IOSR Journal of Mathematics (IOSR-JM)
e-ISSN: 2278-5728, p-ISSN: 2319-765X. Volume 15, Issue 4 Ser. I (Jul – Aug 2019), PP 58-62
www.iosrjournals.org

The Radius, Diameter, Girth and Circumference of the Zero-Divisor Cayley Graph of the Ring $(Z_n, \oplus, \odot)$

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Abstract:*Thezero-divisor Cayley graph $G(Z_n, D_0)$, associated with the ring $(Z_n,\oplus,\odot)$, of residue classes modulo $n \geq 1$, an integer and the set $D_0$of nonzero zero-divisors is studied by Devendra et al. In this paper we present the eccentricity, radius, diameter, girth and circumference of the zero-divisor Cayley graph $G(Z_n, D_0)$.*
Keywords:*Cayley graph, zero-divisor Cayley graph, eccentricity, girth and circumference.*
AMS Subject Classification(2010):*05C12, 05C25, 05C38.*

Date of Submission: 08-07-2019 Date of acceptance: 23-07-2019

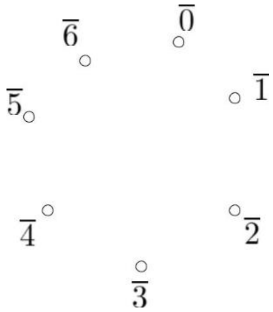
I. Introduction

The Cayley graph $G(X, S)$ associated with the group $(X,.)$ and its symmetric subset S (a subset S of the group $(X,.)$ is called a symmetric subset $s^{-1} \in S$ for every $s \in S$) is introduced to study whether given a group $(X,.)$, there is a graph Γ, whose automorphism group is isomorphic to the group $(X,.)$ [10].Later independent studies on Cayley graphs have been carried out by many researches [5,6]. The Cayley graph $G(X, S)$ associated with the group $(X,.)$ and its symmetric subset S is the graph, whose vertex set is X and the edge set$E = \{(x, y)\text{: either } xy^{-1} \in S, \text{or}, yx^{-1} \in S\}$. If $e \notin S$, where e is the identity element of X, then $G(X, S)$ is an undirected simple graph. Further $G(X, S)$ is $|S| -$ regular and contains $\frac{|X||S|}{2}$ edges [12]. Madhavi [12] introduced Cayley graphs associated with the arithmetical functions, namely, the Euler totient function $\varphi(n)$, the quadratic residues modulo a prime p and the divisor function $d(n), n \geq 1$, an integer and obtained various properties of these graphs.

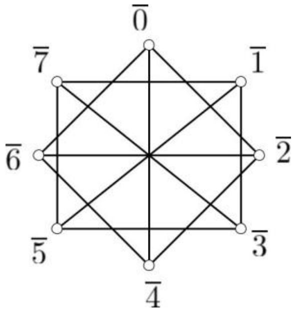
Recent studies on the zero-divisor graphs of commutative rings are carried out by Beck [4], Anderson and Naseer [2], Livingston[11], Anderson and Livingston [1], Smith [13], Tongsuo [14]and others. Given a commutative ring R with unity, they define the zero-divisor graph $\Gamma(R)$ is the graph, whose vertex set is the ring $Z(R)^*$, the set of nonzero divisors of R and the edge set is the set of order pairs (x, y) of elements$x, y \in Z(R)^*$, such that $xy = 0$ and studied the connectedness, the diameter, the girth, the automorphism $\Gamma(R)$ and other properties under conditions on the ring R. Our study differs from their study basically that the zero-divisor graph we consider is the Cayley graph associated with the set of zero-divisors of the ring $(Z_n, \oplus, \odot)$ of residue classes modulo $n \geq 1$, an integer. The terminology and notations that are used in this paper can be found in [7] for graph theory, [9] for algebra and [3] for number theory.

