IOSR Journal of Mathematics (IOSR-JM)
e-ISSN: 2278-5728, p-ISSN: 2319-765X. Volume 15, Issue 5 Ser. IV (sep – Oct 2019), PP 11-18
www.iosrjournals.org

Dynamics of ellipses inscribed in quadrilaterals

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Date of Submission: 07-10-2019 Date of Acceptance: 22-10-2019

I. Introduction

Suppose that we are given a point, *P*, in the interior of a convex quadrilateral, *Q*, in the xy plane. Is there an ellipse, E_0, inscribed in *Q* and which also passes through *P* ? If yes, how many such ellipses ? By inscribed in *Q* we mean that E_0 lies in *Q* and is tangent to each side of *Q*. Looked at in a dynamic sense: Imagine a particle constrained to travel along the path of an ellipse inscribed in *Q*, so that the particle bounces off of each side of *Q* along its path. Of course there are infinitely many such paths. Can we also specify a point in *Q* that the particle must pass through ? If yes, is such a path then unique ? We show below(Theorem 1) that the path is unique when *P* lies on one of the diagonals of *Q*(but does not equal their intersection point), while there are two such paths if *P* does not lie on one of the diagonals of *Q*. If *P* equals the intersection point of the diagonals of *Q*, then no ellipse inscribed in *Q* passes through *P*. Finally, there is a unique ellipse inscribed in *Q* which is tangent at a given point on the boundary of *Q*, assuming, of course, that that point is not one of the vertices of *Q*. For a paper somewhat similar to this one, but involving ellipses inscribed in triangles, see [4].

II. Main Result

Theorem 1: Let *Q* be a convex quadrilateral in the *xy* plane, let $int(Q)$ denote the interior of *Q*, and let $\partial(Q)$ denote the boundary of *Q*. Let D_1 and D_2 denote the diagonals of *Q* and let *IP* denote their point of intersection. Let $P=(x,y)$ be a point in $Q=int(Q)\cup\partial(Q)$.

(i) If $P\in int(Q), P\notin D_1\cup D_2$, then there are exactly two ellipses inscribed in *Q* which pass through *P*.

(ii) If $P\in int(Q)$ and $P\in D_1\cup D_2$, but $P\neq IP$, then there is exactly one ellipse inscribed in *Q* which passes through *P*.

(iii) There is no ellipse inscribed in *Q* which passes through *IP*.

(iv) If $P\in\partial(Q)$, but *P* is not one of the vertices of *Q*, then there is exactly one ellipse inscribed in *Q* which passes through *P*(and is thus tangent to *Q* at one of its sides).

Figures 1 and 2 below illustrate Theorem 1(i) and (ii), respectively.

