IOSR Journal of Mathematics (IOSR-JM)
e-ISSN: 2278-5728, p-ISSN: 2319-765X. Volume 15, Issue 5 Ser. IV (sep – Oct 2019), PP 01-04
www.iosrjournals.org

All Fermat Numbers are Square-free: A Simple Proof

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Abstract: *Fermat numbers are those in the form of $F_n=2^{2^n}+1$ where n is a nonnegative integer. It was introduced by Pierre de Fermat, and the first four of these are primes. The search has been going on to see whether Fermat numbers are square-free, the mathematical term for a multiple of a square. This article tries to give a simple proof that Fermat numbers are square-free.*

Key words: *Integer, square, square-free.*

Date of Submission: 07-10-2019 Date of Acceptance: 22-10-2019

I. Introduction

In mathematics a **Fermat number**[1], named after Pierre de Fermat who first studied them, is a positive integer of the form $F_n=2^{2^n}+1$ where *n* is a nonnegative integer. The first few Fermat numbers are: 3, 5, 17, 257, 65537, 4294967297, 18446744073709551617 etc. Fermat's assumption that all of $F_n=2^{2^n}+1$ to be prime, called **Fermat primes**[1][2], later proved wrong for $n>4$. All known Fermat numbers are square-free. The aim of this article is to give a simple proof that all Fermat numbers are square-free [3]. A **square-free** integer [4], defined in mathematics, is an integer which is divisible by no perfect square other than 1. For example, 10 is square-free but 18 is not, as 18 is divisible by $9 = 3^2$.

II. A different concept of irrational number:

In his book 'A Course on Number Theory' Peter J. Cameron asked a few basic questions on number theory in the introductory chapter. One of them was[5]:

"How closely can we approximate a given irrational number by rational numbers which are not too complicated?"

"How closely can √p be approximated by a rational number? For example, √2 is approximately equal to 141421/100000, but 1393/985 is an even better approximation, and has much smaller numerator and denominator. How does one find such good approximations?"

By the simple calculator supplied with Microsoft windows we could find further approximation, √2=1.4142135623730950488016887242097=14142135623730950488016887242097/10000000000000000000 000000000000. This will not stop here. More and appropriate value can be acquired by further increasing the numbers of digits in the numerator and denominator of the closest approximation reached so far. It would be in the form of 14142135623730950488016887242097…../10000000000000000000000000000000…….. or 1393……/985…….

This concept of expression of irrational number has been employed only in that part of our following proof where irrational number is dealt with. Moreover, an alternative proof of that part, expressing an irrational number as a multiple of a real number with an irrational number, here, *e*, has also been provided.

III. A simple proof:

We have to prove that Fermat number, $F_n = 2^{2^n} + 1 \neq km^2$ When *k, m, n* are positive integers.

It would suffice to prove, $\boldsymbol{km^2 - 2^{2^n} \neq 1}$ **for all possible values of *k, m, n*.**

If $\boldsymbol{km^2 \leq 2^{2^n}}$, $2^{2^n} + 1 > km^2 \Rightarrow 2^{2^n} + 1 \neq km^2 \therefore \boldsymbol{km^2 - 2^{2^n} \neq 1}$

If $\boldsymbol{km^2 > 2^{2^n}}$, $(\sqrt{km})^2 > \left(2^{2^{n-1}}\right)^2$ or, $\sqrt{km} > 2^{2^{n-1}}$

Then, $km^2 - 2^{2^n} = (\sqrt{km})^2 - \left(2^{2^{n-1}}\right)^2$

$=(\sqrt{km} - 2^{2^{n-1}})(\sqrt{km} + 2^{2^{n-1}})$

Obviously, $(\sqrt{km} + 2^{2^{n-1}})>1$

As k is an integer>1, $\sqrt{k}$ is either an integer, or an irrational mixed number.

If $\sqrt{k}$ is an integer, $km^2 - 2^{2^n} =(\sqrt{km} - 2^{2^{n-1}})(\sqrt{km} + 2^{2^{n-1}})>1$

Then, $(\sqrt{km} - 2^{2^{n-1}})$ is the difference between two integers, and $(\sqrt{km} + 2^{2^{n-1}})$ is the sum of two integersand greater than 1.

