of the numerical solution of multilane traffic flow model for three lanes.

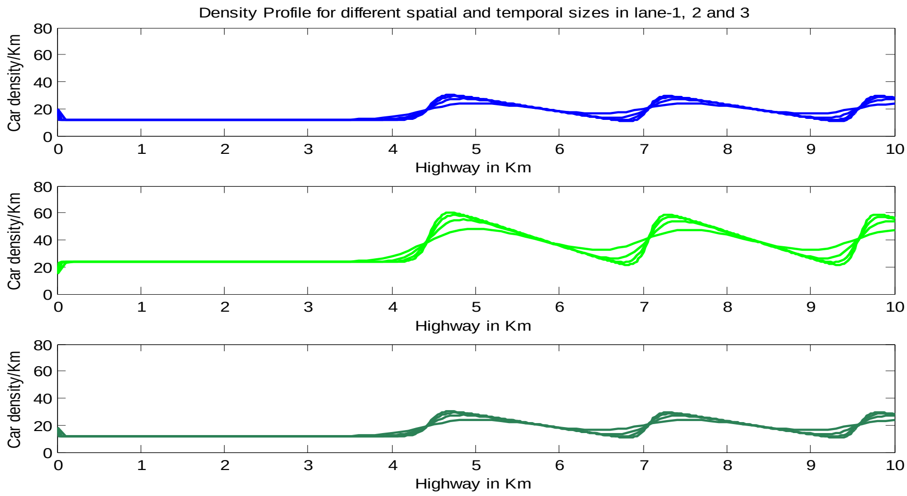

Figure-VII: Density Profile for different spatial and temporal step sizes

We also compute the total traffic flow from the traffic flow of three lanes as our consideration. In that case we add the computed density of three lanes which is presented in figure-VIII according to the distance. The total density of three lanes agrees with density profile for single lane traffic flow model as in same.

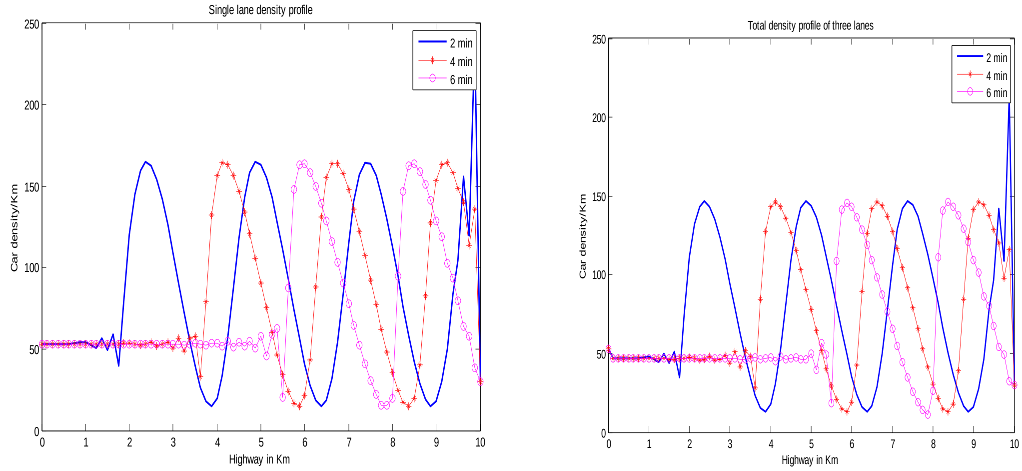

Figure-VIII: Density profile in single lane and Total density of three lanes

The fundamental diagram of traffic flow is one of the important characteristics of fluid dynamic traffic flow model. The computed flux is plotted with respect to the density profile by the flux-density relationship formula (linear case) $q(\rho) = v_{\max}\left(\rho - \dfrac{\rho^2}{\rho_{\max}}\right)$, which is parabolic and concave function in the range $0 \le \rho \le \rho_{\max}$. Figure-IX presents the graphs of flux for three lanes with respect to the density which are known as the fundamental diagram of traffic flow. Consequently, this is a very good qualitative agreement of the second order Lax-Wendroff scheme of multilane traffic flow model for three lanes.

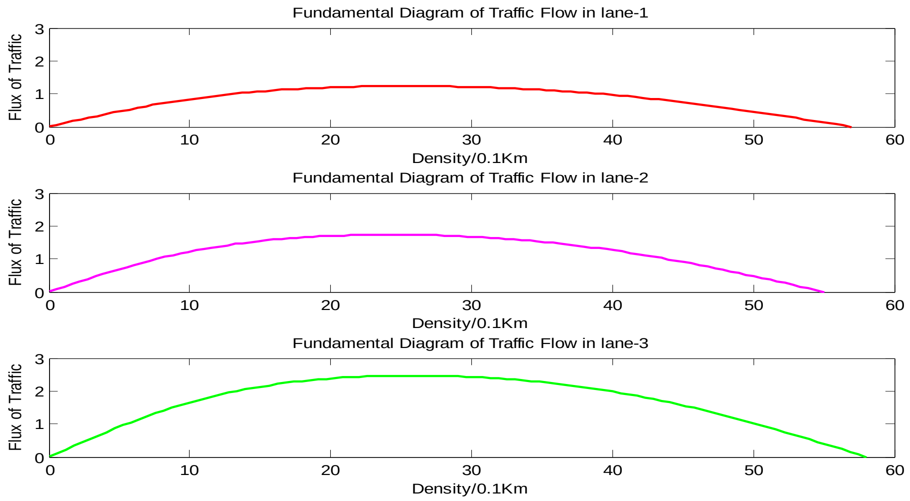

Figure-IX: Traffic flux as a function of density

In this paper, we investigate stability condition and well-posed-ness condition for the numerical scheme. The stability condition for Lax-Wendroff scheme for single lane traffic flow model follows from [17] is guaranteed by the simultaneous conditions $0 < \left(v_{\max}\dfrac{\Delta t}{\Delta x}\right) \le 1 / \left(1 - \dfrac{2\max(\rho_i^0)}{\rho_{\max}}\right)$ and $-\Delta x \le v_{\max}\,\Delta t \max(\rho_i^0) \le \Delta x.$ In case of multilane traffic flow model, experimentally we see that the stability condition of Lax-Wendroff scheme also remain unchanged for our considered multilane traffic flow model.

V. Conclusion

We have described the derivation of second order Lax-Wendroff scheme for the numerical solution of multilane traffic flow model as an IBVP. We have developed a computer programming code for the numerical scheme in order to perform numerical simulation with respect to various traffic flow parameters. We have shown that the numerical result based on the Lax-Wendroff difference scheme agrees with basic some qualitative behavior of multilane traffic flow model for three lanes. This qualitative behavior agreement verified the implementation of the multilane traffic model for three lanes with sufficient accuracy. Also we have shown that the numerical solution converges for decreasing the temporal and spatial grid sizes which is very good agreement of the numerical solution. We also present that the sum of the flow of three lanes is in same nature as the flow of the single lane traffic flow model. We have presented fundamental diagram of multilane traffic flow model for three lanes provided by Lax-Wendroff scheme that is also a very good agreement of the qualitative behavior of the numerical scheme.

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