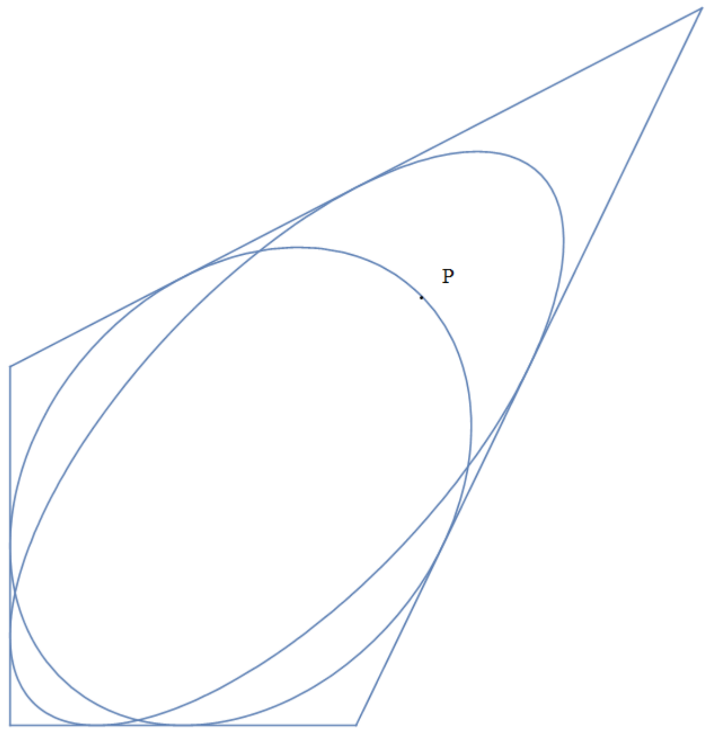


Figure 1

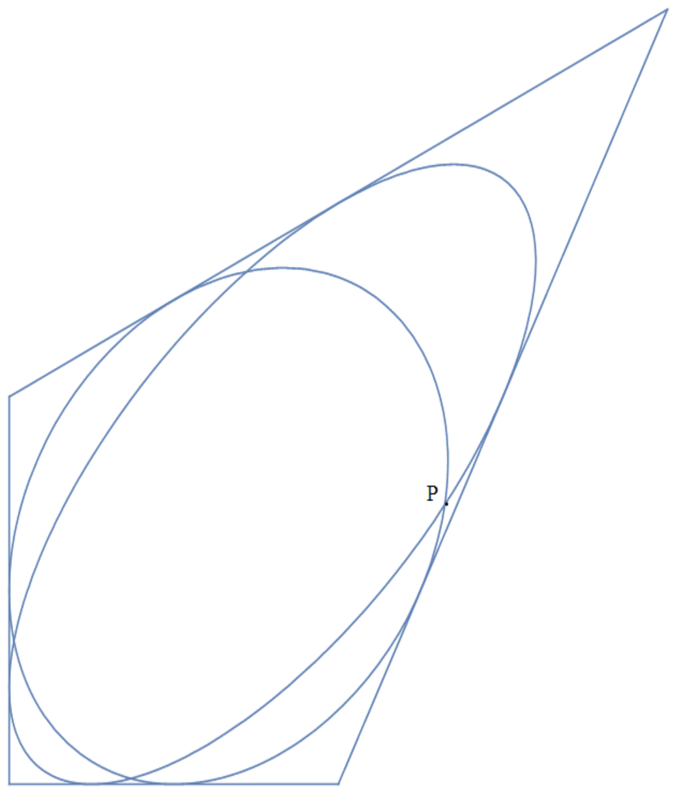


Figure 2

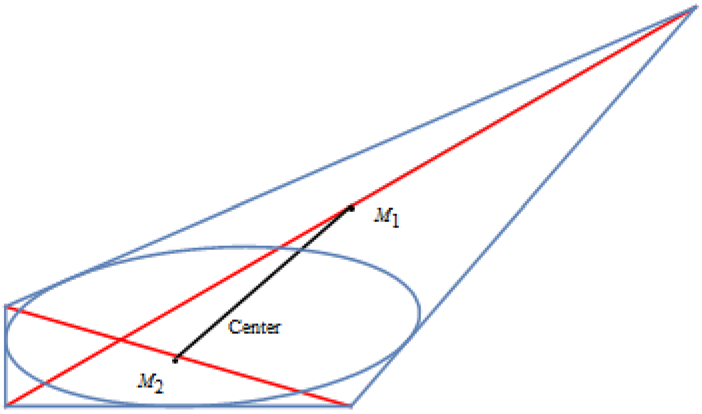
By Theorem 1 we have the following:
Corollary: If two ellipses inscribed in a convex quadrilateral intersect at a point, then that point of intersection cannot lie on either diagonal of the quadrilateral.

III. Preliminary Results

A problem, often referred to in the literature as Newton's problem, was to determine the locus of centers of ellipses inscribed in a convex quadrilateral, *Q*, in the *xy* plane. Chakerian [1] gives a partial solution of Newton's problem using orthogonal projection, which is the solution actually given by Newton.

Newton's Theorem: Let M_1 and M_2 be the midpoints of the diagonals of a convex quadrilateral, *Q*. If E_0 is an ellipse inscribed in *Q*, then the center of E_0 must lie on the open line segment, *Z*, connecting M_1 and M_2.

