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Generalized Additive Functional Equation in Digital Spatial Image Crypto Techniques System

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Abstract: *Authors derive solution in general of a generalized additive functional equation, investigate its stabilities in modular space using fixed point theory and specially introduce its application in digital spatial image crypto techniques system using MATLAB*

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I. Introduction

For the detailed study on Ulam problem and its recent developments now called generalized Hyers-Ulam-Rassias stability, one can refer [1], [3], [4], [8], [9], [10] and [11]. The definitions related to our main theorem related to modular space can be referred in [5].

Authors in [7] in 2017, obtained refined stability results and investigate modular stability for functional equations by direct method.

Authors, in this work mainly introduce a following generalized functional equation

$$\begin{aligned} &d\{g(dx+y)+g(dx-y)\}+g(x+dy)+g(x-dy) \\ &=g(x+y)+g(x-y)+2d^2g(x) \end{aligned} \tag{1.1}$$

of additive type for $d \neq 0, \pm 1$ and obtain its general solution and investigate its stabilities in modular space by using fixed point theory. Also, authors apply (1.1) in digital spatial image crypto techniques system using MATLAB.

This paper contains five sections includes introduction, general solution, stability, application of (1.1) and conclusion, respectively.

II. General solution

In this section, we obtain general solution of (1.1). Let X and Y be vector spaces.

Theorem 2.1. *If $g : X \to Y$ satisfies (1.1), then g is additive and odd.*

Proof. Let g fulfills (1.1). Substituting (x, y) by $(0, 0)$ in (1.1) leads $g(0) = 0$. Setting $y = 0$ in (1.1) gives

$$g(dx) = dg(x), \ \forall x \in X. \tag{2.1}$$

Therefore g is additive. Inserting $x = 0$ in (1.1) leads $g(-y) = -g(y)$, $\forall y \in X$ by (2.1). Therefore, g is odd. □

III. ULAM Stabilities of (1.1)

we determine the various Ulam stabilities of (1.1) in modular space by using fixed point theory. Let

$$D_d g(x,y) := d\{g(dx+y)+g(dx-y)\}+g(x+dy)+g(x-dy) \\ -\{g(x+y)+g(x-y)\}-2d^2g(x)$$

for $g: L \to X_m$, X_m-complete modular space and $x, y \in L$ with $d \neq 0, \pm 1$.

Theorem 3.1. *A function $s: L^2 \to [0,+\infty)$ such that*

$$\lim_{n\to\infty} \frac{1}{d^n} s\{d^n x, d^n y\} = 0, \tag{3.1}$$

and

$$s\{dx, dy\} \leq dvs(x,y) \tag{3.2}$$

$\forall x, y \in L$ with $v < 1$. If $g: L \to X_m$ fulfills the condition

$$m\left(D_d g(x,y)\right) \leq s(x,y), \tag{3.3}$$

$\forall x, y \in L$ and $g(0) = 0$. Then there exists $\alpha_d : L \to X_m$ a unique additive mapping such that

$$m(\alpha_d(x) - g(x)) \leq \frac{1}{d(1-v)} s(x,0), \ \forall x \in L. \tag{3.4}$$

Proof. We consider $M = \{h : L \to X_m\}$ and define the function m' on M as follows,

$$m'(g') =: \inf\{e > 0 : m(h(x)) \leq es(x,0), \forall x \in L\}.$$

One can easily prove m' is a convex modular on M satisfies the Fatou property and $M_{m'}$ is $m-$complete, see [12]. Consider the function $\theta : M_{m'} \to M_{m'}$ defined by

$$\theta q(x) = \frac{1}{d} q(dx), \tag{3.5}$$

for all $x \in L$ and $q \in M_{m'}$. Let $q, t \in M_{m'}$ and $e \in [0,1]$ an arbitrary constant such that $m'(q-t) < e$. By definition of m', we get

$$m(q(x) - t(x)) \leq es(x,0) \tag{3.6}$$

$\forall x \in L$. Equations (3.2) and (3.6) leads

$$m\left(\frac{q(dx)}{d} - \frac{t(dx)}{d}\right) \leq \frac{1}{d} m(q(dx) - t(dx)) \leq \frac{1}{d} es(dx,0) \leq evs(x,0),$$

for all $x \in L$. Hence, $m'(\theta q - \theta t) \leq vm'(q-t)$, for all $q, t \in M_{m'}$, so θ is a $m'-$contraction. Substituting $y = 0$ in (3.3), we obtain

$$m\left(\frac{g(dx)}{d} - g(x)\right) \leq \frac{1}{d}s(x,0),\ \forall x \in L. \tag{3.7}$$