II. The Zero-Divisor Cayley Graph And Its Properties

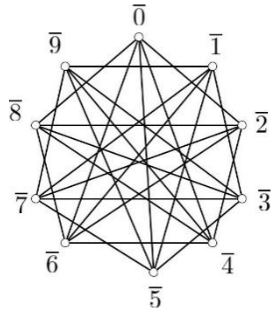
Consider the ring$(Z_n, \oplus,\odot)$of integers modulo n, $n \geq 1$, an integer, which is a commutative ring with unity. In [8], it is established that the set D_0 of nonzero zero-divisors in the ring $(Z_n, \oplus,\odot)$ is a symmetric subset of the group $(Z_n, \oplus)$ and the zero-divisor Cayley graph $G(Z_n, D_0)$ is the graph, whose vertex set is Z_n and the edge set is the set of ordered pairs (u, v)such that $u, v \in Z_n$ and either $u - v \in D_0$ or $v - u \in D_0$.This graph is$(n - \varphi(n) - 1)$ $-$regular and its size is$\frac{n}{2}(n - \varphi(n) - 1)$.
The graphs $G(Z_7, D_0), G(Z_8, D_0)$ and $G(Z_{10}, D_0)$ are given below :



$G(Z_7, D_0)$



$G(Z_8, D_0)$



$G(Z_{10}, D_0)$

We state below the main results that are established in [8] for the zero-divisor Cayley graph $G(Z_n, D_0)$.

Lemma 2.1: (Lemma 2.10, [8]) For a prime p, the graph $G(Z_p, D_0)$ contains only isolated vertices.

Lemma 2.2: (Theorem 3.7, [8]) For a prime p and an integer $r > 1$, the graph $G(Z_{p^r}, D_0)$ contains p disjoint components, each of which is complete subgraph of $G(Z_{p^r}, D_0)$.

Lemma 2.3: (Theorem 4.4, [8]) Let $n > 1$ be an integer, which is not a power of a single prime. Then the graph $G(Z_n, D_0)$ is a connected graph.

III. Eccentricity, Radius And Diameter Of The Zero-Divisor Cayley Graph

Definition 3.1: Let $G(V, E)$ be a graph with the vertex set V and edge set is E. The **distance** $d(u, v)$ between two vertices u and v in the graph G is defined as the length of the shortest path joining them, if any. If there is no path joining the vertices u and v in graph $G(V, E)$, then it is defined by $d(u, v) = \infty$.

Definition 3.2: Let $G(V, E)$ a graph. The **eccentricity** $e(v)$ of a vertex $v \in V(G)$ is defined as $e(v) = \max\{d(u, v): u \in V(G)\}$.

Definition 3.3: Let $G(V, E)$ be a graph. The **radius** $r(G(V, E))$ and the **diameter** $d(G(V, E))$ of the graph $G(V, E)$ are respectively defined as $r(G(V, E)) = min\{e(v): v \in V\}$ and $d(G(V, E)) = max\{e(v): v \in V\}$.

Example 3.4: Consider the graph $G(V, E)$, where $V = \{a, b, c, d, e, f\}$ and
$E = \{(a, b), (a, d), (b, c), (b, d), (c, e), (d, f), (e, f)\}$,
whose diagram is given below. The following table gives $d(u, v)$ for all vertices in V and eccentricity $e(v)$ of a vertex $v \in V(G)$, radius $r(v)$ and diameter $d(v)$ of a vertex $v \in V(G)$.

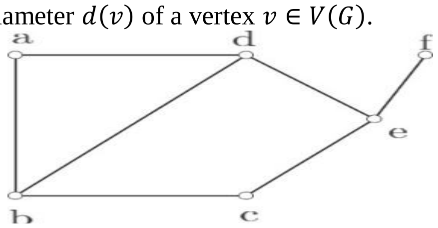


Fig. 3.1

$d(u,v)$	a	b	c	d	e	f
a	0	1	2	1	2	3
b	1	0	1	1	2	3
c	2	1	0	2	1	2
d	1	1	2	0	1	2
e	2	2	1	1	0	1
f	3	3	2	2	1	0
$e(v)$	3	3	2	2	2	3
$r(G(V,E))$	$min = \{3,3,2,2,2,3\} = 2$					
$d(G(V,E))$	$max = \{3,3,2,2,2,3\} = 3$					

Theorem 3.5: If $n = p^r, r > 1$ is a integer, then the eccentricity of any vertex $v \in Z_{p^r}$ is ∞.