The figure below illustrates Newton's Theorem, which we use to help with deriving the general equation of an ellipse inscribed in *Q*(see Proposition 1).

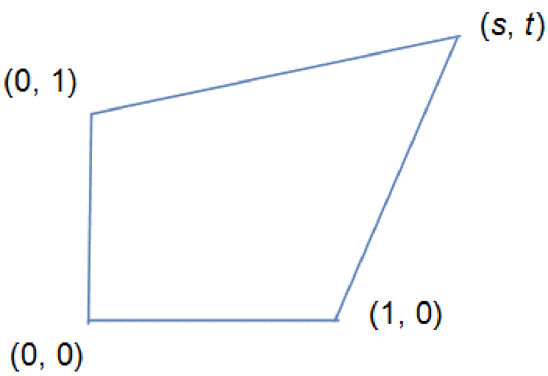


We now state the following result about when a quadratic equation in *x* and *y* yields a nondegenerate ellipse.

Lemma 1: The equation $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$, with $A, C > 0$, is the equation of an ellipse if and only if $\Delta > 0$ and $\delta > 0$, where $\Delta = 4AC - B^2$ and $\delta = CD^2 + AE^2 - BDE - F\Delta$

Remark: The condition $\delta > 0$ implies that the equation defines a curve and not just a single point or the empty set. The condition $\Delta > 0$ implies that the equation defines an ellipse [2].

We shall prove Theorem 1 below when *Q* is not a parallelogram. We leave the details when *Q* is a parallelogram for the reader. Let *Q* be a convex quadrilateral with vertices $A_1, A_2, A_3,$ and A_4, starting with $A_1 =$ lower left corner and going clockwise. Then there is an affine transformation which sends $A_1, A_2,$ and A_4 to the points $(0,0), (0,1),$ and $(1,0)$, respectively. It then follows that $A_3 = (s,t)$ for some $s, t > 0$; Thus it suffices to consider the quadrilateral, $Q_{s,t}$, with vertices $(0,0), (0,1),$ (s,t), and $(1,0)$.`



Since $Q_{s,t}$ is convex, $s + t > 1$; Also, if Q has a pair of parallel vertical sides, first rotate counterclockwise by

90°, yielding a quadrilateral with parallel horizontal sides. Since we are assuming that *Q* is not a parallelogram, we may then also assume that $Q_{s,t}$ does not have parallel vertical sides and thus $s \neq 1$. The midpoints of the diagonals of $Q_{s,t}$ are $M_1 = \left(\frac{1}{2}, \frac{1}{2}\right)$ and $M_2 = \left(\frac{1}{2}s, \frac{1}{2}t\right)$, and the line through M_1 and M_2 has equation

$$y = L(x) = \frac{s - t + 2x(t-1)}{2(s-1)}$$

Any point on the open line segment connecting M_1 and M_2 has the form $(h, L(h)), h \in I = \frac{1}{2}$ and $\frac{1}{2}s$.

Now suppose that E_0 is an ellipse inscribed in $Q_{s,t}$. How does one find the equation of E_0 and the points of tangency of E_0 with $Q_{s,t}$? We sketch the derivation of the equation and points of tangency now.First, since E_0 has center $(h, L(h), h \in I$ by Newton's Theorem, one may write the equation of E_0 in the form

$$(x-h)^2 + B(x-h)\big(y - L(h)\big) + C\big(y - L(h)\big)^2 + F = 0. \qquad (1)$$

Throughout we let *J* denote the open interval $(0,1)$; Now suppose that E_0 is tangent to $Q_{s,t}$ at the points $P_\zeta = (\zeta, 0)$ and $P_v = (0, v)$, where $\zeta, v \in J$;Differentiating(1)with respectto *x* andplugging in P_ζ *and* P_v yields

$$\zeta - h = \frac{BL(h)}{2} \qquad (2)$$

$$v - L(h) = \frac{Bh}{2C}.$$

Plugging in P_ζ *and* P_v into (1) yields

$(\zeta - h)^2 - BL(h)(\zeta - h) + C\big(L(h)\big)^2 + F = 0$ and

$h^2 - Bh\big(v - L(h)\big) + C\big(v - L(h)\big)^2 + F = 0$; By (2) we have $F = \frac{h^2}{4C}\left(B^2 - 4C\right)$ and

