*IOSR Journal of Mathematics (IOSR-JM)*
*e-ISSN: 2278-5728, p-ISSN: 2319-765X. Volume 15, Issue 5 Ser. II (sep – Oct 2019), PP 57-65*
*www.iosrjournals.org*

# Mathematical Sanctity of the Golden Ratio

Nafish Sarwar Islam
*(Department of Industrial and Production Engineering, American International University - Bangladesh)*

***Abstract:*** *The frequency of appearance of the Golden Ratio (Φ) in nature implies its importance as a cosmological constantand sign of beingfundamental characteristic of the Universe.Except than Leonardo Da Vinci's 'Monalisa' it appears on the sunflower seed head, flower petals, pinecones, pineapple, tree branches, shell, hurricane, tornado, ocean wave, and animal flight patterns. It is also very prominent on human body as it appears on human face, legs, arms, fingers, shoulder, height, eye-nose-lips, and all over DNA molecules and human brain as well. It is inevitable in ancient Egyptian pyramids and many of the proportions of the Parthenon. But very few of us are aware of the fact that it is part and parcel for constituting black hole's entropy equations,black hole's specific heat change equation,also it appears atKomar Mass equation ofblack holes and Schwarzschild–Kottler metric - for null-geodesics with maximal radial acceleration at the turning point of orbits [1, 2, 3, 4].But here in this paper the discussion is limited to the exhibition of mathematical aptitude of Golden Ratio a.k.a. the Devine Proportion.*

***Keywords:*** *Golden Ratio, Devine Proportion, Cosmological Constant, Fundamental Constant of Nature*

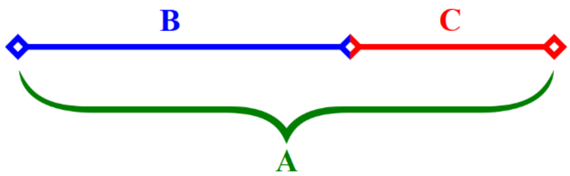
---------------------------------------------------------------------------------------------------------------------------------------

Date of Submission: 12-09-2019 Date of Acceptance: 27-09-2019

---------------------------------------------------------------------------------------------------------------------------------------

## I. Introduction

By definition, two quantities are in the golden ratio if their ratio is the same as the ratio of their sum to the larger of the two quantities. Let's say, straight line A is divided into two segments, into B and C in such manner that:




A/B = B/C = Φ; or, B.B = A.C; But we know that A = B + C. So, B.B = [B + C].C; that is, B.B = B.C + C.C. Now if we divide this equation by C.C, [ie, C-Square], we will find that, B.B/C.C = B.C/C.C + C.C/C.C. Which means that, (B/C)-Square = B/C + 1; ie, Φ^2 = Φ + 1 or, [Φ^2 – Φ – 1] = 0. Solving this quadratic equation will give us Φ = 1.618033988749895... the most irrational number which we denote by the Greek alphabet Phi.

Golden Ratio can be expressed in so many different ways. One of the most common expression is given below:

$$\Phi = 1 + \cfrac{1}{1 + \cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cdots}}}}$$

From this expression it can be formulated into Φ = $(1 + \frac{1}{\Phi})$, that is Φ^2 = (Φ + 1) or, [Φ^2 – Φ – 1] = 0.

Also another most common expression of Golden Ratio is: $\Phi = \sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\cdots}}}}}$ From this expression it can be formulated into Φ = $\sqrt{1+\ \Phi}$ ; that is, Φ^2 = (Φ + 1) or, [Φ^2 – Φ – 1] = 0.

## II. Golden Ratio Φ in Arithmetic Numerals

As we can see the quadratic equation [Φ^2 – Φ – 1] = 0 gives the root value equal to the golden ratio, it can be written as Φ = [Φ^2 – 1]. Hence, Φ = (Φ + 1).(Φ – 1). ie, 1.618034 = 2.618034 X 0.618034. Another interesting fact about that equation is, (Φ + 1) = 2.618034 = Φ^2 & (Φ – 1) = 0.618034 = $\frac{1}{\Phi}$ . So, [Φ^2.$\frac{1}{\Phi}$ ] = Φ.

So, [Φ^2 –Φ– 1] = 0
Or, 2Φ^2 –2Φ –2 = 0
Or, 2Φ^2 –(+1)Φ + (–1)Φ –2 = 0
Or, 2Φ^2 –($\sqrt{5}$ + 1)Φ + ($\sqrt{5}$ – 1)Φ –2 = 0
Or, Φ[2Φ – ($\sqrt{5}$ + 1)] + [($\sqrt{5}$ – 1)/2].[2Φ –$\frac{4}{(\sqrt{5}-1)}$] = 0
Or, Φ[2Φ – ($\sqrt{5}$ + 1)] + [($\sqrt{5}$ – 1)/2].[2Φ –$\frac{4(\sqrt{5}+1)}{(\sqrt{5}+1).(\sqrt{5}-1)}$] = 0
Or, Φ[2Φ – ($\sqrt{5}$ + 1)] +[($\sqrt{5}$ – 1)/2].[2Φ –$\frac{4(\sqrt{5}+1)}{\{(\sqrt{5})^2-(1)^2\}}$]= 0
Or, Φ[2Φ – ($\sqrt{5}$ + 1)] + [($\sqrt{5}$ – 1)/2].[2Φ –$\frac{4(\sqrt{5}+1)}{(5-1)}$] = 0
Or, Φ[2Φ – ($\sqrt{5}$ + 1)] + [($\sqrt{5}$ – 1)/2].[2Φ –4($\sqrt{5}$ + 1)/(4)] = 0
Or, Φ[2Φ – ($\sqrt{5}$ + 1)] + [($\sqrt{5}$ – 1)/2].[2Φ –($\sqrt{5}$ + 1)] = 0
Or, [2Φ – ($\sqrt{5}$ + 1)] × [Φ + {($\sqrt{5}$ – 1)/2}]= 0
Or, 2[Φ – {($\sqrt{5}$ + 1)/2}] × [Φ + {($\sqrt{5}$ – 1)/2}]= 0
Or, [Φ – {($\sqrt{5}$ + 1)/2}] × [Φ + {($\sqrt{5}$ – 1)/2}]= 0
So, either [Φ – {($\sqrt{5}$ + 1)/2}] = 0 or else, [Φ + {($\sqrt{5}$ – 1)/2}]= 0
Which means, Φ = $\frac{1\pm\sqrt{5}}{2}$ ; that is, +1.618034 or, –0.618034.