Proof: By the Remark 3.1[8], the zero-divisor Cayley graph $G(Z_{p^r}, D_0)$ is a disjoint union of the following$p$-components$C_0, C_1, \ldots, C_{p-1}$, each of it is a complete sub graph of the graph $G(Z_{p^r}, D_0)$.

$$C_0 = \{\bar{0},\ \bar{p},\ 2\bar{p}, \ldots, i\bar{p}, \ldots, j\bar{p}, \ldots, (p^{r-1} - 1)\bar{p}\},$$
$$C_1 = \{\bar{1},\ \bar{p} + \bar{1},\ 2\bar{p} + \bar{1}, \ldots, i\bar{p} + \bar{1}, \ldots, j\bar{p} + \bar{1}, \ldots, (p^{r-1} - 1)\bar{1}\},$$
$$\vdots$$
$$C_k = \{\bar{k},\ \bar{p} + \bar{k},\ 2\bar{p} + \bar{k}, \ldots,\ i\bar{p} + \bar{k}, \ldots, j\bar{p} + \bar{k}, \ldots, (p^{r-1} - 1)\bar{k}\},$$
$$\vdots$$
$$C_{p-1} = \{\overline{p-1}, \bar{p} + \overline{p-1}, 2\bar{p} + \overline{p-1}, \ldots, i\bar{p} + \overline{p-1}, \ldots, (p^{r-1} - 1)\overline{p-1}\}.$$

Let $v \in G(Z_{p^r}, D_0)$.
Then$v \in C_i$, for some $i, 0 \le i \le p-1$. For any $u \in Z_{p^r}$, the following two cases will arise.
Case (i): Let $u \in C_i$. Then by the Lemma 3.4[8], C_i is a complete subgraph of $G(Z_{p^r}, D_0)$, so that v and u are adjacent in $G(Z_{p^r}, D_0)$and$d(v,u) = 1$.
Case (ii): Let $u \notin C_i$. Then $u \in C_j$, for some $j \ne i, 0 \le j \le p-1$. By the Lemma 3.5[8], C_i and C_j are edge disjoint subgraphs of $G(Z_{p^r}, D_0)$. So there is no edge between v and u so that $d(v,u) = \infty$.Thus
$e(v) = max\{1, \infty\} = \infty$. ■

Theorem 3.6: If $n > 1$ is a integer, where n is not a power of single prime, then the eccentricity of a any vertex $v \in G(Z_n, D_0)$ is 2.
Proof: By the Theorem 4.4[8], $G(Z_n, D_0)$ is connected and by the Remark 4.1[8], the vertex set V is the union of the subsets $V_0, V_1, \dots, V_{p-1}$ of vertices, where

$$V_0 = \left\{0, 1\overline{p_1}, 2\overline{p_1}, \dots, i\overline{p_1}, \dots, \left(\frac{n-p_1}{p_1}\right)\overline{p_1}\right\},$$
$$V_1 = \left\{\overline{p_2}, 1\overline{p_1} + \overline{p_2}, 2\overline{p_1} + \overline{p_2}, \dots, i\overline{p_1} + \overline{p_2}, \dots, \left(\frac{n-p_1}{p_1}\right)\overline{p_1} + \overline{p_2}\right\},$$
$$\vdots$$
$$V_{p_1-1} = \left\{(p_1-1)\overline{p_2}, \dots, i\overline{p_1} + (p_1-1)\overline{p_2}, \dots, \left(\frac{n-p_1}{p_1}\right)\overline{p_1} + (p_1-1)\overline{p_2}\right\}.$$