$F = \frac{L^2(h)}{4}\left(B^2 - 4C\right)$; Using both expressions for *F* gives

$$C = \frac{h^2}{L^2(h)}. \qquad (3)$$

Now by (2) again,

$$B = \frac{2(\zeta - h)}{L(h)}. \qquad (4)$$

(2), (3), and (4) then imply that

$$v = \frac{\zeta\, L(h)}{h}. \qquad (5)$$

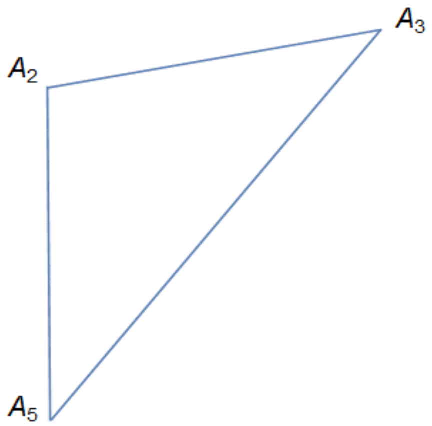
Substituting (3) and (4) into $F = \frac{h^2}{4C}\left(B^2 - 4C\right)$ yields $F = \zeta^2 - 2\zeta h$; (1) then becomes

$$(x-h)^2 + \frac{2(\zeta - h)}{L(h)}(x-h)\big(y - L(h)\big) + \frac{h^2}{L^2(h)}\big(y - L(h)\big)^2 + \zeta^2 - 2\zeta h = 0. \quad (6)$$

Finally, we want to find *h* in terms of ζ, which makes the final equation simpler than expressing everything in terms of *h*. One way to do this is to use the following well-known Theorem of Marden [5].

Marden's Theorem: Let $F(z)=\frac{t_1}{z-z_1}+\frac{t_2}{z-z_2}+\frac{t_3}{z-z_3}$, $\sum_{k=1}^{3} t_k = 1$, and let Z_1 and Z_2 denote the zeros of $F(z)$. *Let* L_1, L_2, L_3 be the line segments connecting z_2 & z_3, z_1 & z_3, *and* z_1 & z_2, respectively.If $t_1t_2t_3>0$, then Z_1 and Z_2 are the foci of an ellipse, E_0, which is tangent to L_1, L_2, and L_3 at the points $\zeta_1, \zeta_2, \zeta_3$, where $\zeta_1=\frac{t_2z_3+t_3z_2}{t_2+t_3}, \zeta_2=\frac{t_1z_3+t_3z_1}{t_1+t_3}$, and $\zeta_3=\frac{t_1z_2+t_2z_1}{t_1+t_2}$, respectively.

Using $A_2=(0,1), A_3=(s,t)$, and $A_5=\left(0,-\frac{t}{s-1}\right)$, and applying Marden's Theorem to the triangle $\triangle A_2A_3A_5$, one can show that E_0 is tangent to $Q_{s,t}$ at the point $\left(\frac{s-2h}{2(t-1)h+s-t},0\right)$.



Many of the details of this can be found in [3]. Hence $\zeta=\frac{s-2h}{2(t-1)h+s-t}$, which implies that

$$h=\frac{1}{2}\frac{\zeta(t-s)+s}{\zeta(t-1)+1} \tag{7}$$

Substituting for *h* in (6) using (7) and simplifying gives

$$t^2x^2+\left(4\zeta^2(t-1)t+2\zeta t(s-t+2)-2st\right)xy+$$
$$\left(\zeta(t-s)+s\right)^2y^2-2\zeta t^2x-2\zeta t\left(\zeta(t-s)+s\right)y+\zeta^2t^2=0, \zeta\in J. \tag{8}$$