Alternate Solution - 1:

Given, [Φ^2 – Φ – 1] = 0
Or, [Φ^2 –1.618Φ + 0.618Φ – 1] = 0
Or, Φ(Φ – 1.618) + 0.618(Φ - $\frac{1}{0.618}$) = 0
Or, Φ(Φ – 1.618) + 0.618(Φ – 1.618)= 0
Or, (Φ – 1.618).(Φ + 0.618) = 0
ie, (Φ–1.618) = 0 or, (Φ+0.618) = 0
So, Φ = 1.618034 or, Φ = – 0.618034.

Alternate Solution - 2:

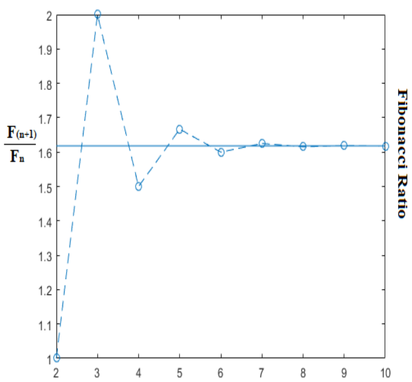
Given, [Φ^2 – Φ – 1] = 0
Or, [ax^2 + bx + c] = 0
Or, x = $\frac{-b\pm\sqrt{b^2-4ac}}{2a}$ ,a = 1, b = – 1, c = – 1
So, we can say that, Φ = $\frac{1\pm\sqrt{5}}{2}$ ;
that is, +1.618034 or, –0.618034.

The reason of getting two values are, by definition if we take the ratio of larger to shorter, then it will give us the $+_{ve}$ value, [ie, (Larger/Shorter) = 1.618034 = {($\sqrt{5}$ + 1)/2}]. But if we take the ratio of shorter to larger, then it will give us the $-_{ve}$ value, [ie, (Shorter/Larger) = 0.618034 = {($\sqrt{5}$ – 1)/2}]. Now, we can see that, (Larger/Shorter) × (Shorter/Larger) = 1. So, 1.618034 × 0.618034 = 1. Or in other way we can also prove that, [($\sqrt{5}$ + 1)/2] × [($\sqrt{5}$ – 1)/2] = [{($\sqrt{5}$)^2 – 1^2)}/(2 × 2)] = [(5 – 1)/4] = 1

## III. Golden Ratio Φ in Algebra

It has been observed that Golden Ratio appears at Fibonacci sequence as well. The Fibonacci sequence$F_n$ is such that each number is the sum of the two preceding ones, starting from 0 & 1; that is, $F_n = F_{(n-1)} + F_{(n+2)}$. So, $F_n$= 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, … $F_n$, $F_{(n+1)}$, $F_{(n+2)}$, … (up to infinity). One of the most frequently rediscovered facts about the Fibonacci sequence is if we tabulate these numbers in a column, shifting the decimal point one place to the right for each successive number, the sum equals 1/$F_{12}$, 1/89, as indicated below:

Sum of:
0.0
0.01
0.001
0.0002
0.00003
0.000005
0.0000008
0.00000013
0.000000021
0.0000000034
0.00000000055
0.000000000089
<u>… …ETC… …</u>
0.01123595505618… = 1/89




Another fun fact of Fibonacci Sequence is (Last digit of $F_{60}$), (Last digit of $F_{61}$), (Last digit of $F_{62}$), … (up to infinity) = Fibonacci Sequence itself. So, the reason for bringing up this mysterious sequence is it has an uncanny relationship with the Golden ratio Φ.

It has been observed that, the golden ratio can be approximated by a process of successive dividing of each term in the Fibonacci sequence by the previous term. And with each successive division, the result comes closer and closer to Φ. ie, $F_{(n+1)}/F_n = \Phi$. For example. 89/55 = 1.6181818181...Very close to Φ; as shown in the graph above. Because, let, $F_n$ = 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89,... A, B, C... (up to infinity). Say, B/A = X. So, C/B ≈ X, as well. Hence, B/A = C/B. But, C = (A + B). That is, B/A = (A + B)/B.Or, B/A = (A/B + 1). Which means, X = (1/X + 1). Or, X^2 = X + 1; ie [X^2 – X – 1] = 0. Hence, X = Φ.

Again, by forming matrix, we can say that, $\begin{bmatrix} C \\ B \end{bmatrix} = \begin{bmatrix} (A+B) \\ B \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}.\begin{bmatrix} A \\ B \end{bmatrix}$. Here say, Æ = $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$. So, the characteristic equation will be, |Æ – λ.I| = 0; where λ is eigenvalue of Æ, & I is a (2×2) identity matrix.

So, $\begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} - \lambda.\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$=0; » $\begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} + \begin{vmatrix} -\lambda & 0 \\ 0 & -\lambda \end{vmatrix}$=0; » $\begin{vmatrix} (1-\lambda) & 1 \\ 1 & -\lambda \end{vmatrix}$=0 ; »[–λ.(1 – λ) – (1×1)] = 0. So,

[λ^2 – λ – 1] = 0. And here Æ is a (2×2) binary matrix. And similar to this matrix, the highest probability of any non-trivial eigenvalues that show up in binary matrixes is (like this one), Φ.Furthermore the quadratic equation,

[Φ^2 – Φ – 1] = 0; can be represented as, $\begin{vmatrix} 1 & \Phi \\ \Phi & (\Phi+1) \end{vmatrix}$ = 0. And again, $\begin{vmatrix} \Phi & 1 \\ \Phi & (\Phi-1) \end{vmatrix}$ = 0. [2]

## IV. Golden Ratio Φ in Trigonometry

From the inception of the idea of Golden Ratio, mathematicians all across the globe attempted to come up with equations corelating pi and phi. Personally, I figured two pi-phi relations: (i) $6\Phi^2 \approx 5\pi$ & (ii) $\Phi \approx \frac{7\pi}{5e}$; by myself. But nothing beats the pi-phi relation cos(π/5) = Φ/2. Here beneath goes the mathematical evidence of the claim.

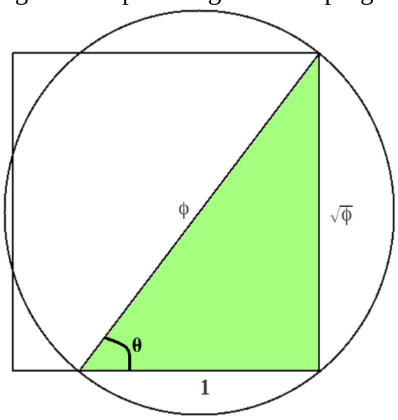
Let's say, a = cos(π/5) and b = cos(2π/5). Hence, b = cos(2π/5) = cos(π/5 +π/5). Which means, term 'b' can be expressed as; b = cos(π/5).cos(π/5) – sin(π/5).sin(π/5) = $\cos^2(\pi/5) - \sin^2(\pi/5) = \cos^2(\pi/5) - [1 - \cos^2(\pi/5)]$. That is, b = $2\cos^2(\pi/5) - 1 = 2[\cos(\pi/5)]^2 - 1$. ie, $\boxed{b = 2a^2 - 1}$… [equation (i)].