Substituting x by xd in (3.7), we obtain

$$m\left(\frac{g(d^2x)}{d} - g(dx)\right) \leq \frac{s}{d}(dx,0), \tag{3.8}$$

$\forall x \in L$. Equation (3.7) and (3.8), gives

$$\begin{aligned} & m\left(\frac{g(d^2x)}{d^2} - g(x)\right) \\ & \leq m\left(\frac{g(d^2x)}{d^2} - \frac{g(dx)}{d}\right) + m\left(\frac{g(dx)}{d} - g(x)\right) \\ & \leq \frac{s(xd,0)}{d^2} + \frac{s(x,0)}{d},\ \forall x \in L. \end{aligned} \tag{3.9}$$

Generalizing by induction,

$$\begin{aligned} m\left(\frac{g(d^nx)}{d^n} - g(x)\right) & \leq \sum_{i=1}^{n} \frac{1}{d^i}s(d^{i-1}x,0) \\ & \leq \frac{1}{vd}s(x,0)\sum_{i=1}^{n} v^i \\ & \leq \frac{1}{d(1-v)}s(x,0),\ \forall x \in L. \end{aligned} \tag{3.10}$$

Equation (3.10) leads

$$\begin{aligned} & m\left(\frac{g(d^nx)}{d^n} - \frac{g(d^cx)}{d^c}\right) \\ & \leq \frac{1}{2}m\left(2\frac{g(d^nx)}{d^n} - 2g(x)\right) + \frac{1}{2}m\left(2\frac{g(d^cx)}{d^c} - 2g(x)\right) \\ & \leq \frac{\kappa}{2}m\left(\frac{g(d^nx)}{d^n} - g(x)\right) + \frac{\kappa}{2}m\left(\frac{g(d^cx)}{d^c} - g(x)\right) \\ & \leq \frac{\kappa s(x,0)}{d(1-v)},\ \forall x \in L, \end{aligned} \tag{3.11}$$

$n, c \in N$. Thus

$$m'(\theta^n g - \theta^c g) \leq \frac{\kappa}{d(1-v)},$$

which gives the boundedness of an orbit of θ at g. The sequence $\{\theta^n g\}$ is $m'-$converges to $\alpha_d \in M_{m'}$. By $m'-$contractivity of θ, we get

$$m'(\theta^n g - \theta\alpha_d) \leq vm'(\theta^{n-1}g - \alpha_d).$$

Allowing $n \to \infty$ and by Fatou property of m', we get

$$\begin{aligned} m'(\theta\alpha_d - \alpha_d) &\leq \lim_{n\to\infty} \inf m'(\theta\alpha_d - \theta^n g) \\ &\leq v \lim_{n\to\infty} \inf m'(\alpha_d - \theta^{n-1}g) = 0. \end{aligned}$$

Therefore, α_d is a fixed point of θ. Changing (x, y) as $(d^n x, d^n y)$ in (3.3), it leads

$$m\left(\frac{1}{d^n} D_d g(d^n x, d^n y)\right) \leq \frac{1}{d^n} s(d^n x, d^n y), \ \forall x, y \in L. \tag{3.12}$$

Allowing $n \to \infty$ also by Theorem 2.1, α_d is additive and using (3.10), we arrive (3.4). For the uniqueness of α_d, consider another additive mapping $\alpha : L \to X_m$ satisfying (3.4). Then, α is a fixed point of θ.

$$m'(\alpha_d - \alpha) = m'(\theta\alpha_d - \theta\alpha) \leq vm'(\alpha_d - \alpha) \tag{3.13}$$

