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Generalized of Integral Type C^*-Valued Contraction with Fixed Point Theorem

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Abstract: *In this present paper we have proved some Generalized of Integral Type C* -Valued Contraction with Fixed Point. Also we provided an example to support our main result.*

Keywords: *Metric Space, Algebra Valued Metric Space,C* -Algebra Valued Metric Space, Fixed Point Result, Contraction Mapping,Branciari Contraction Function.*

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I. Introduction

In 1922 French mathematician S. Banach, introduced a result which was also called as Banach contractive theorem or Principle, which is stated as follows;

The study of fixed point theory plays an important role in applications of many branches of mathematics. Finding a fixed point of contractive mappings becomes the center of strong research activity. There are some researchers who have worked about the fixed point of contractive mappings see [1,20] an important result regarding a contraction mapping, known as the Banach contraction principle generalized the famous Banach contraction principle in metric spaces. They replace an ordered Banach space for the real numbers and proved some fixed point theorems of contractive mappings in metric space.

Banach contraction principle plays an important role for solving non linear problems. Kannan [6] used the Banach contractive principle for analyzing new type of contractive condition.In 2002,Branciare [3] introduced the concept of integral type contractive mapping to generalized the concept of Banach contraction principle.In 2010,F.Khojasteh et al.[7] used the Branciare integral type contractive mapping for the cone metric space and proved some fixed point theorems

In 2002, Branciari [3] introduced the notion of integral type contractive mappings in complete metric spaces and study the existence of fixed points for mappings which are defined on complete metric space satisfying integral type C^* - Valued contraction. . Many researchers studies various contractions and a lot of fixed point theorems are proved in different spaces; see [1,12-17,18,19,20,21,22,23,24,25].

In this paper we presented some fixed point theorems of Integral type C^* - Valued contractive mappings .Moreover, we present suitable example that support our main result.

In this section we introduce the integral type C^* - valued contractive mapping for the C^* - algebra valued metric spaces and prove some fixed point theorems.

II. Preliminaries

Here we introduced some basic definitions, notations and results of C^* - algebra

1. A $*-algebra$ $\mathcal{A}$ is a complex algebra with linear involution * such that $y^{**}= y$ and $(yz)^* = z^*y^*$, for any $y,z \in \mathcal{A}$.

2.If * -algebra together with complete sub multiplicative norm satisfying $\|y^*\| =\| y\|$ for all $y \in \mathcal{A}$, then * - algebra is said to be a Banach * -algebra.

3.A C^*-algebra is a Banach*- algebra such that $\|y^*y\|= \| y\|^2$ for all $y \in \mathcal{A}$.

4.An element of $\mathcal{A}$ is called positive element, if $\mathcal{A}_+ = \{y^* = y / y \in \mathcal{A}\}$ and $\sigma(y) \subset R_+$, where $\sigma(y)$ is the spectrum of an element $y \in \mathcal{A}$,i.e $\sigma(y) = \{\lambda \in R : \lambda I - y$ is not invertible$\}$ There is a natural partial ordering on $\mathcal{A}_+$ given by $y \le z$ if and only if y-z $\in \mathcal{A}_+$.

Definition 2.1. Suppose that X be a nonempty set, and the mapping $d : X \times X \to \mathbb{A}$ is satisfying the following conditions:

1. $d(y,z) \ge 0$ for all $y,z \in X$ and $d(y,z) = 0 \Leftrightarrow y = z$;
2. $d(y,z) = d(z,y)$ for all $y,z \in X$;
3. $d(y,z) \le d(y,x) + d(x,z)$ for all $x,y,z \in X$.

Then d is C^*-algebra valued metric on X, and $(X,\mathbb{A},d)$ is C^*-algebra valued metric space.

It is clear that C^*-algebra valued metric spaces is the generalization of the metric space by substituting $\mathbb{A}$ instead of $\mathbb{R}$.

Definition 2.2

Let $(X,\mathbb{A},d)$ is C^*-algebra valued metric space and let $\{y_n\}$ be a sequence in X. If

I). for any $\in \succ 0$,there is N such that for all n>N , $\| d(y_n,y) \|$ then the sequence **{** $y_n\}$ is said to be convergent, and we denote it as $\lim_{n\to\infty} y_n = y$

II). for any $\in \succ 0$,there is N such that for all n>N , $\| d(y_m,y_n) \| \le \in$ then the sequence **{** $y_n\}$ is said to be Cauchy sequence.

