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A Note on a New Extension of Extended Gamma and Beta Functions and their Properties

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Abstract: *Al-Gonah and Mohammed (A New Extension of Extended Gamma and Beta Functions and their Properties, Journal of Scientific and Engineering Research 5(9), 2018, 257-270) introduced a new extension of Gamma and Beta functions. In this note, we will show that a problem has been encountered regarding the Gamma function integral representations. We also studied certain results of the Gamma and Beta functions such as beta distribution, new defined Gauss and Confluent hypergeometric functions with their properties.*

Keyword: *Classical Gauss and Confluent Hypergeometric functions, Generating function, Mittag-Leffler function*

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I. Introduction

Al-Gonah et al [1] introduced the following extended gamma and beta functions in equations (1.1) and (1.2) as follows:

$$\Gamma_{\rho}^{(\alpha,\beta,\gamma)}(\tau)=\int_{0}^{1}t^{\tau-1}E_{\alpha,\beta}^{\gamma}\left(-t-\frac{\rho}{t}\right)dt \tag{1.1}$$

$$\left(\mathrm{Re}(\alpha)\succ 0,\mathrm{Re}(\beta)\succ 0,\mathrm{Re}(\gamma)\succ 0,\mathrm{Re}(\rho)\succ 0,\mathrm{Re}(\tau)\succ 0\right)$$

and

$$B_{\rho}^{(\alpha,\beta,\gamma)}(\tau,\mu)=\int_{0}^{1}t^{\tau-1}(1-t)^{\mu-1}E_{\alpha,\beta}^{\gamma}\left(-\frac{\rho}{t(1-t)}\right)dt \tag{1.2}$$

$$\left(\mathrm{Re}(\alpha)\succ 0,\mathrm{Re}(\beta)\succ 0,\mathrm{Re}(\gamma)\succ 0,\mathrm{Re}(\rho)\succ 0,\mathrm{Re}(\tau)\succ 0,\mathrm{Re}(\mu)\succ 0\right)$$

If $\beta=\gamma=1$, equation (1.2) reduces to extended beta function introduced in 2018 by Shadab et al [5] using Mittag-Leffler as follow:

$$B_{\rho}^{\alpha}(\tau,\mu)=\int_{0}^{1}t^{\tau-1}(1-t)^{\mu-1}E_{\alpha}\left(-t-\frac{\rho}{t}\right)dt \tag{1.3}$$

$$\left(\mathrm{Re}(\alpha)\succ 0,\mathrm{Re}(p)\succ 0,\mathrm{Re}(\tau)\succ 0,\mathrm{Re}(\rho)\succ 0\right)$$

If $\alpha=\beta=\gamma=1$, equation (1.2) reduces extended beta function introduced by Chaudhry et al [2-3] in 1997 given by:

$$B_{\rho}(\tau,\mu)=\int_{0}^{1}t^{\tau-1}(1-t)^{\mu-1}\exp\left(-\frac{\rho}{t(1-t)}\right)dt, \tag{1.4}$$

$$\left(\mathrm{Re}(\tau)\succ 0,\mathrm{Re}(\mu)\succ 0,\mathrm{Re}(\rho)\succ 0\right)$$

If $\alpha=\beta=\gamma=1$ and $\rho=0$, equation (1.2) reduces to the Euler beta function [5] also known as classical beta defined by:

$$B(\tau,\mu)=\int_{0}^{1}t^{\tau-1}(1-t)^{\mu-1}dt=\frac{\Gamma(\tau)\Gamma(\mu)}{\Gamma(\tau+\mu)},\ \mathrm{Re}(\tau)\succ 0 \text{ and } \mathrm{Re}(\mu)\succ 0 \tag{1.5}$$

Al-Gonah and Mohammed also gives the following integral representations of function [1, p.262, **Theorem** 3.1, **Remark** 3.1-3.2 and **Corollary** 3.1]:

$$\Gamma_{\rho}^{(\alpha,\beta,\gamma)}(\tau)=-\int_{-1}^{0}\frac{t^{\tau-1}}{(t+1)^{\tau+1}}E_{\alpha,\beta}^{\gamma}\left(\frac{-t^2-\rho(t+1)^2}{t(t+1)}\right)dt \tag{1.6}$$

$$\Gamma_{\rho}^{(\alpha,\beta,\gamma)}(\tau)=-\int_{0}^{1}\frac{t^{\tau-1}}{(t-1)^{\tau+1}}E_{\alpha,\beta}^{\gamma}\left(\frac{-t^2-\rho(t-1)^2}{t(t-1)}\right)dt \tag{1.7}$$