Now we use Lemma 1 to show that (8) gives the equation of an ellipse. First,Δ simplifies to $16t^2(1-\zeta)\zeta\left(\zeta(t-1)+1\right)\left(s+\zeta(t-1)\right)>0$ since $\zeta\in J$, $s,t>0$, and $s+t>1$;Similarly, δ simplifies to $\zeta^2(\zeta-1)^2\left(s+\zeta(t-1)\right)^2>0$; Note that by (7), any ellipse with equation given by (8) has center $\left(h,L(h)\right)=C_\zeta=\left(\frac{1}{2}\frac{\zeta(t-s)+s}{\zeta(t-1)+1},\frac{1}{2}\frac{t}{(t-1)\zeta+1}\right)$; This leads to the following result, some of which we have already proven.

Proposition 1: (i) E_0 is an ellipse inscribed in $Q_{s,t}$ if and only if the general equation of E_0 is given by (8) for some $\zeta\in J$. Furthermore, (8) provides a one-to-one correspondence between ellipses inscribed in $Q_{s,t}$ and points $\zeta\in J$.

(ii) If E_0 is an ellipse given by (8) for some $\zeta\in J$, then E_0 is tangent to the four sides of $Q_{s,t}$ at the points

$\zeta_1 = \left(0, \dfrac{\zeta t}{\zeta(t-s)+s}\right), \zeta_2 = \left(\dfrac{(1-\zeta)s^2}{\zeta(t-1)(s+t)+s}, \dfrac{t(s+\zeta(t-1))}{(\zeta(t-1)(s+t)+s)}\right),$

$\zeta_3 = \left(\dfrac{s+\zeta(t-1)}{\zeta(s+t-2)+1}, \dfrac{(1-\zeta)t}{\zeta(s+t-2)+1}\right)$, and $\zeta_4 = (\zeta, 0)$, going clockwise and starting with the leftmost side.

Proof: First, the derivation given above proves that if E_0 is an ellipse inscribed in $Q_{s,t}$, then the general equation of E_0 is given by (8) for some $\zeta \in J$. Now it is clear geometrically that if E_1 and E_2 are distinct ellipses with the same center and which are each inscribed in a convex quadrilateral, *Q*, then *Q* must be a parallelogram. Chakerian mentions this in [1], but no proof is cited or given. One way to prove this is as follows:By using nonsingular affine transformations, one may assume that E_1 is the unit circle and that E_2 has major and minor axes parallel to the x and y axes. We leave the rest of the details to the reader. Since $Q_{s,t}$ is not a parallelogram, there is a one-to-one correspondence between ellipses inscribed in $Q_{s,t}$ and points $\zeta \in J$ and completes theproof of (i). Second, if E_0 is an ellipse with equation given by (8), then using basic calculus techniques it is easy to show that E_0 is inscribed in $Q_{s,t}$ and is tangent to the four sides of $Q_{s,t}$ at $\zeta_1 - \zeta_4$, which proves (ii).

Lemma 2: Let

$$g(x,y) = (ys+(1-y)t)^2 + 4t(t-1)xy, \qquad (9)$$
$$(s,t) \in G = \{(s,t) : s,t>0, s+t>1, s \neq 1\}.$$

Then $g(x,y) > 0$ for any $(x,y) \in int(Q_{s,t})$.

Proof: Suppose that $(x,y) \in int(Q_{s,t})$.

Since $ys+(1-y)t$ is a linear function of *y* which is positive at $y=0$ (yields $t>0$) and at $y=1$ (yields $s>0$),

$$ys+(1-y)t > 0, y \in J. \qquad (10)$$

Hence $g(x,y) > 0$ if $t \geq 1$. Assume now that $s>1$ and $t<1$: By completing the square we have