From (3.13), we get $\alpha_d = \alpha$. This ends the proof. □

Corollary 3.2. *Let X be a Banach space, $s : L^2 \to [0, +\infty)$ a function such that*

$$\lim_{n\to\infty} \frac{1}{d^n} s\{d^n x, d^n y\} = 0, \tag{3.14}$$

and

$$s\{dx, dy\} \leq dvs(x, y), \ \forall x, y \in V, v < 1. \tag{3.15}$$

If $g : L \to X$ fulfills the condition

$$\|D_d g(x, y)\| \leq s(x, y), \ \forall x, y \in L, g(0) = 0, \tag{3.16}$$

there exists $\alpha_d : L \to X$ a unique additive mapping with

$$\|\alpha_d(x) - g(x)\| \leq \frac{1}{d(1-v)} s(x, 0), \ \forall x \in L. \tag{3.17}$$

Theorem 3.3. *Let $s : L^2 \to [0, +\infty)$ a function such that*

$$\lim_{n\to\infty} \kappa^n s\left(\frac{x}{d^n}, \frac{y}{d^n}\right) = 0, \tag{3.18}$$

and

$$s\left(\frac{x}{d}, \frac{y}{d}\right) \leq \frac{vs(x, y)}{d}, \ \forall x, y \in L, v < 1. \tag{3.19}$$

If $g : L \to X_m$ fulfills the condition

$$m\left(D_d g(x,y)\right) \le s(x,y), \forall x,y \in L, g(0)=0. \tag{3.20}$$

there exists $\alpha_d : L \to X_m$ *a unique additive mapping with*

$$m(\alpha_d(x) - g(x)) \le \frac{v}{d(1-v)} s(x,0), \tag{3.21}$$

for all $x \in L$.

Proof. we prove this Theorem by considering x by $\frac{x}{d}$ in (3.5) of Theorem3.1 and remaining part of the proof is same as Theorem 3.1. □

Corollary 3.4. *Let* $s : L^2 \to [0,+\infty)$ *a function with*

$$\lim_{n\to\infty} \theta^n s\left(\frac{x}{d^n}, \frac{y}{d^n}\right) = 0, \tag{3.22}$$

and

$$s\left(\frac{x}{d}, \frac{y}{d}\right) \le \frac{v}{d} s(x,y), \ \forall x,y \in L, v<1. \tag{3.23}$$

If $g : L \to X$ *fulfills the condition*

$$\|D_d g(x,y)\| \le s(x,y), \ \forall x,y \in L, g(0)=0, \tag{3.24}$$

there exists $\alpha_d : L \to X$ *a unique additive mapping with*

$$\|\alpha_d(x) - g(x)\| \le \frac{v}{d(1-v)} s(x,0), \ \forall x \in L. \tag{3.25}$$

We obtain Hyers-Ulam and Hyers-Ulam-Rassias stabilities of (1.1) in the following corollaries.

Corollary 3.5. *Let* X *be a Banach space, a real number* $\epsilon \ge 0$ *and a function* $s : L^2 \to [0,+\infty)$ *with*

$$\lim_{n\to\infty} \frac{1}{d^n} s(d^n x, d^n y) = 0, \tag{3.26}$$

and

$$s\{dx, dy\} \le dvs(x,y), \ \forall x,y \in L, v<1. \tag{3.27}$$

If $g : L \to X$ *fulfills the inequality*

$$\|D_d g(x,y)\| \le \epsilon, \ \forall x,y \in L, g(0)=0, \tag{3.28}$$

there exists $\alpha_d : L \to X$ *a unique additive mapping defined by* $\alpha_d(x) = \lim_{n\to\infty} \frac{g(d^n x)}{d^n}$ *with*

$$\|\alpha_d(x) - g(x)\| \leq \frac{\epsilon}{d-1}, \tag{3.29}$$

for all $x \in L$ *and* $d \neq 0, \pm 1$.