$$\Gamma_{\rho}^{(\alpha,\beta,\gamma)}(\tau)=(a-b)\int_{a}^{b}\frac{(t-a)^{\tau-1}}{(t-b)^{\tau+1}}E_{\alpha,\beta}^{\gamma}\left(\frac{-(t-a)^2-\rho(t-b)^2}{(t-a)(t-b)}\right)dt \tag{1.8}$$

$$\Gamma_{\rho}^{(\alpha,\beta,\gamma)}(\tau)=-2\int_{-1}^{1}\frac{(t+1)^{\tau-1}}{(t-1)^{\tau+1}}E_{\alpha,\beta}^{\gamma}\left(\frac{-(t+1)^2-\rho(t-1)^2}{(t+1)(t-1)}\right)dt \tag{1.9}$$

$$\Gamma_{\rho}^{(\alpha,\beta,\gamma)}(\tau)=-\int_{0}^{z}\frac{(t+1)^{\tau-1}}{(t-z)^{\tau+1}}E_{\alpha,\beta}^{\gamma}\left(\frac{-t^2-\rho(t-z)^2}{t(t-z)}\right)dt \tag{1.10}$$

$$\Gamma_{\rho}^{(\alpha,\beta,\gamma)}(\tau)=2\int_{-\infty}^{\infty}e^{2t\tau}E_{\alpha,\beta}^{\gamma}\left(e^{t}\cosh t+\rho e^{-2t}\right)dt \tag{1.11}$$

Substituting $t=\frac{u}{u+1}$ and $t=\frac{u}{u-1}$ into equation (1.1) yield (1.6) - (1.7). Putting $t=\frac{u-a}{b-a}$, $t=\frac{u+1}{2}$ and $t=\frac{u}{z}$ into equation (1.7) gives (1.8) - (1.10) and putting $t=\tanh u$ into (1.9) gives (1.11)

In this paper we give correct integral representations in equations (1.6) – (1.11) and other properties of the extended beta function in (1.2) in the following theorem.

II. Integral Representations

Theorem 1: The following integral representations holds true:

$$\Gamma_{\rho}^{(\alpha,\beta,\gamma)}(\tau)=\int_{0}^{\infty}\frac{t^{\tau-1}}{(t+1)^{\tau+1}}E_{\alpha,\beta}^{\gamma}\left(\frac{-t^2-\rho(t+1)^2}{t(t+1)}\right)dt \tag{2.1}$$

$$\Gamma_{\rho}^{(\alpha,\beta,\gamma)}(\tau)=\int_{0}^{\infty}\frac{t^{\tau-1}}{(t-1)^{\tau+1}}E_{\alpha,\beta}^{\gamma}\left(\frac{-t^2-\rho(t-1)^2}{t(t-1)}\right)dt \tag{2.2}$$

$$\Gamma_{\rho}^{(\alpha,\beta,\gamma)}(\tau)=(a-b)\int_{a}^{\infty}\frac{(t-a)^{\tau-1}}{(t-b)^{\tau+1}}E_{\alpha,\beta}^{\gamma}\left(\frac{-(t-a)^2-\rho(t-b)^2}{(t-a)(t-b)}\right)dt \tag{2.3}$$

$$\Gamma_{\rho}^{(\alpha,\beta,\gamma)}(\tau)=-2\int_{-1}^{\infty}\frac{(t+1)^{\tau-1}}{(t-1)^{\tau+1}}E_{\alpha,\beta}^{\gamma}\left(\frac{-(t+1)^2-\rho(t-1)^2}{(t+1)(t-1)}\right)dt \tag{2.4}$$

$$\Gamma_{\rho}^{(\alpha,\beta,\gamma)}(\tau)=z\int_{0}^{\infty}\frac{t^{\tau-1}}{(t-z)^{\tau+1}}E_{\alpha,\beta}^{\gamma}\left(\frac{-t^2-\rho(t-z)^2}{t(t-z)}\right)dt \tag{2.5}$$

$$\Gamma_{\rho}^{(\alpha,\beta,\gamma)}(\tau)=2\int_{0}^{\infty}e^{2t\tau}E_{\alpha,\beta}^{\gamma}\left(e^{t}\sinh t+\rho\,\mathrm{cs}ch\;t\,e^{-2t}\right)dt \tag{2.6}$$

Proof

In equation (1.1) putting $t=\frac{u}{u+1}$, then $dt=\frac{du}{(u+1)^2}$ when $t=0:u=0$ and $t\to\infty:u\to\infty$. Therefore equation (2.1) follows.