$\dfrac{g(x,y)}{(s-t)^2} = \left(y + \left(\dfrac{t}{s-t}\right)\left(\dfrac{2(t-1)x}{s-t}+1\right)\right)^2 + 4t^2(1-t)x\ \dfrac{(t-1)x+s-t}{(s-t)^4}$. Using similar reasoning,

$$(t-1)x+s-t>0, x \in J.$$

Hence $g(x,y) > 0$ if $s>1$ and $t<1$. Finally, assume that $s<1$ and $t<1$: $\dfrac{\partial g(x,y)}{\partial x} = 4ty(t-1) \neq 0$, which implies that *g* has no critical points in $int(Q_{s,t}) = S_1 \cup S_2$,

where $S_1 = \{(x,y) : 0 < x \leq s, 0 < y < L_2(x)\}, S_2 = \{(x,y) : s \leq x < 1, 0 < y < L_3(x)\}$, and

$$L_2(x) = 1 + \frac{t-1}{s}x, \qquad (11)$$
$$L_3(x) = \frac{t}{s-1}(x-1).$$

We now check *g* on $\partial(Q_{s,t})$. $g(x,0) = t^2 > 0$, $g(x, L_2(x)) = (s^2 - (1-t)(s+t)x)^2 / s^2$; Since $x \leq s$

for $(x,y)\in S_1$,
$s^2-(1-t)(s+t)x\geq s^2-(1-t)(s+t)s=\ st(s+t-1)>0$, and hence nonzero. Thus
$g(x,L_2(x))>0$; $g(x,L_3(x))=t^2((s+t-2)x+1-t)^2/(s-1)^2$; Since $s+t-2<0$ and $s\leq x$ for $(x,y)\in S_2$, $(s+t-2)x+1-t\leq(s+t-2)s+1-t=\ (s-1)(s+t-1)<0$, and hence nonzero. Thus $g(x,L_3(x))>0$; Finally, $g(0,y)=(ys+(1-y)t)^2>0$ by (10).

Proof of Theorem 1: For fixed x,y,s,t, one can rewrite the left hand side of (8) as the following polynomial in ζ : $p(\zeta)=p_2\zeta^2+p_1\zeta+p_0$, where $p_2=g(x,y)$,
$p_1=2t(s-t+2)xy-2sy^2(s-t)-2sty-2t^2x$, $p_0=(sy-tx)^2$, and $g(x,y)$ is from Lemma 2.
Evaluating p at the endpoints of J yields

$$p(0)=\ (sy-tx)^2\geq 0, \qquad (12)$$
$$p(1)=\ t^2(x+y-1)^2\geq 0.$$

Now a simple computation yields, in simplified form, the discriminant of p:

$$p_1^2-4p_2p_0= \qquad (13)$$
$$-16s(s-1)t^2xy(y-L_2(x))(y-L_3(x)).$$

Also, $p'(\zeta_0)=0$, where $\zeta_0=-\dfrac{p_1}{2p_2}$. Another simple computation yields $p(\zeta_0)=-\dfrac{p_1^2-4p_2p_0}{4p_2}$, which implies, by (13), that

$$p(\zeta_0)=\frac{4s(s-1)t^2xy(y-L_2(x))(y-L_3(x))}{p_2}. \qquad (14)$$

We now assume throughout that $s>1$ and thus $I=\left(\frac{1}{2},\frac{1}{2}s\right)$. The case $s<1$ follows similarly and we omit the details. Suppose that $(x,y)\in int(Q_{s,t})=S_1\cup S_2$, where $S_1=\{(x,y):0<x\leq 1,0<y<L_2(x)\}$, L_2 and L_3 given in (11). By (14), $p(\zeta_0)<0$. Summarizing:

$$(x,y)\in int(Q_{s,t}) \text{ and } p'(\zeta_0)=0 \text{ implies that } p(\zeta_0)<0. \qquad (15)$$