Let v be any vertex of $G(Z_n, D_0)$. Then
$v = \overline{ip_1 + lp_2} \in V_l$, for some l, $0 \le l \le p_1 - 1$ and for some $i, 0 \le i \le \left(\frac{n-p_1}{p_1}\right) - 1$.
For any vertex $u \in G(Z_n, D_0)$, the following two cases will arise.
Case (i): Let $u \in V_l$. Then by the Lemma 4.3 [8], V_l is a complete subgraph of $G(Z_n, D_0)$, so that v and u are adjacent in $G(Z_n, D_0)$ and $d(v,u) = 1$.
Case (ii): Let $u \notin V_l$. Then $u = \overline{jp_1 + kp_2} \in V_k$, for some $k \ne l$, $0 \le k \le p_1 - 1$ and for some j, $0 \le j \le \left(\frac{n-p_1}{p_1}\right) - 1$. We may assume that $i < j$. Let $w = \overline{ip_1 + kp_2} \in V_k$. Since $i < j \le \left(\frac{n-p_1}{p_1}\right)$, it follows that$w \in V_k$. That is, $u, w \in V_k$. Now V_k being a complete subgraph of $G(Z_n, D_0), u$ and w are adjacent, so that $d(w,u) = 1$.
Further $v - w = \overline{ip_1 + lp_2} - \overline{ip_1 + kp_2} = \overline{(l-k)p_2}$, which is a zero divisor in the ring $(Z_n, \oplus, \odot)$. So there is an edge between v and w, so that $d(v,w) = 1$.So
$d(v,u) = d(v,w) + d(w,u) = 1 + 1 = 2$ and $e(v) = max\{1,2\} = 2$.

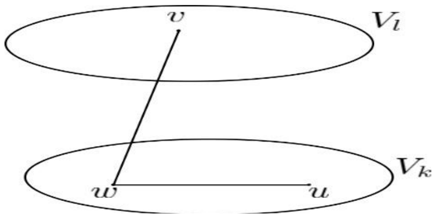


Fig. 3.2

■

Theorem 3.7: If$n = p^r, r > 1$ is a integer, then the radius and diameter of any vertex $v \in Z_{p^r}$ is ∞.
Proof: By the Theorem 3.5, the eccentricity$e(v) = \infty$, for every vertex$v$ in $G(Z_n, D_0)$. So
$r(G(Z_n, D_0)) = min\{\infty\} = \infty$ and $d(G(Z_n, D_0)) = max\{\infty\} = \infty$. ■

Theorem 3.8: If $n > 1$is a integer, where n is not a power of single prime, then

$$r(G(Z_n, D_0)) = d(G(Z_n, D_0)) = 2.$$

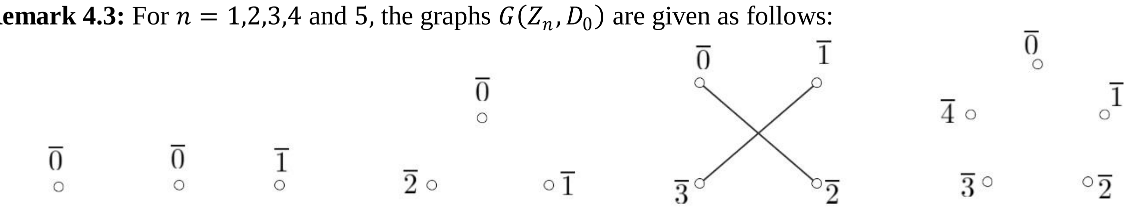
Proof: By the Theorem 3.6, the eccentricity $e(v) = 2$, for any vertex v in $G(Z_n, D_0)$, so that

$$r(G(Z_n, D_0)) = min\{2\} = 2 \text{ and } d(G(Z_n, D_0)) = max\{2\} = 2.$$ ■

IV. The Girth And The Circumference Of The Zero-Divisor Cayley Graph

Definition 4.1: The length of the smallest cycle in the graph $G(V, E)$is called the **girth** of the graph $G(V, E)$ and it is denoted by $g(G(V, E))$ and the length of the largest cycle in the graph $G(V, E)$ is called the **circumference** of the graph $G(V, E)$ and it is denoted by $c(G(V, E))$. If the graph $G(V, E)$ has no cycles then the girth and the circumference are undefined.
Remark 4.2: If p is a prime, then the graph $G(Z_p, D_0)$ has no edges. So that the girth and the circumference of graph $G(Z_p, D_0)$ is undefined.