So, $km^2 - 2^{2^n} =(\sqrt{km} - 2^{2^{n-1}})(\sqrt{km} + 2^{2^{n-1}})>1$

$\therefore km^2 - 2^{2^n} \neq 1$

If $\sqrt{k}$ is an irrationalmixed number, $\sqrt{km}$ is also an irrational mixed number, **let $\sqrt{km}= d\frac{c}{b}$, $a=2^{2^{n-1}}$ and b,c,d, are positive integers>1, $b>c$ and co-primes, as $\sqrt{km}>2^{2^{n-1}}$, $d\frac{c}{b} > a$ and $d \geq a$.**

(This $d\frac{c}{b}$ is not $d.\frac{c}{b}= d \times \frac{c}{b}$ rather a mixed number $d\frac{c}{b} = d + \frac{c}{b} = \frac{bd+c}{b}$ like $1\frac{1}{2} = \frac{3}{2}$or irrational mixed number like $1\frac{12134\ldots}{25321\ldots.}$. It is the same concept as we write irrational numbers in decimals, unlimited digits after decimal points, in case of trying to write irrational number in fraction or mixed number the numerator and denominator of the fraction part should have unlimited digits. So, c and b have unlimited digits. e.g. **π**=3.14159265358979=3+0.**141592**65358979 and a close fraction to **π** is $\frac{22}{7} = 3\frac{1}{7}$=3+$\frac{1}{7}$=3+0.**142857** with a little bit of fine tuning we get $3\frac{1011}{7141}$=3+$\frac{1011}{7141}$=3+0.**141576**80997059 closer to **π**, further fine tuning gets $3\frac{101111\ldots}{714101\ldots}$ =3+ $\frac{101111\ldots}{714101\ldots}$ =3+0.**141592**01569526 further close to **π**. We may compare $\frac{c}{b}$ with $\frac{e}{\pi} = \frac{2.71828182845905\ \ldots..}{3.14159265358979\ \ldots..} = \frac{271828182845905\ \ldots..}{314159265358979\ \ldots..}$, though not exactly in the same mould, *c,b* are integer numerator and denominator with unlimited digits to make fraction part of an irrational mixed number, whereas *e* and **π** are irrational mixed number themselves, while in ratio their decimal point has been conveniently shifted to the right by same number of digits; for ease of mathematical operation we can use *c* and *b* like *e* and **π** in their denoted appearances, or like letting $\sqrt{2}=p$ doing all the operations and putting values finally, the final result will not differ.**)**

In this case,

$km^2 - 2^{2^n} =(\sqrt{km} - 2^{2^{n-1}})(\sqrt{km} + 2^{2^{n-1}})$

$= \left(d\frac{c}{b} - a\right)\left(d\frac{c}{b} + a\right)$

$$= \frac{db + c - ba}{b} \times \frac{db + c + ba}{b}$$

$= \frac{(db+c)^2 - (ba)^2}{b^2}$

$= \frac{d^2b^2 + c^2 + 2dbc - b^2a^2}{b^2}$

$=d^2 - a^2 + \frac{2dc}{b} + \frac{c^2}{b^2}$

$=(d-a)(d+a) + (\frac{2dc}{b} + \frac{c^2}{b^2})$

When $d>a$, $(d-a)(d+a)$is a positive integer. $(\frac{2dc}{b} + \frac{c^2}{b^2})$ could either be a positive fraction, a mixed number or an integer.

In any of these cases, $(d-a)(d+a) + (\frac{2dc}{b} + \frac{c^2}{b^2})>1$

$km^2 - 2^{2^n} =(\sqrt{km} - 2^{2^{n-1}})(\sqrt{km} + 2^{2^{n-1}})$

$=(d-a)(d+a) + (\frac{2dc}{b} + \frac{c^2}{b^2})>1$

$\therefore km^2 - 2^{2^n} \neq 1$

When $d=a$, $km^2 = (\sqrt{km})^2 = \left(d\frac{c}{b}\right)^2 = \left(d + \frac{c}{b}\right)^2 = d^2 + 2d.\frac{c}{b} + \left(\frac{c}{b}\right)^2 = d^2 + \frac{c}{b}\left(2d + \frac{c}{b}\right)$

$=a^2 + \frac{c}{b}\left(2a + \frac{c}{b}\right) = 2^{2^n} + \frac{c}{b}\left(2.2^{2^{n-1}} + \frac{c}{b}\right)$