In equation (1.1) putting $t=\frac{u}{u-1}$, then $dt=-\frac{du}{(u-1)^2}$ when $t=0:u=0$ and $t\to\infty:u\to\infty$. Therefore equation (2.2) follows.

In equation (2.2) putting $t=\frac{u-a}{b-a}$, then $dt=\frac{du}{b-a}$ when $t=0:u=a$ and $t\to\infty:u\to\infty$. Therefore equation (2.3) follows.

In equation (2.2) putting $t=\frac{u+1}{2}$, then $dt=\frac{du}{2}$ when $t=0:u=-1$ and $t\to\infty:u\to\infty$. Therefore equation (2.4) follows.

In equation (2.2) putting $t=\frac{u}{z}$, then $dt=\frac{du}{z}$ when $t=0:u=0$ and $t\to\infty:u\to\infty$. Therefore equation (2.5) follows.

In equation (2.4) putting $t=\tanh u$, then $dt=\operatorname{sech}^2 u du$ when $t=0:u=0$ and $t\to\infty:u\to\infty$. Therefore equation (2.6) follows.

2.1 The Beta Distribution of $B_p^{(\alpha,\beta,\gamma)}(a,b)$

In this section, we define the extended beta distribution using $B_\rho^{(\alpha,\beta,\gamma)}(a,b)$, where a and b satisfy the conditions $-\infty\prec a\prec\infty$, $-\infty\prec b\prec\infty$ and $p\succ 0$ as:

$$f(t)=\begin{cases}\dfrac{1}{B_\rho^{(\alpha,\beta,\gamma)}(a,b)}t^{a-1}(1-t)^{b-1}E_{\alpha,\beta}^{\gamma}\left(-\dfrac{\rho}{t(1-t)}\right), & 0\prec t\prec 1\\ 0, & \text{elsewhere}\end{cases} \tag{2.7}$$

$$\left(a,b\in R,\rho,\alpha,\beta,\gamma\in R^+\right)$$

If σ is any real number, then we have σ th moment of X

$$E\left(X^\sigma\right)=\frac{B_\rho^{(\alpha,\beta,\gamma)}(a+\sigma,b)}{B_\rho^{(\alpha,\beta,\gamma)}(a,b)},\quad \left(a,b\in R,\rho,\alpha,\beta,\gamma\in R^+\right) \tag{2.8}$$

In particular, if $\sigma=1$ we obtain the mean of the distribution as:

$$\mu=E(X)=\frac{B_\rho^{(\alpha,\beta,\gamma)}(a+1,b)}{B_\rho^{(\alpha,\beta,\gamma)}(a,b)},\ \left(a,b\in R,\rho,\alpha,\beta,\gamma\in R^+\right) \tag{2.9}$$

While the variance of the distribution as:

$$\delta^2=E\left(X^2\right)-\left\{E(X)\right\}^2$$

$$\delta^2=\frac{B_\rho^{(\alpha,\beta,\gamma)}(a,b)B_\rho^{(\alpha,\beta,\gamma)}(a+2,b)-\left\{B_\rho^{(\alpha,\beta,\gamma)}(a+1,b)\right\}^2}{\left\{B_\rho^{(\alpha,\beta,\gamma)}(a,b)\right\}^2} \tag{2.10}$$

The moment generating function of the distribution is given by:

$$M(t)=\frac{1}{B_\rho^{(\alpha,\beta,\gamma)}(a,b)}\sum_{n=0}^{\infty}B_\rho^{(\alpha,\beta,\gamma)}(a+n,b)\frac{t^n}{n!} \tag{2.11}$$

The cumulative distribution of equation (2.7) can be given by:

$$F(x)=\frac{B_{\rho,x}^{(\alpha,\beta,\gamma)}(a,b)}{B_{\rho}^{(\alpha,\beta,\gamma)}(a,b)} \tag{2.11}$$

where

$$B_{\rho,x}^{(\alpha,\beta,\gamma)}(a,b)=\int_0^x t^{a-1}(1-t)^{b-1}E_{\alpha,\beta}^{\gamma}\left(-\frac{\rho}{t(1-t)}\right)dt,\left(a,b\in R,\rho,\alpha,\beta,\gamma\in R^+\right) \tag{2.12}$$

is the incomplete extended beta function

III. Extended Gauss and Confluent Hypergeometric Functions

In this section, we extend the gauss and confluent hypergeomentric functions by making use of the extended beta function in equation (1.2):

$$F_{\rho}^{(\alpha,\beta,\gamma)}(a,b;c;z)=\sum_{n=0}^{\infty}(a)_n\frac{B_{\rho}^{(\alpha,\beta,\gamma)}(a+n,c-b)}{B(b,c-b)}\frac{z^n}{n!} \tag{2.13}$$