Again, cos(4π/5) = cos[(5π – π)/5] = cos(π – π/5) = cos(π).cos(π/5) – sin(π).sin(π/5) = – cos(π/5) = – a. As we know that, sin(π) = 0 and cos(π) = – 1. Hence, – a = cos(4π/5) = cos(2π/5 +2π/5). Hence, we can say that, – a = cos(2π/5).cos(2π/5) – sin(2π/5).sin(2π/5) = $\cos^2(2\pi/5) - \sin^2(2\pi/5) = \cos^2(2\pi/5) - [1 - \cos^2(2\pi/5)]$. That is, – a = $2\cos^2(2\pi/5) - 1 = 2[\cos(2\pi/5)]^2 - 1$. ie, $\boxed{-a = 2b^2 - 1}$… [equation (ii)].

Now if we deduct eqn. (ii) from eqn. (i), we get that; (b + a) = $(2a^2 - 1) - (2b^2 - 1) = 2a^2 - 1 - 2b^2 + 1$. That is (a + b) = $2a^2 - 2b^2 = 2(a^2 - b^2)$ = 2(a + b).(a – b). That is, (a – b) = (a + b)/[2(a + b)] = ½ or, $\boxed{b = a - ½}$.

Putting this value in equation (i) gives us a – ½ = $2a^2 - 1$, or, $2a^2 - a - 1 + ½ = 0$, that is $2a^2 - a - ½ = 0$. So, that is, $4a^2 - 2a - 1 = 0$. If we would put the value of b in equation (ii), we would've got, –a = $2(a - ½)^2 - 1$. That is to say, $2(a^2 - a + ¼) - 1 + a = 0$.Which means, $4a^2 - 2a - 1 = 0$, the same. So, a = cos(π/5) = $\frac{1 \pm \sqrt{5}}{4}$ = Φ/2.

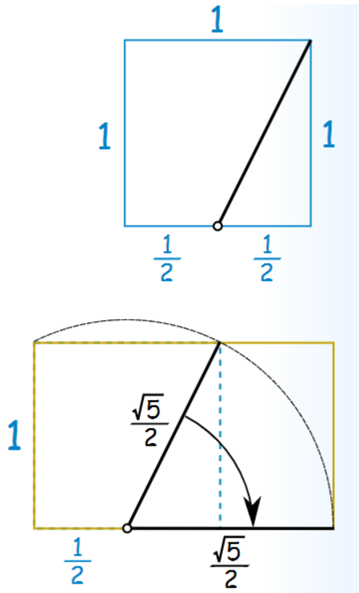
Based on the concept of Pythagoras a right-angle triangle was made known as the Kepler Triangle, which is named after the German mathematician and astronomer Johannes Kepler (1571–1630). The edge lengths in a precise geometric progression in which the common ratio is $\sqrt{\Phi}$; and the geometric progression goes like 1: $\sqrt{\Phi}$ : Φ. Here, length of the hypotenuse of the right-angle triangle is Φ and so the other two arms have lengths of 1 and $\sqrt{\Phi}$.So, Pythagoras $(\Phi)^2 = (\sqrt{\Phi})^2 + 1$; or, $\Phi^2 = \Phi + 1$ ie,[Φ^2 – Φ – 1] = 0 [5].




The picture shown beside is a Kepler triangle. If ϴ is the angle between hypotenuseΦ & base 1, then following relations can be drawn as well: (i) sinϴ = $\sqrt{\Phi}/\Phi = 1/\sqrt{\Phi}$ (ii) cosϴ = 1/Φ (iii) tanϴ = $\sqrt{\Phi}$. Hence, we can say that, ϴ = $\sin^{-1}(1/\sqrt{\Phi}) = \cos^{-1}(1/\Phi) = \tan^{-1}\sqrt{\Phi}$ = 0.9 rad = 51.83°. So, the other angle is (180 – 90° – 51.83°) = 38.17° = 2/3 rad (roughly). Another interesting fact of this diagram is, here we have a circle with a diameter of Φ and we have a square with sides of $\sqrt{\Phi}$. Though it is not possible to square a circle, we can see Sketch of the "Vitruvian Man" by Leonardo Vinci shows these two geometric figures have perimeter very close to each other. So, the circle and the square have closely equal perimeter. Now, perimeter of the square is four times its arms, viz. $4\sqrt{\Phi}$. And perimeter of the circle is 2π.radius = π.Diameter = π.Φ. Hence, we can say,$\pi.\Phi \approx 4\sqrt{\Phi}$, ie, π = $4/\sqrt{\Phi}$. It fit for an error that's less than 0.1%. Which brings us to another pi-phi relationship.

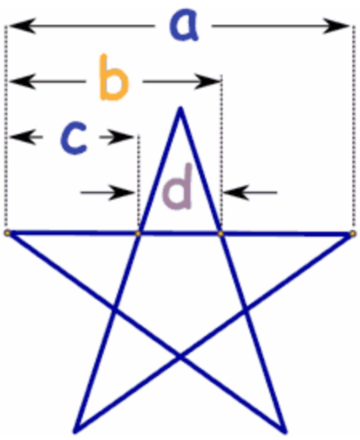
## V. Golden Ratio Φ in Geometry

Now here in this final segment of discussion we will get to know how we can draw the golden ratio as well as the geometric interpretation of it.

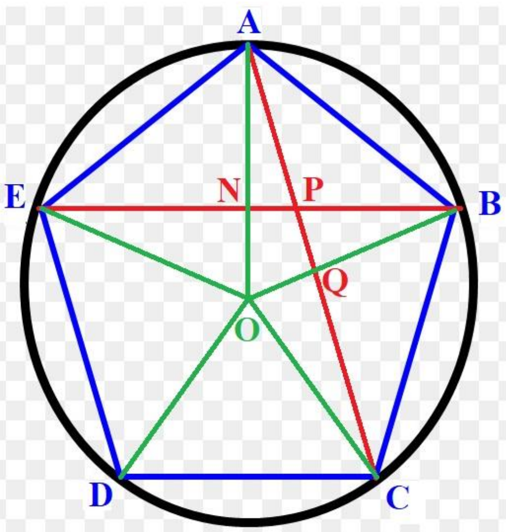



Here is one way to draw a rectangle with the Golden Ratio [6]:
First draw a square of unit length, that is the length is one. Place a dot at half way along one side& draw a line from that point to an opposite corner. So, the line will have a length of $\sqrt{(1)^2 + (½)^2}$ = $\sqrt{[1 + ¼]}$ = $\sqrt{5/4}$ = $\sqrt{5}$/2; Now either we add this value with ½ or we deduct this value from ½ to get the golden ratio. So, we turn that line so that it runs along the square's side and then we extend the square to be a rectangle with the Golden Ratio as shown in diagram. Notice that the arm of the rectangle is (½ + $\sqrt{5}$/2) while the additional extended portion is (½ – $\sqrt{5}$/2). So, we get both Φ and – 1/Φ from this diagram.