Corollary 3.6. *Let* L *and* X *a linear space and a Banach space, respectively. If* $g : L \to X$ *fulfills the inequality*

$$\|D_d g(x, y)\| \leq \epsilon \left(\|x\|^p + \|y\|^p\right), \tag{3.30}$$

$\forall x, y \in L$ *and* $g(0) = 0$ *with* $0 \leq p < 1$ *or* $p > 1$ *some* $\epsilon \geq 0$, *there exists* $\alpha_d : L \to X$ *a unique additive mapping defined by* $\alpha_d(x) = \lim_{n\to\infty} \frac{g(d^n x)}{d^n}$ *with*

$$\|\alpha_d(x) - g(x)\| \leq \frac{\epsilon}{|d - d^p|} \|x\|^p, \ \ \forall x \in L, \ d \neq 0, \pm 1. \tag{3.31}$$

IV. Functional Equations Based Spatial Image Crypto Technique

The term remote sensing takes on a specific implication dealing with space-borne imaging systems used to remotely sense the surface. Remote sensing is defined as data collected from a distance without visiting or interacting directly. When the distance between the object and viewer is large, or rather small, remote sensing approach suggests the use of spatial image. In modern days, the image based cryptographic techniques have advocated new and efficient ways to develop secure spatial image encryption techniques, see [2], [6]. In this research work, functional equations are used to improve the level of security in spatial image encryption. We apply functional equation (1.1) in digital spatial image crypto techniques system using MATLAB. An elementary idea is to encrypt the digital spatial image by applying the left hand side of (1.1). As the result, the intricate cypher image is obtained. See figures 1 and 2.

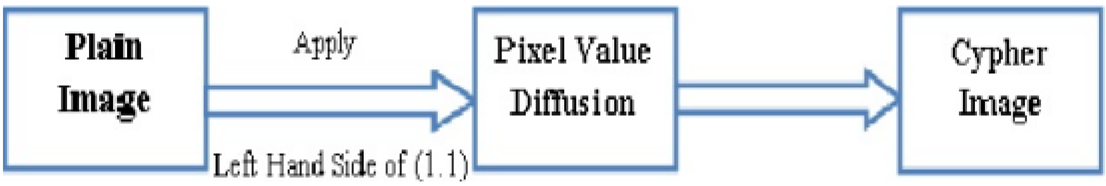

Figure 1. Encryption

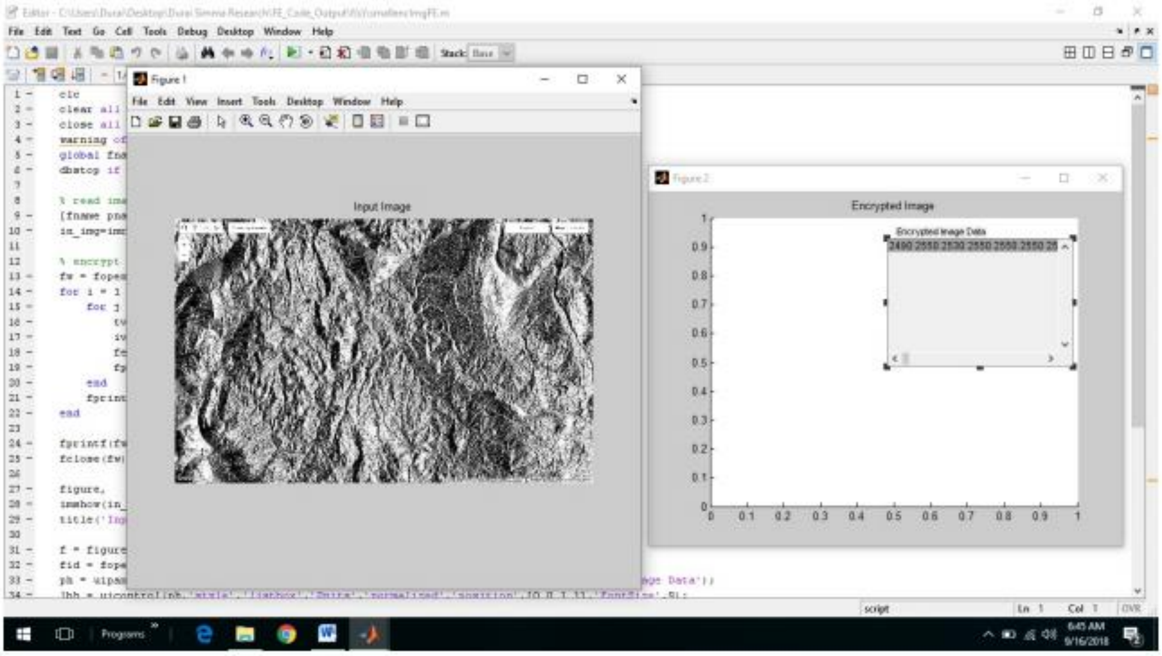

Figure 2. Image Encryption

When cypher image reaches the receiver, he must use right hand side of (1.1) as a key. On entering the accurate key, the MATLAB code decrypts the entire image and provides original image to the receiver. See figures 3 and 4.