For given $P=(x,y)\in int(Q_{s,t})$, by Proposition 1(i), the number of distinct ellipses inscribed in $Q_{s,t}$ which pass through P equals the number of distinct roots of $p(\zeta)=0$ in J. To prove (i), suppose that $P\notin D_1\cup D_2$. Then $x+y-1\neq 0\neq sy-tx$, which implies, by (12), that $p(0)>0$ and $p(1)>0$. By (15), $p(\zeta)$ has two distinct roots in J. To prove (ii), suppose that $P\in D_1\cup D_2$, but $P\neq IP$. Then either $x+y-1=0$ or $sy-tx=0$, but not both, which implies, by (12), that $p(0)>0$ and $p(1)>0$, or $p(0)>0$ and $p(1)=0$. By (15), $p(\zeta)$ has one root in J. Finally, to prove (iii), if $P=IP$, then by (12), $p(\zeta)$ vanishes at both endpoints of J, which implies that p has no roots in J. The proof of (iv) follows from the proof of Proposition 1(ii) and we leave the details to the reader.

Examples: (1) $s=\frac{1}{2},\ t=\frac{3}{4},\ x=\frac{1}{3},$ and $y=\frac{3}{4}$. Then $P=\left(\frac{1}{3},\frac{3}{4}\right)\in int(Q_{s,t}),\ P\notin D_1\cup D_2$. By Theorem 1(i), there are exactly two ellipses, E_1 and E_2, inscribed in $Q_{s,t}$ and which pass through P;
$256p(\zeta)\ =\ 33\zeta^2-36\zeta+4$, which has roots $\frac{6}{11}\pm\frac{8}{33}\sqrt{3}\in J$. Letting $\zeta=\frac{6}{11}-\frac{8}{33}\sqrt{3}$ in (8) yields the equation of E_1:

$$27\left(1099-152\sqrt{3}\right)x^2+16\left(1477-444\sqrt{3}\right)y^2+24\left(1117-1062\sqrt{3}\right)xy+$$
$$36\left(524\sqrt{3}-1065\right)x+48\left(-773+398\sqrt{3}\right)y$$
$$=36\left(-481+272\sqrt{3}\right).$$

Letting $\zeta=\frac{6}{11}+\frac{8}{33}\sqrt{3}$ in (8) yields the equation of E_2:

$$27\left(1099+152\sqrt{3}\right)x^2+16\left(1477+444\sqrt{3}\right)y^2+24\left(1117+1062\sqrt{3}\right)xy-$$
$$36\left(1065+524\sqrt{3}\right)x-48\left(773+398\sqrt{3}\right)y$$
$$=-36\left(481+272\sqrt{3}\right).$$

(2) $s=4,\ t=2,\ x=\frac{1}{2},$ and $y=\frac{1}{4}.$ Then $P=\left(\frac{1}{2},\frac{1}{4}\right)\in D_1\cup D_2,\ P\neq IP=\left(\frac{2}{3},\frac{1}{3}\right)$. By Theorem 1(ii), there is exactly one ellipse, E_0, inscribed in $Q_{s,t}$ which passes through *P*;

$p(\zeta)=(1/4)\zeta(29\zeta-28)$, which has roots 0 and $\frac{28}{29}$; Letting $\zeta=\frac{28}{29}$ in (8) yields the equation of E_0 : $15979x^2+17100y^2+27588xy-30856x-31920y+16240=1344.$

References

[1]. G. D. Chakerian, A Distorted View of Geometry, MAA, Mathematical Plums, Washington, DC, 1979, 130-150.

[2]. Jack Goldberg, "Matrix Theory with Applications", McGraw-Hill, 1991.

[3]. Alan Horwitz, Ellipses of maximal area and of minimal eccentricity inscribed in a convex quadrilateral, Australian Journal of Mathematical Analysis and Applications, 2(2005), Issue 1, Article 4, 1-12.

[4]. Alan Horwitz, Dynamics of ellipses inscribed in triangles, Journal of Science, Technology and Environment, Volume 5, Issue 1 (2016), 1-21.

[5]. Morris Marden, The Location of the Zeros of the Derivative of a Polynomial, The American Mathematical Monthly, Vol. 42, No. 5 (May, 1935), pp. 277-286.

Alan Horwitz. " Dynamics of ellipses inscribed in quadrilaterals." IOSR Journal of Mathematics (IOSR-JM) 15.5 (2019): 11-18.