Remark 4.3: For $n = 1,2,3,4$ and 5, the graphs $G(Z_n, D_0)$ are given as follows:



$G(Z_1, D_0)$ $G(Z_2, D_0)$ $G(Z_3, D_0)$ $G(Z_4, D_0)$ $G(Z_5, D_0)$

Fig. 4.1

One can observe that there are no cycles in the above graphs, so that the girth and the circumference are undefined. Thus for $n \leq 5$, the terms girth and circumference of the graph $G(Z_n, D_0)$ are undefined.

Theorem 4.4: If $n > 5$ is not a prime, then $g\big(G(Z_n, D_0)\big)$ is 3.

Proof: Let $n > 5$ be not a prime and let p_1 be the least prime divisor of n. Then $\overline{p_1}, \overline{2p_1} \in D_0$. For the vertices $\overline{0}, \overline{p_1}, \overline{2p_1} \in G(Z_n, D_0)$, we have $\overline{2p_1} - \overline{p_1} = \overline{p_1} \in D_0$, $\overline{p_1} - \overline{0} = \overline{p_1} \in D_0$ and $\overline{2p_1} - \overline{0} = \overline{2p_1} \in D_0$, so that $(\overline{0}, \overline{p_1}), (\overline{p_1}, \overline{2p_1})$ and $(\overline{2p_1}, \overline{0})$ are edges in $G(Z_n, D_0)$. So $(\overline{0}, \overline{p_1}, \overline{2p_1}, \overline{0})$ is a 3-cycle and $g\big(G(Z_n, D_0)\big)$ is 3. ■

Theorem 4.5: If $n = p^r$, where p is a prime and $r > 1$ an integer and let $n > 5$, then$c\big(G(Z_n, D_0)\big)$ is $\frac{n}{p}$.

Proof: Let $n > 5$ be a power of single prime say $n = p^r$, p a prime and $r > 1$ aninteger. By the Theorem 3.7[8], $G\big(Z_{p^r}, D_0\big)$ is decomposed into p-components, $C_0, C_1, C_2, \ldots C_{p-1}$, where

$$C_k = \{\bar{k},\ \bar{p} + \bar{k},\ 2\bar{p} + \bar{k}, \ldots,\ i\bar{p} + \bar{k}, \ldots, j\bar{p} + \bar{k}, \ldots, (p^{r-1} - 1)\bar{k}\},$$

for some k, $0 \leq k \leq p - 1$. Now

$$C = \big(\bar{k},\ \bar{p} + \bar{k},\ 2\bar{p} + \bar{k}, \ldots,\ i\bar{p} + \bar{k}, \ldots, j\bar{p} + \bar{k}, \ldots, (p^{r-1} - 1)\bar{k}, \bar{k}\big)$$

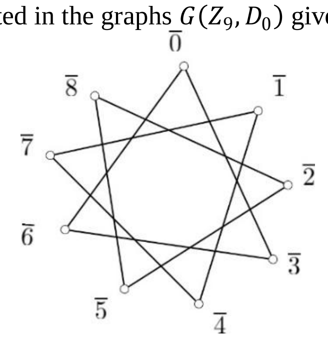
is a cycle of length $\frac{n}{p}$, which is also a cycle of maximum length. So

$$c\big(G(Z_n, D_0)\big) = \frac{n}{p}.$$ ■

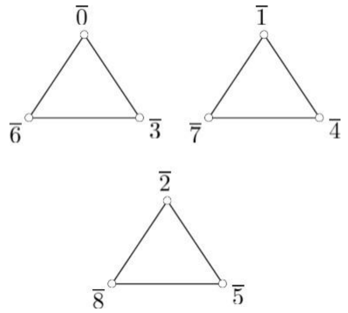
Example 4.6: Consider the graph $G(Z_9, D_0)$. Here $p = 3$. This graph has 3-components $\{\overline{0}, \overline{3}, \overline{6}\}, \{\overline{1}, \overline{4}, \overline{7}\}$ and $\{\overline{2}, \overline{5}, \overline{8}\}$ each of which is a triangle. Further these are the only cycles in $G(Z_9, D_0)$. Since a triangle is a cycle of length 3. It follows that

$$g\big(G(Z_9, D_0)\big) = c\big(G(Z_9, D_0)\big) = 3.$$

This fact is exhibited in the graphs $G(Z_9, D_0)$ given below.