$$\left(\alpha,\beta,\gamma\in R^+,\rho\in R_0^+,|z|\prec 1,\mathrm{Re}(c)\succ\mathrm{Re}(b)\succ 0\right)$$

and

$$\Phi_{\rho}^{(\alpha,\beta,\gamma)}(b;c;z)=\sum_{n=0}^{\infty}\frac{B_{\rho,x}^{(\alpha,\beta,\gamma)}(a+n,c-b)}{B(a,c-b)}\frac{z^n}{n!} \tag{2.14}$$

$$\left(\alpha,\beta,\gamma\in R^+,\rho\in R_0^+,\mathrm{Re}(c)\succ\mathrm{Re}(b)\succ 0\right)$$

3.1 Integral Representations of $F_{\rho}^{(\alpha,\beta,\gamma)}(a,b;c;z)$ and $\Phi_{\rho}^{(\alpha,\beta,\gamma)}(b;c;z)$

In this subsection, we obtain the integral representations of the extended Gauss and Confluent hypergeometric functions in the following theorems:

Theorem 2: the following integral representations in equations (2.15) and (2.16) holds true:

$$F_{\rho}^{(\alpha,\beta,\gamma)}(a,b;c;z)=\frac{1}{B(b,c-b)}\int_0^1 t^{a-1}(1-t)^{c-b-1}(1-zt)^{-a}E_{\alpha,\beta}^{\gamma}\left(-\frac{\rho}{t(1-t)}\right)dt \tag{2.15}$$

$$\left(\left|\arg(1-z)\right|\prec\pi;p,\alpha,\beta,\gamma\in R^+,\rho=0,\mathrm{Re}(c)\succ\mathrm{Re}(b)\succ 0\right)$$

and

$$\Phi_{\rho}^{(\alpha,\beta,\gamma)}(b;c;z)=\frac{1}{B(b,c-b)}\int_0^1 t^{a-1}(1-t)^{c-b-1}e^{zt}E_{\alpha,\beta}^{\gamma}\left(-\frac{\rho}{t(1-t)}\right)dt \tag{2.16}$$

$$\left(\rho,\alpha,\beta,\gamma\in R^+,\rho=0,\mathrm{Re}(c)\succ\mathrm{Re}(b)\succ 0\right)$$

Proof

Substituting the definition of $B_{\rho}^{(\alpha,\beta,\gamma)}(\tau,\mu)$ in equation (1.2) into (2.15) and interchanging the order of integral and summation, we obtain:

$$F_{\rho}^{(\alpha,\beta,\gamma)}(a,b;c;z)=\frac{1}{B(b,c-b)}\int_0^1 t^{a-1}(1-t)^{c-b-1}E_{\alpha,\beta}^{\gamma}\left(-\frac{\rho}{t(1-t)}\right)\sum_{n=0}^{\infty}\frac{(a)_n(zt)^n}{n!}dt$$

using generalized binomial expansion, we obtain:

$$F_{\rho}^{(\alpha,\beta,\gamma)}(a,b;c;z)=\frac{1}{B(b,c-b)}\sum_{n=0}^{\infty}(a)_n\int_0^1 t^{a+n-1}(1-t)^{c-b-1}E_{\alpha,\beta}^{\gamma}\left(-\frac{\rho}{t(1-t)}\right)\frac{z^n}{n!}dt \tag{2.17}$$

on applying the definition of the extended beta function in equation (1.2), we obtain the desired result. Following similar argument, we obtain equation (2.16).

3.2 Derivative Formulae

Theorem 3: The following derivative formulae holds true:

$$\frac{d^n}{dz^n}\left\{F_\rho^{(\alpha,\beta,\gamma)}(a,b;c;z)\right\}=\frac{(a)_n(b)_n}{(c)_n}F_\rho^{(\alpha,\beta,\gamma)}(a+n,b+n;c+n;z), \tag{2.18}$$

$$\left(\rho,\alpha,\beta,\gamma\in R^+,n\in N_0\right)$$

$$\frac{d^n}{dz^n}\left\{\Phi_\rho^{(\alpha,\beta,\gamma)}(b;c;z)\right\}=\frac{(b)_n}{(c)_n}\Phi_\rho^{(\alpha,\beta,\gamma)}(b+n;c+n;z),\left(\rho,\alpha,\beta,\gamma\in R^+,n\in N_0\right) \tag{2.19}$$

Proof

Differentiating extended Gauss and Confluent hypergeometric functions in equations (2.13) and (2.14) with respect to z using the fact that:

$$B(b,c-b)=\frac{c}{b}B(b+1,c-b)\text{ and }(a)_{n+1}=a(a+1) \tag{2.20}$$

we obtain the integral representations of equations (2.18) - (2.19), when $n=1$. Recursive applying the same process to the identities, led to the required result.