III). C^*-algebra valued metric space is said to be complete if every Cauchy sequence in X with respect to A is convergent.

Theorem 2.3 Let (X,d) be a complete metric space, $\delta \in (0,1)$ and $f : X \to X$,then f is said to be a contractive mapping such that for all y,z$\in X$, $d(fy, fz) \le \delta d(y,z)$ then f has a unique fixed point.
Recently in 2014,Z.Ma et al.[9] established the notion of C^* - algebra valued metric spaces,and proved some fixed point theorems for contractive and expansive mappings.For more details and basic definitions of C^* - algebra we refer [2,4,5,8,11].

Example 2.4. Let X = R and A = M_2(R). Define d(y,z) = $\begin{pmatrix} |y-2z| & 0 \\ 0 & \delta|y-2z| \end{pmatrix}$ for all y,z $\in$ R and $\delta \ge 0$.

It is essay to verify that d is a C^*-algebra valued metric space and (X , $M_2(\mathbb{R})$, d) is complete C^*-algebra valued metric spaces.

Example:2.5

Let Y=[0,1], $\mathbb{G} = \mathbb{R}^2$ and p >1 be a constant.Take $\mathbb{K} = \{(u,v) \in \mathbb{G}: u,v \ge 0\}$. We define $d : Y \times Y \to \mathbb{G}$ as $d(u,v) = (|u-v|^p, |u-v|^p)$. Then (Y,d) is a complete C^*- valued metric space

Suppose $H : Y \times Y$ as

$$Hu = \frac{1}{2}u - \frac{1}{4}u^2 \quad \text{for all} \quad u \in Y \text{ and } \varphi(k) = 1 \text{ where } k \in \mathbb{R}$$

in fact

$$\int_0^{d(Hu,Hv)} \varphi\, d_{\mathbb{K}} = \int_0^{(|u-v|^p,|u-v|^p)} \varphi\, d_{\mathbb{K}}$$

$$= \int_0^{\left(|\frac{1}{2}(u-v)-\frac{1}{4}(u-v)(u+v)|^p,\frac{1}{2}(u-v)-\frac{1}{4}(u-v)(u+v)|^p\right)} \varphi\, d_{\mathbb{K}}$$

$$= \int_0^{\left(|u-v|^p|\frac{1}{2}-\frac{1}{4}(u+v)|^p,|u-v|^p|\frac{1}{2}-\frac{1}{4}(u+v)|^p\right)} \varphi\, d_{\mathbb{K}}$$

$$\leq \frac{1}{2^p}\int_0^{(|u-v|^p,|u-v|^p)} \varphi\, d_{\mathbb{K}}$$

$$\leq \frac{1}{2^p}\int_0^{d(u,v)} \varphi\, d_{\mathbb{K}}$$

Hence $0 \in Y$ is the unique fixed of H.

Definition 2.6. Let $(X,\mathbb{A},d)$ be a C^*-valued metric spaces. A mapping f from X into X is said to be a C^*-valued contractive if there exists an $c \in \mathbb{A}$ with $\|c\| < 1$ such that $d(fy, fz) \leq c^* d(y,z)c$,

$\forall\ y,z \in X$.

Branciari in 2002, introduced the general integral type contraction which stated as follows.

Let ψ be the class of all mappings ψ from $\mathbb{R}_+$ into $\mathbb{R}_+$ which is Lebesgue integrable, summable on each compact subset of $\mathbb{R}_+$,non negative and for each $\in\ > 0$, $\int_0^{\in} \psi(z)dz > 0$.

Lemma2.7: Let (X,d) be a complete metric space, $\delta \in (0, 1)$ and let $h : X \to X$ be a mapping such that for each $y,z \in X$,

$$\left(\int_0^{d(f(x),f(y))} \phi d_p \leq \alpha \int_0^{d(x,y)} \phi d_{p,}\right) \qquad \text{(A)}$$

where φ from $\mathbb{R}_+$ into $\mathbb{R}_+$ is a Lebesgue-integrable mapping which is summable (i.e., with finite integral) on each compact subset of $\mathbb{R}_+$ nonnegative and such that for each $\in\ > 0$ $\int_0^{\in} \varphi(z)dz \succ 0$ .Then h has a unique fixed point $y \in X$ such that for each $y \in X$, $\lim_{n\to\infty} h^n y = y$.

Motivated by the work of Z. Ma et al. [9] and Branciari[3], we introduce the following definition.