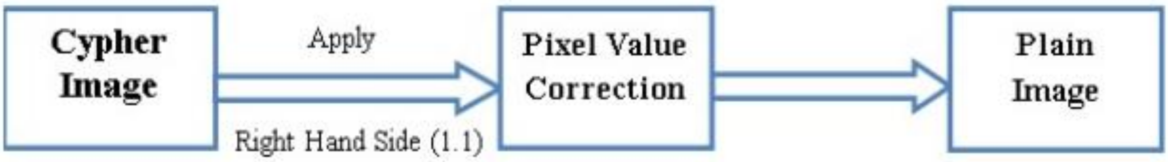

Figure 3. Decryption

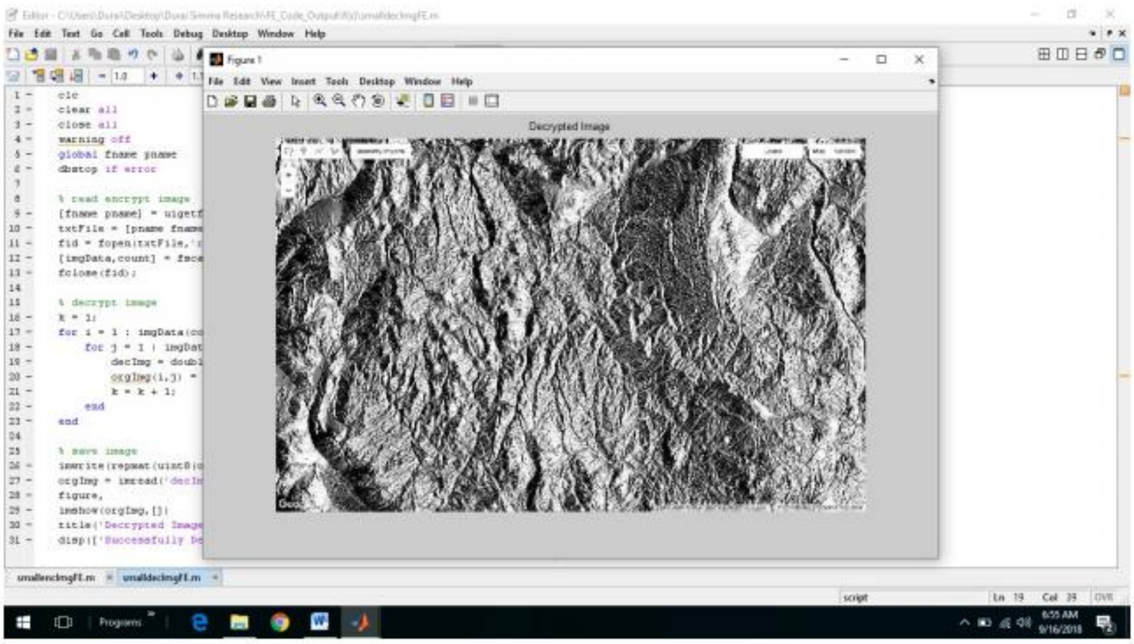

Figure 4. Image Decryption

4.1. Security Analysis. The distinctive approach in applying functional equations on spatial image crypto technique is, we use two different keys with same solutions that are LHS of functional equations for encrypting and RHS of functional equations for decrypting, whereas, traditional systems like DES, Triple- DES, RSA and IDEA use single key for both encryption and decryption. This uniqueness of functional equation progresses the security level of transmitting spatial image and overwhelmed traditional techniques limitations. A statistical analysis shows that the tactic for image crypto technique provides an effective and secure way for real time spatial image encryption and transmission from the cryptographic viewpoint.

V. Conclusion

We introduced a generalized additive functional equation, obtained its general solution and stabilities in modular space by using fixed point theory. Also, we applied (1.1) in digital spatial image crypto techniques system using MATLAB.

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