Another interesting geometrical expression of the golden ratio can be obtain at perfect pentagon shown below. Here in this diagram a/b = b/c = c/d = Φ.

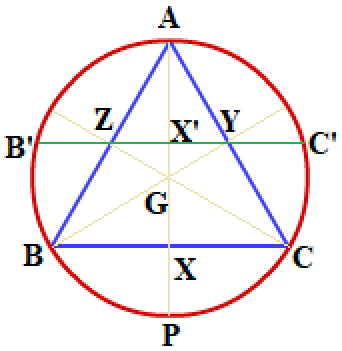



Then to prove the claim we need to change the diagram a little bit. We need to draw a polygon inscribed inside a circle consisting five arms. Besides, the assumptions will be all the five arms of the 'polygon' will have equal length. Let's suppose ABCDE is the polygon; then AB = BC = CD = DE = AE. Hence, <A = <B = <C = <D = <E. And all the angles of the pentagon are equal to be: ϴ = [{(n – 2)×180$^o$}/n]. That is to say, [{(5 – 2)×180$^o$}/5] = [(3×180$^o$)/5] = (3×36$^o$) = 108$^o$. Having said that, it is noticeable that a perfect pentagon will inscribe inside a circle, and the five points will divide the circle into 360$^o$/5 = 72$^o$. Noticeably, (72$^o$ + 108$^o$) = 180$^o$, also 72$^o$ = (36$^o$×2), 108$^o$ = (36$^o$×3) & 180$^o$ = (36$^o$×5). Also from trigonometric expression we derived cos(π/5) = cos(36$^o$) = Φ/2. AB = BC = CD = DE = AE &<A = <B = <C = <D = <E = 108$^o$. As well as, AO = BO = CO = DO = EO, where O is the center of the circle. Join B & E. Line BE intersects line AO at point N. So, AN┴BE, as well as NE = NB = ½BE. Join A & C. Line AC intersects line BE at point P and line BO at point Q. Also, BQ┴AC; which means, AQ = CQ = ½AC. We need to prove that, $\frac{a}{b}=\frac{b}{c}=\frac{c}{d}$= Φ.




Now at triangle AEB; AE = BE. As, <BAE = 108$^o$, so other two angles; <AEB = <ABE = (180$^o$ – 108$^o$)/2 = 72$^o$/2 = 36$^o$. In ΔAEN; <ANE = 90$^o$,<NAE = 108$^o$/2 = 54$^o$. So <AEN will be equal to (180$^o$ – 90$^o$ – 54$^o$) = 36$^o$. As, <AEN = <AEB. So, in ΔAEN; cos<AEN = NE/AE, viz, 2cos<AEN = 2NE/AE. So, 2cos(36$^o$) = (NE+NE)/AE viz, 2cos(π/5) = (NE+BN)/AE. That is, 2×Φ/2 = BE/AE; viz, BE/AE = Φ. So, now all we will need to prove is AE = PE to prove the pentagon relation stated before. If we can prove AE = PE, then BE/PE will be equal Φ. Now at triangle ABC; AB = BC. As, <ABC = 108$^o$, so other two angles; <BAC = <BCA = (180$^o$ – 108$^o$)/2 = 72$^o$/2 = 36$^o$. In ΔAPB; <BAP = <BAC = 36$^o$&<ABP = <ABE = 36$^o$. Which means, AP = BP &<APB = (180$^o$ – 36$^o$ – 36$^o$) = (180$^o$ – 72$^o$). So, <APE = (180$^o$ – <APB) = [180$^o$ – (180$^o$ – 72$^o$)] = 72$^o$. Now in ΔAEP; <APE = 72$^o$&<AEP = <AEB = 36$^o$; Hence, <PAE = [180$^o$ – <APE – <AEP] = [180$^o$ – 72$^o$ – 36$^o$] = 72$^o$. So in ΔAEP; <PAE = <APE = 72$^o$. So we can say, AE = PE. So we can say [BE]/[AE] = [BE]/[PE] = Φ. That is,$\frac{a}{b}=\frac{b}{c}=\frac{c}{d}$= Φ. Hence, we can conclude by saying that the line BE is divided at golden ratio at point P.

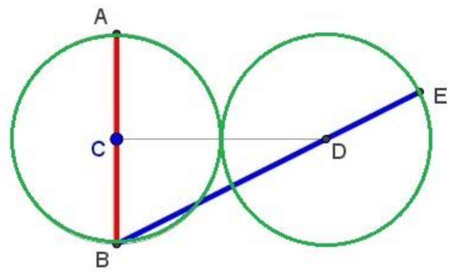
Golden ratio can be expressed geometrically via an equilateral triangle inscribed inside a circle as well. In the figure below ΔABC is an equilateral triangle inscribed in a circle with center G and radius of AG = BG = CG. Now extend AG, that intersects BC at point X, & extend BG, that intersects AC at point Y, and extend CG, that intersects AB at point Z. Join Z & Y and extend in both directions to intersect the circle at point B' & C'. From this construction we will see that, ZY/B'Z = ZY/C'Y = B'Y/ZY = C'Z/ZY = Φ.