For, $km^2 - 2^{2^n} = 1$,
$2^{2^n} + \frac{c}{b}\left(2.2^{2^{n-1}} + \frac{c}{b}\right) - 2^{2^n} = 1$
$\Rightarrow \frac{c}{b}\left(2.2^{2^{n-1}} + \frac{c}{b}\right) = 1$
$\Rightarrow \left(2.2^{2^{n-1}} + \frac{c}{b}\right) = \frac{b}{c}$
$\Rightarrow 2.2^{2^{n-1}} = \frac{b}{c} - \frac{c}{b} = \frac{b^2 - c^2}{bc} = \frac{(b-c)(b+c)}{bc}$

$bc \nmid (b-c)$ and $bc \nmid b+c)$ so, $bc \nmid (b-c)(b+c)$ moreover, b, c are co-primes as per assumption.

So, $2.2^{2^{n-1}} \neq \frac{(b-c)(b+c)}{bc}$

$\therefore \boldsymbol{km^2 - 2^{2^n} \neq 1}$

In all possible cases $km^2 - 2^{2^n} \neq 1$
Or, $F_n = 2^{2^n} + 1 \neq km^2$
Fermat numbers are square-free.

If we try to think and express more conventionally:
When $\sqrt{k}$ is an irrational mixed number, $\sqrt{km}$ is also an irrational mixed number, let
$\sqrt{km} = d + je$, when $j \in \mathbb{R}^+ < 1$ and e representing an irrational number, irrational $je < 1$, as our previous assumption, $d, a \in \mathbb{N}$ and $\boldsymbol{d \geq a}$. Then,
$km^2 - 2^{2^n} = (\sqrt{km} - 2^{2^{n-1}})(\sqrt{km} + 2^{2^{n-1}})$
$= (d + je - a)(d + je + a)$

Here, $(d + je + a) > 1$,
d and a are positive integers and when $d \neq a$, $d - a \neq 0$, $d - a \geq 1$
$\therefore (d + je - a) = (d - a + je) > 1$
$\therefore km^2 - 2^{2^n} = (\sqrt{km} - 2^{2^{n-1}})(\sqrt{km} + 2^{2^{n-1}})$
$= (d + je - a)(d + je + a) > 1$

When $d = a$, and $d - a = 0$ then,
$\therefore km^2 - 2^{2^n} = (\sqrt{km} - 2^{2^{n-1}})(\sqrt{km} + 2^{2^{n-1}})$
$= (d + je - a)(d + je + a) = (d - a + je)(d + je + a) = je(d + je + a)$
Let us assume, $je(d + je + a) = 1$,
Then, $d + je + a = \frac{1}{je}$
$\Rightarrow d + a = \frac{1}{je} - je = \frac{1 - (je)^2}{je}$ which is absurd as the sum of integers has to be an integer.
So, $je(d + je + a) \neq 1$
Consequently, $km^2 - 2^{2^n} \neq 1$
Though our assumption was $\boldsymbol{d \geq a}$,and it is needless to prove the case when $a > d$, but even when $a > d$ then,
$km^2 - 2^{2^n} = (\sqrt{km} - 2^{2^{n-1}})(\sqrt{km} + 2^{2^{n-1}})$
$= (d + je - a)(d + je + a)$
$= (d + je)^2 - a^2 = d^2 - a^2 + 2dje + (je)^2 =$ an integer+ irrational number
So, $km^2 - 2^{2^n} \neq 1$

So, in all possible cases, in every way, $km^2 - 2^{2^n} \neq 1$
Or, $F_n = 2^{2^n} + 1 \neq km^2$
Fermat numbers are square-free. ■

Corollary: When a Fermat number can be factorized into prime factors, no prime factor will be repeated.

IV. Conclusion

Since Pierre de Fermat introduced Fermat numbers, a positive integer of the form $F_n=2^{2^n}+1$ where n is a nonnegative integer, it has drawn attention of the mathematicians. Fermat's original claim that all Fermat numbers are primes prove wrong for $n>4$. It is also considered that all Fermat numbers are square-free, but not proven yet. In this article a simple proof that all Fermat numbers are square-free has been provided. In most parts the proof was direct.

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S. A. Kader. "All Fermat Numbers are Square-free: A Simple Proof." IOSR Journal of Mathematics (IOSR-JM) 15.5 (2019): 01-04.