The graph $G(Z_9, D_0)$



The components of $G(Z_9, D_0)$

Fig.4.2

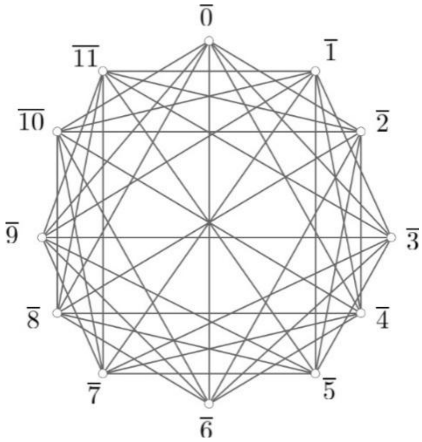
Theorem 4.7: If $n > 1$, is an integer and if$n$is not a power of a single prime, then $c\big(G(Z_n, D_0)\big)$ is n.

Proof: Let $n$be not a power of single prime and let p_1 be the least prime divisor of n. One can see that the following cycle

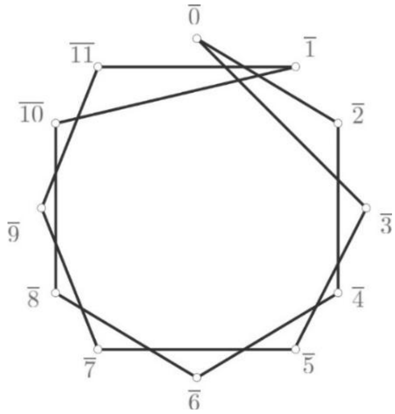
$$H = \left(\overline{0}, \overline{p_1}, .. \overline{ip_1}, \overline{p_1\left(\frac{n-p_1}{p_1}\right)}, \overline{\left(\frac{n-p_1}{p_1}\right)p_1} + \overline{p_2}, \ldots, \overline{p_1} + \overline{p_2}, \overline{p_2}, \ldots, \overline{p_1\left(\frac{n-p_1}{p_1}\right) + 2p_2}, \ldots, \overline{(p_1 - 1)p_2}, \overline{0}\right)$$ is

a Hamilton cycle in $G(Z_n, D_0)$ of length n. Since a Hamilton cycle is a cycle of maximum length n, it follows that $c\big(G(Z_n, D_0)\big) = n$. ■

Example4.8: Consider the graph $G(Z_{12}, D_0)$. Here $n = 12 = 2^2 \times 3$. So the graph is a connected graph. In this graph $(\overline{0}, \overline{2}, \overline{4}, \overline{0})$ is a cycle of length 3, so that $g\big(G(Z_{12}, D_0)\big) = 3$. Further $\big(\overline{0}, \overline{2}, \overline{4}, \overline{6}, \overline{8}, \overline{10}, \overline{1}, \overline{11}, \overline{9}, \overline{7}, \overline{5}, \overline{3}, \overline{0}\big)$ is a Hamilton cycle in $G(Z_{12}, D_0)$ which is of length 12, so that $c\big(G(Z_{12}, D_0)\big) = 12$. The graph $G(Z_{12}, D_0)$ and the above Hamilton cycle is given below.



The graph $G(Z_{12}, D_0)$



A Hamilton cycle in (Z_{12}, D_0)

Fig.4.3

Acknowledgements

The authors express their thanks to Prof. L. Nagamuni Reddy for his valuable suggestions during the preparation of this paper.

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Jangiti Devendra. "The Radius, Diameter, Girth and Circumference of the Zero-Divisor Cayley Graph of the Ring (Z_n,⊕, ⊙)." IOSR Journal of Mathematics (IOSR-JM) 15.4 (2019): 58-62