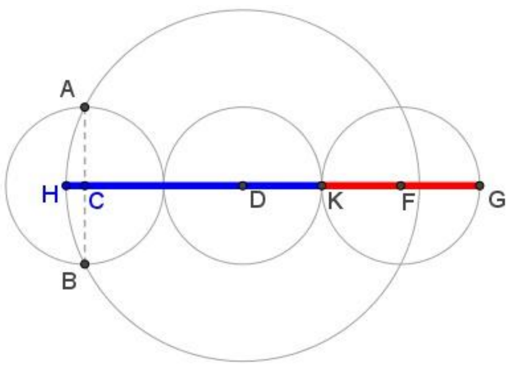

Let us assume, ZY = a, & YC' = b. We need to prove that a/b = (a + b)/a. Now here, AB = BC = CA & AG = BG = CG = 2GX = 2GY = 2GZ as well as AX = BY = CZ. So, AX⊥BC, BY⊥AC & CZ⊥AB. Also we know that, points X, Y, and Z are midpoints of BC, CA,and AB respectively viz, AZ = BZ = BX = CX = CY = AY = ½AB = ½BC = ½CA. Now BC||B'C'. So, ΔAZY;<AZY = <ABY &<AYZ = <ACB &<ZAY = <BAC. So, ΔAZY is also equilateral as all these angles are $60^o$ viz AZ = ZY = AY = a. Now, as product of segments of two intersecting cords of a circle are equal. So, at point Y, [B'Y.YC' = AY.YC]. ie, (B'Z + ZY).YC' = AY.YC. As B'Z = YC' so, (YC' + ZY).YC' = AY.YC = AY.AY = ZY.ZY; [as ΔAZY is equilateral]. Assume that, YC' = b & ZY = a; Hence, (b + a).b = a.a; or, a/b = (b + a)/a = b/a + a/a = b/a + 1. This expression it can be formulated into $\Phi = (1 + \frac{1}{\Phi})$, that is Φ^2 = (Φ + 1) or, [Φ^2 – Φ – 1] = 0 (proved).

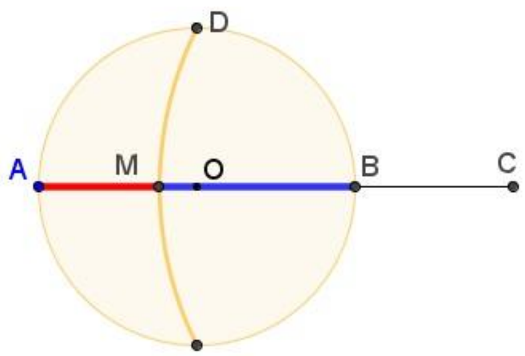
There are other ways to prove it as well like in ΔABP; <BAP = $30^o$, <ABP = $90^o$, hence, <APB = $60^o$. Now sin<APB = AB/AP. Say the arms of the equilateral triangles are of length aand radius of circle is R. Hence, sin<APB = sin$60^o$ = a/2R; viz, R = a/(2sin$60^o$). Now as sin$60^o$ = $\sqrt{3}$/2, so, R = a/$\sqrt{3}$. Also in ΔABX, we see, sin<ABX = sin$60^o$ = $\sqrt{3}$/2 = AX/AB = (AG + GX)/AB = (R + ½R)/a = (2R + R)/2a = 3R/2a; viz, R = a/$\sqrt{3}$. Now, PX' = (AP – AX') = (AP – AZ.sin<AZX) = (2R – ½AB.sin$60^o$) = (2a/$\sqrt{3}$ – ½a.$\sqrt{3}$/2) = (2a/$\sqrt{3}$ – a.$\sqrt{3}$/4); ie, PX' = a(2/$\sqrt{3}$ – $\sqrt{3}$/4) = a[(8 – 3)/4$\sqrt{3}$] = 5a/4$\sqrt{3}$. That is, GX' = PX'–PG = PX'–R = 5a/4$\sqrt{3}$ – a/$\sqrt{3}$ = a/4$\sqrt{3}$ Now in ΔB'X'G; $(B'X')^2$ = $(B'G)^2$– $(GX')^2$= $R^2$ – $(a/4\sqrt{3})^2$ = $(a/\sqrt{3})^2$ – $(a/4\sqrt{3})^2$ = [$a^2$/3 – $a^2$/48] = $5a^2$/16.Hence,B'X' = ($\sqrt{5}$a/4). But, X'Y = YC' = ½a. So, B'Y/YC' = (B'X' + X'Y)/YC' = $[\frac{\sqrt{5}a}{4} + \frac{a}{2}]/(\frac{a}{2})$. So the ratio becomes equal to (1 + $\sqrt{5}$)/2 = 1.618034 =Φ.

The golden ratio is available in numerous other images, another example can be the diagram shown below. Here we can see that, two equal circles with center C & D are tangent to each other. AB is the diameter of the circle with center C. CD⊥AB. Join B & D. BD crosses the circle with center D in two points, let E be further one from B.

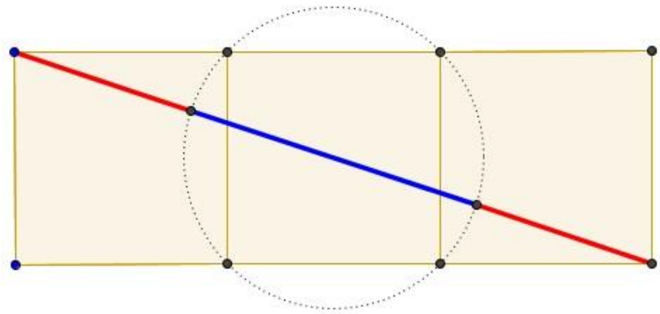


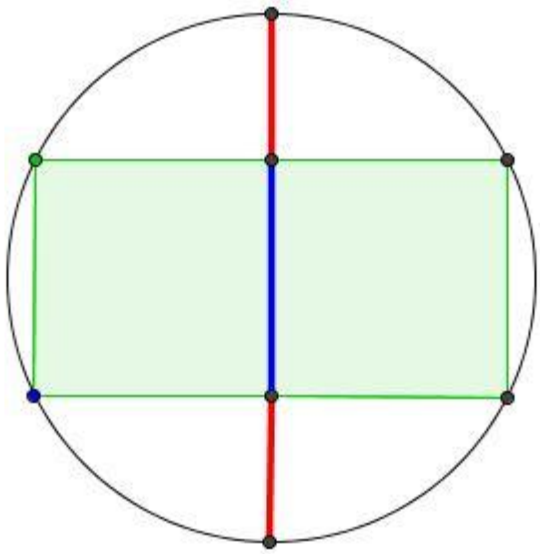

Let the radius of both circles r. In ΔBCD, $(BD)^2 = (CD)^2 + (BC)^2$. But, AC = BC = ½AB = DE = ½CD = r. So, $(BD)^2 = (2r)^2 + (r)^2$. Viz, $(BD)^2 = 4(r)^2 + (r)^2 = 5(r)^2$. Suggests, BD = $\sqrt{5}$r. Now in here, BE = BD + DE. So, BE = ($\sqrt{5}$r + r). Again AB = 2r. So, the ratio BE/AB = ($\sqrt{5}$r + r)/2r = (1 + $\sqrt{5}$)/2 = 1.618034 =Φ. Not only this but also this concept can be nicely modified into a construction with four circle which is shown in the diagram below (left). As well as another most straightforward construction of the golden ratio with this concept has been devised by Nguyen Thanh Dung shown in the diagram below (right) [7].