3.3 Transformation and Summation Formulae

Theorem 4: The following transformation formulae holds true:

$$F_\rho^{(\alpha,\beta,\gamma)}(a,b;c;z)=(1-z)^{-a}F_\rho^{(\alpha,\beta,\gamma)}\left(a,c-b;b;\frac{z}{1-z}\right),\left(\rho,\alpha,\beta,\gamma\in R^+\right) \tag{2.21}$$

$$\Phi_\rho^{(\alpha,\beta,\gamma)}(b;c;z)=e^z\Phi_\rho^{(\alpha,\beta,\gamma)}(b;c;-z),\left(\rho,\alpha,\beta,\gamma\in R^+\right) \tag{2.22}$$

Proof

Replacing t by $1-t$ in equation (2.15) and rewriting

$$\left[1-z(1-t)\right]^{-a}=(1-z)^{-a}\left(1+\frac{z}{1-z}t\right)^{-a}$$

Yields:

$$F_\rho^{(\alpha,\beta,\gamma)}(a,b;c;z)=\frac{(1-z)^{-a}}{B(b,c-b)}\int_0^1 t^{c-b-1}(1-t)^{b-1}\left(1+\frac{z}{1-z}t\right)^{-a}E_{\alpha,\beta}^{\gamma}\left(-\frac{\rho}{t(1-t)}\right)dt \tag{2.23}$$

Using equation (2.15) in (2.23) yield the result in (2.21). Following similar argument we obtain equation (2.22)

Theorem 5: The following summation formulae holds true:

$$F_\rho^{(\alpha,\beta,\gamma)}(a,b;c;1)=\frac{B_\rho^{(\alpha,\beta,\gamma)}(a,c-a-b)}{B(b,c-b)},\left(\rho,\alpha,\beta,\gamma\in R^+,\rho=0,\mathrm{Re}(c-a-b)\succ 0\right) \tag{2.24}$$

Proof

Inserting $z=1$, in equation (2.15) and using definition of extended beta function in (1.2), we obtain the desired result.

3.4 Generating Function for $F_\rho^{(\alpha,\beta,\gamma)}(a,b;c;z)$

Theorem 6: The following generating function for $F_p^{(\alpha,\beta,\gamma)}(a,b;c;z)$ holds true:

$$\sum_{n=0}^{\infty}(a)_n F_\rho^{(\alpha,\beta,\gamma)}(a+n,b;c;z)\frac{t^n}{n!}=(1-t)^{-a}F_\rho^{(\alpha,\beta,\gamma)}\left(a,c-b;b;\frac{z}{1-t}\right) \tag{2.25}$$

Proof

Let $\Im$ be the left side of equation (2.25), then

$$\Im=\sum_{n=0}^{\infty}(a)_n\left\{\sum_{k=0}^{\infty}(a+n)_k\frac{B_\rho^{(\alpha,\beta,\gamma)}(b+k,c-b)}{B(b,c-b)}\frac{z^k}{k!}\right\}\frac{t^n}{n!} \tag{2.26}$$

From the relation $(a)_n(a+n)_k=(a)_k(a+k)_n$, equation (2.26) gives:

$$\Im = \sum_{k=0}^{\infty} (a)_k \frac{B_{\rho}^{(\alpha,\beta,\gamma)}(b+k,c-b)}{B(b,c-b)} \left\{ \sum_{n=0}^{\infty} (a+k)_n \frac{t^n}{n!} \right\} \frac{z^k}{k!} \tag{2.27}$$

Applying generalized binomial theorem to the inner summation in equation (2.27), we get:

$$\Im = \sum_{k=0}^{\infty} (a)_k \frac{B_{\rho}^{(\alpha,\beta,\gamma)}(b+k,c-b)}{B(b,c-b)} (1-t)^{-a-k} \frac{z^k}{k!}$$

$$\Im = (1-t)^{-a} \sum_{k=0}^{\infty} (a)_k \frac{B_{\rho}^{(\alpha,\beta,\gamma)}(b+k,c-b)}{B(b,c-b)} \frac{\left(\frac{z}{1-t}\right)^k}{k!} \tag{2.28}$$

Using the definition of the extended Gauss hypergeometric function in equation (2.28), we obtain the desired result.

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