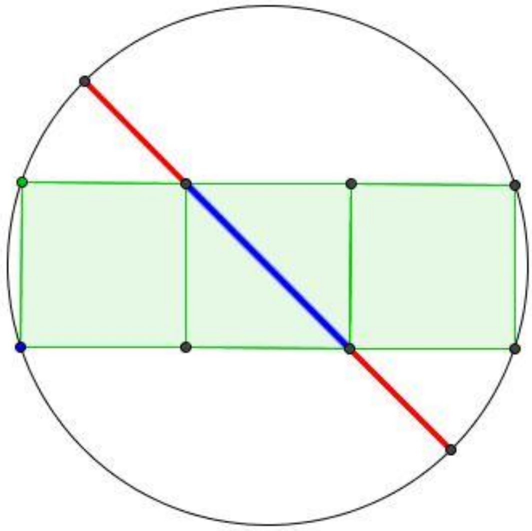


Tran Quang Hung [7] has devised another configuration of a 1×3 rectangle with a circle that producesgolden ratio. But there is a not immediately obvious relation between the case of 1×2 and 1×3 rectangles. If we consider the arms of the square of the diagram to be a, then the red plus blue line becomes equal to $\sqrt{(3a)^2 + (a)^2} = a\sqrt{10}$; while the blue one becomes equals to the diameter $\sqrt{(a)^2 + (a)^2} = a\sqrt{2}$ (1[st]-bottom-left). So, the ratio becomes equal to $\sqrt{5}$.



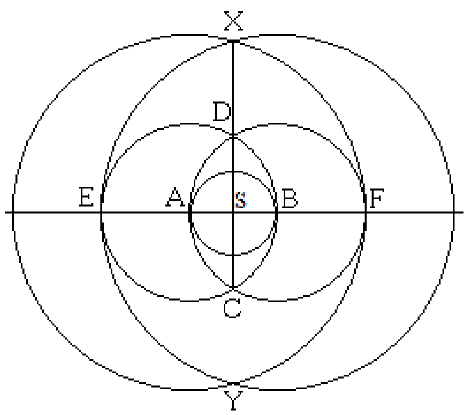




In all these three diagrams the above-mentioned relationship can be observed. And as we know that the value of $\Phi = \frac{1 \pm \sqrt{5}}{2}$; therefore, golden ratio is found in all these three diagrams as well. Golden ratio has also observed in the constructions that involves a rhombus and a regular hexagon. Before going into that discussion it will be better to converse about another very elegant way of obtaining the golden ratio, offered in a (2002) article byK.Hofstetter [7]. It's shown in the diagram provided below. Here, it will be convenient to denote S(R) the circle with center S through point R. For the construction, let A and B be two points. Circles A(B) and B(A) intersect in C and D and cross the line AB in points E and F. Circles B(E) and A(F) intersect in X and Y, as in the diagram. Because of the symmetry, points X, D, C, Y are collinear. The fact is CX/CD = Φ.

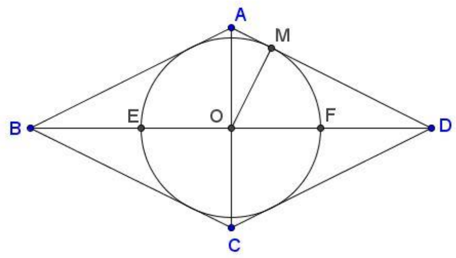
Assume for simplicity that AB = 2. Then CD = $2\sqrt{3}$, &CX = $\sqrt{15} + \sqrt{3}$. Hence,the ratio of CX & CD:

(CX)/(CD) = $(\sqrt{15} + \sqrt{3})/2\sqrt{3}$
= $(\sqrt{5} + 1)/2$
= Φ.




Notice that the whole construction can be accomplished with compass only. This much simplicity as well as diversity has made golden ratio this much widespread and this is the reason of calling it in different other names like the golden mean or golden section (Latin: section aurea). Similarly some other names includeextreme mean ratio, medial section, divine cut proportion, divine section (Latin: section divina), golden cut, golden proportionand golden number [5].Hence, now we will discuss how this ratio has also observed in constructions involvinga rhombus and a regular hexagon.

Let, ABCD is a rhombus with 2AC=BD.The inscribed circle has a center O.Also, E and F are the points of intersection of the circle with BD. Then, the point F divides DE in the golden ratio.

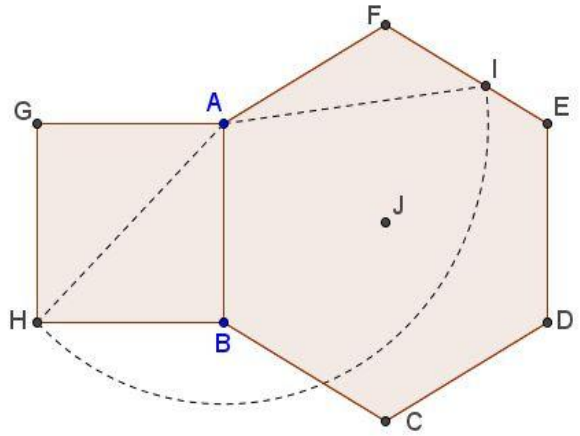



Now, let M be the point of tangency of (O) with AD. So, OM⊥AD. Hence, ΔMOD & ΔAOD are similar as we can see<ADO = <MDO Therefore, (MD)/(MO) = (OD)/(OA) =2;viz, MD=2(OM) =EF. From the property of a tangent, $DM^2 = DF \cdot DE$.

Or, $EF^2 = DF \cdot DE$;

Or, (FD)/(FE) = (EF)/(ED), as required.

Tran Quang Hung [7] has devised another configuration of golden ratio Φ in a hexagon. Square ABHG is constructed outside the hexagon ABCDEF. A circle with center at A, radius AH cuts EF at I in golden ratio as shown in the diagram below.




Say AB = BC = CD = DE = EF = AF = GA = BH = a Therefore, AH = AI = $\sqrt{2}$a.Set α=<FAI, β=<AIF. Now by applying the Law of Sines in ΔAIF:

AF/sinβ=FI/sinα=AI/sin120º. Now as AF = a and AI = $\sqrt{2}$a, so, this can be rewritten as:

a/sinβ= ($\sqrt{2}$a)/($\sqrt{3}$/2) = (2$\sqrt{2}$)a/$\sqrt{3}$ = (2$\sqrt{6}$)a/3

Thus we can say: sinβ= 3/(2$\sqrt{6}$) = 6/(4$\sqrt{6}$)= $\sqrt{6}$/4. Hence, cosβ=$\sqrt{1-(sin\beta)^2}$ = $\sqrt{1-6/16}$= $\sqrt{10}$/4.

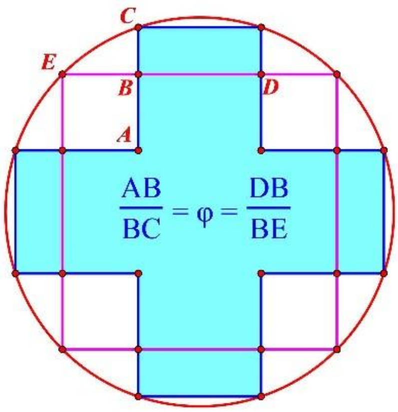
Now, observe that (α+β) =60º, so we can imply that sinα=sin(60º−β)=sin60º·cosβ−cos60∘·sinβ. Thus

sinα= $[\frac{\sqrt{3}}{2}.\frac{\sqrt{10}}{4} - \frac{1}{2}.\frac{\sqrt{6}}{4}] = [\frac{\sqrt{30}}{8} - \frac{\sqrt{6}}{8}] = \frac{\sqrt{6}}{8}.(\sqrt{5}-1)$ Now

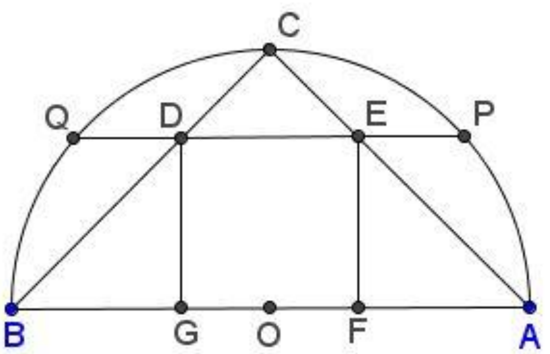
AF/sinβ=FI/sinα; or, (2$\sqrt{6}$)a/3 = FI/[$\frac{\sqrt{6}}{8}$.($\sqrt{5}$ – 1)] or,

FI/a = ($\sqrt{5}$ – 1)/2 viz, a/FI = FE/FI = ($\sqrt{5}$ + 1)/2 = Φ.

Not only these there are several other numerous geometrical figures where golden ratio is observed. The following is a new invention of Bui Quang Tuan [7]. In the diagram given below the cross consists of five equal square. Here, let S be the side of the inscribed square, C the side of any of the five squares that compose the cross, then $S^2=5C^2$. From this expression the following relationship can be obtained as mentioned in the image.

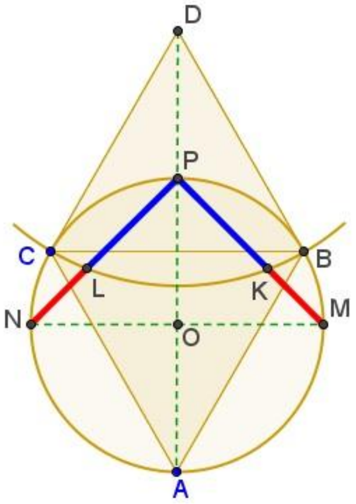



In 2015 Tran Quang Hung has found once more the golden ratio in a combination of a semicircle, a square, & a right isosceles triangle [7]. Given a right isosceles triangle ABC and its circumcircle, inscribed a square DEFG with a side FG along the hypotenuse AB. Let the side DE extended beyond E intersect the circumcircle at P.Then the point E divides DP in the golden ratio.



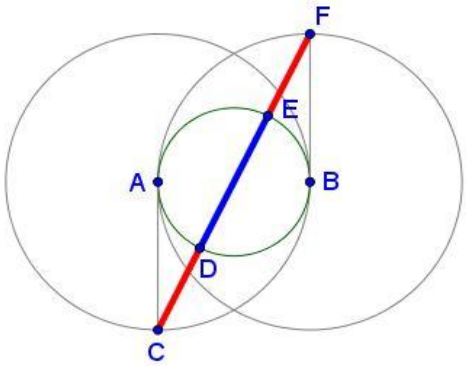

From the similarity of the isosceles right triangles DEC and AEF, we have (DE)/(CE) = (AE)/(EF).It thus follows; $DE^2 = DE \cdot EF = AE \cdot EC$.If the line DE intersects the semicircle again at Q, then EQ=DP. By the intersecting chords theorem,AE·CE=EP·EQ=EP·DP.Therefore, $DE^2 = EP \cdot DP$, meaning that E divides DP in the golden ratio.

We are going to conclude our discussion for this segment with an example of Tran Quang Hung [7]. Let ABC be an equilateral triangle inscribed in circle (O). D is reflection of A through BC. MN is diameter of (O) parallel to BC.AD meets (O) again at P.Then, circle (D)and passing through B,C divides PM,PN in golden ratio.

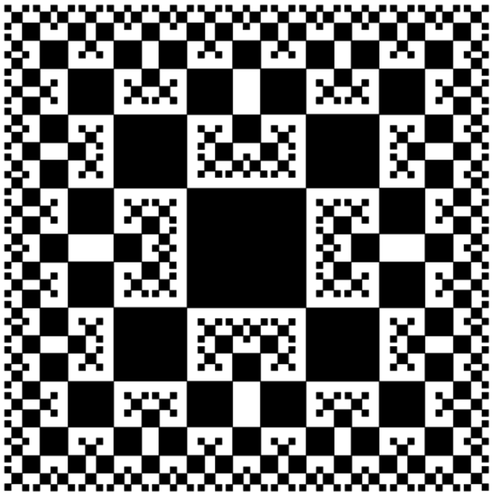



Here, OB=OC = OA = ON = OM = OP = PD = PC = PB = R (say). So, BC=CD=BD = AB = AC = DL = DK =$\sqrt{3}$R and,NP=MP = $\sqrt{R^2 + R^2}$ = $\sqrt{2}$R.In ΔDLP, <DPL=135ºSay, PL=x.
For a triangle with sides a,b,c & angle μ opposite to c, as per law of cosine
$c^2 = a^2 + b^2 - 2ab\cdot\cos(\mu)$; viz, $DL^2 = PD^2 + PL^2 - 2.PD.PL.\cos135^o$;
so, $(\sqrt{3}R)^2 = (R)^2 + (x)^2 - 2.R.x.\cos(180^o - 45^o)$. Now, as $\cos45^o = \frac{1}{\sqrt{2}}$;
so, $(x)^2 + \sqrt{2}.R(x) - 2(R)^2 = 0$; or $2x = -\sqrt{2}.R \pm\sqrt{2R^2 + 8R^2}$
With positive value, $x = \frac{\sqrt{10}-\sqrt{2}}{2}.R = \frac{\sqrt{5}-1}{\sqrt{2}}.R$ = PL.
Now, $NL = NP - PL = \sqrt{2}R - \frac{\sqrt{5}-1}{\sqrt{2}}.R = \frac{3-\sqrt{5}}{\sqrt{2}}.R$.
Now by taking ratio, $PL/NL = [\frac{\sqrt{5}-1}{\sqrt{2}}.R]/[\frac{3-\sqrt{5}}{\sqrt{2}}.R] = \frac{\sqrt{5}-1}{3-\sqrt{5}}$.
viz, $PL/NL = \frac{\sqrt{5}-1}{3-\sqrt{5}}.\frac{\sqrt{5}+1}{\sqrt{5}+1}.\frac{\sqrt{5}+1}{\sqrt{5}+1} = \frac{(5-1).(\sqrt{5}+1)}{(3-\sqrt{5}).(5+2\sqrt{5}+1)} = \frac{(4).(\sqrt{5}+1)}{2(3-\sqrt{5}).(3+\sqrt{5})}$.
Thus we can prove that, $PL/NL = \frac{(\sqrt{5}+1)}{2} = \Phi$.

The following (right) construction of the golden ratio Φ has appeared in the Mathematical Gazette, volume 101,number 551, July 2017, page 303 constructed by John Molkach [7].There are two unit circles (A) & (B).The circle (O) has a 2R diameter of AB and tangent to both circles. Vertical segments AC & BF are tangent to circle B&circle A, respectively. So, AB = AC = BF = 1. CF crosses (O) in D and E, as shown in the diagram below. John proves that CE = Φ.
To prove that let us consider, CE=x.
Then, by Intersecting Secants rules
$CA^2 = CD\times CE$. Or, $(1)^2=(x-1).x$ Viz $x^2 - x - 1=0$




## VI. Golden ratio in Fractals



It is not so much that the golden ratio is “related to a fractal,” as fractal patternsare based on anynumber. Fractal patterns created using golden ratio, however, are optimized in a way that does not occur with any other number. As an example, in the image below the fractal pattern expands using the golden ratio.According to Mario Livia [8]: some of the greatest mathematical minds of all ages, from Pythagoras and Euclid in ancient Greece, through the medieval Italian mathematician Leonardo of Pisa & the Renaissance astronomer Johannes Kepler, to the present-day scientific figures such as Oxford physicist Roger Penrose, have spent endless hours over this simple ratio and its properties. The Biologists, musicians, historians, architects,psychologists, artist and even mystics have pondered debated the basis of its ubiquity and appeal. In fact, it is fair to say that golden ratio has inspired thinkers of all disciplines like no other number in mathematics.

## VII. Conclusion

Mathematicians since Euclid have studied the properties of the golden ratio, including its appearance in dimensions of a regular pentagon and in a golden rectangle, which may be cut into a square and a smaller rectangle with that of the same aspect ratio. The golden ratio has also been used to analyze the proportions of natural objects as well as man-made systems such as financial markets, in some cases based on dubious fits to data. The golden ratio appears in some patterns in nature, including the spiral arrangement of leaves and other plant parts.Some twentieth-century artists and architects, including Le Corbusier and Salvador Dalí, have proportioned their works to approximate the golden ratioespecially in the form of the golden rectangle, in which the ratio of the longer side to the shorter is the golden ratiobelieving this proportion to be aesthetically pleasing.

A Fibonacci spiral which approximates the golden spiral, using Fibonacci sequence square sizes up to 55. The spiral is drawn starting from the inner 1×1 square and continues outwards to successively larger squares.

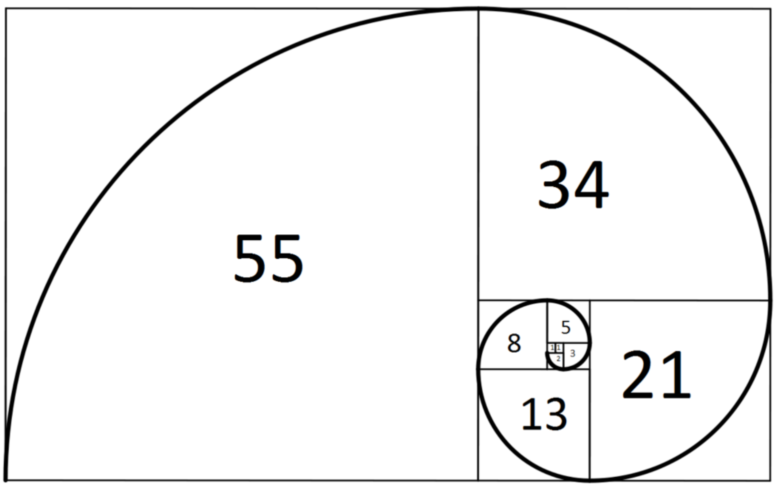



Figure: Fibonacci Spiral Drawn by Nafish Sarwar Islam using MATLAB

## References


[1]. C. Rovelli and F. Vidotto, "Covariant Loop Quantum Gravity", Cambridge University Press 1 edition, 2014

[2]. Klee Irwina, Marcelo M. Amaralab, Raymond Aschheima, Fang Fanga, "Quantum Walk on a Spin Network and theGolden Ratio as the Fundamental Constantof Nature", Minkowski Institute Press, Published on 29 April 2017

[3]. Salvatore Giandinoto,"Superluminal Transportation of High Energy Particles Through Wormholes Using the Phi-Based Solution to the Schrödinger Wave Equation, the theorem of Residues and the Cauchy Integral Formula", Journal Published on 2007

[4]. Norman Cr., Marco Ol. &VillanuevaJ. R., "The Golden Ratio in Schwarzschild–Kottler Black Holes", The European Physical Journal, Received: 16 January 2017 / Accepted: 27 January 2017 / Published online: 23 February 2017

[5]. Weblink: https://en.wikipedia.org/wiki/Golden_ratio; collected on 2019

[6]. Weblink:https://www.mathsisfun.com/numbers/golden-ratio.html; collected on 2019

[7]. Weblink: https://www.cut-the-knot.org/do_you_know/GoldenRatio.shtml, collected on 2019

[8]. Daniel Winter & Martin Alexander Jones, "Fractal Golden Ratio/Phase Conjugation: Cause of Gravity", TheGeneralScienceJournal, 2013


Nafish Sarwar Islam. "MathematicalSanctityofthe Golden Ratio." IOSR Journal of Mathematics (IOSR-JM) 15.5 (2019): 57-65.