Definition 2.8. *Let* $(X,\mathbb{A},d)$ be a C^*-valued metric space. A mapping $h : X \to X$ is said to be a integral C^*-valued contraction mapping on X if there exists an $c \in \mathbb{A}$ with $\|c\| < 1$ such that

$$\left(\int_0^{d(f(x),f(y))} \phi d_p \leq c^* \int_0^{d(x,y)} \phi d_{p,}\right)$$

for all $y,z \in X$ and $\varphi \in \Psi$.

Now we define a subclass of integral type C^*-valued contraction which we will use in our main result. We call this class a sub additive integral type C^*-contraction.Let Θ be the set of all mappings $\psi \in \Psi$. satisfying the following:

$$\int_0^{a+b} \psi(z)dz \le \int_0^{a} \psi(z)dz + \int_0^{b} \psi(z)dz, \forall a,b \ge 0$$

3. Main Theorem:

Let (X,𝔸,d) be complete C^*-algebra valued metric space, if there exists $c \in \mathbb{A}$ with $\|c\| < 1$ and $f : X \to X$ be a integral C^*-valued contractive mapping such that for all $x, y \in X$,

$$\left(\int_0^{d(f(x),f(y))} \phi d_p \le \alpha \int_0^{d(x,y)} \phi d_{p,} \right) \tag{3.1}$$

For some $\alpha \in (0,1)$, then f has a unique fixed point in X.

Proof: Let x $x_1 \in P$. choose $x_{n+1} = f(x_n)$ we have $\int_0^{d(x_{n+1},x_n)} \phi d_p = \int_0^{d(f(x_n),f(x_{n-1}))} \phi d_p$

$$\le \alpha \int_0^{d(x_n,x_{n-1})} \phi d_p \tag{3.2}$$

$$\le \alpha^{n-1} \int_0^{d(x_2,x_1)} \phi d_p$$

Since $\alpha \in (0,1)$

$$\text{thus} \quad \lim_{n\to\infty} \int_0^{d(x_{n+1},x_n)} \phi d_p = 0 . \tag{3.3}$$

If $\lim_{n\to\infty} d(x_{n+1}, x_n) \ne 0$ and this is a contradiction,so $\lim_{n\to\infty} d(x_{n+1}, x_n) = 0$

We now show that (x_n) is a Cauchy sequence. Due to this,we show that

$\lim_{m,n\to\infty} d(f(x_m), f(x_n)) = 0$.

By triangle inequality

$$\int_0^{d(f(x_m),f(x_n))} \phi d_p \le \int_0^{d(f(x_n),f(x_{n+1}))+d(f(x_{n+1}),f(x_{n+2}))+\dots\dots\dots\dots+d(f(x_{m-1}),f(x_m))} \phi d_p$$

Fixed point theory and applications and by sub- additivity of ϕ we get

$$\int_0^{d(f(x_m),f(x_n))} \phi d_p \le \int_0^{d(f(x_n),f(x_{n+1}))} \phi d_p + \dots\dots\dots + \int_0^{d(f(x_{m-1}),f(x_m))} \phi d_p$$

$$\le (\alpha^n + \alpha^{n-1} + \dots\dots\dots + \alpha^{m-1}) \int_0^{d(x_2,x_1)} \phi d_p \le \frac{\alpha^n}{1-\alpha} \int_0^{d(x_2,x_1)} \phi d_p \to 0.$$

Thus

$$\lim_{m,n\to\infty} d(f(x_n), f(x_m)) = 0$$

This means that $\{x_n\}_{n\in N}$ is a Cauchy sequence and since X is a complete cone metric space, thus $\{x_n\}_{n\in N}$ is convergent to $x_0 \in X$. Finally, since $\int_0^{d(x_{n+1}, f(x_0))} \phi d_p = \int_0^{d(f(x_n), f(x_0))} \phi d_p \le \int_0^{d(x_n, x_0)} \phi d_p$,

Thus $\lim_{n\to\infty} d(x_{n+1}, f(x_0)) = 0$. This means that $f(x_0) = x_0$. If x_0, y_0 are two distinct

fixed points of f then $\int_0^{d(x_0, y_0)} \phi d_p = \int_0^{d(f(x_0), f(y_0))} \phi d_p \le \int_0^{d(x_0, y_0)} \phi d_p$

which is a contradiction. Thus f has a unique fixed point $x_0 \in X$.

Corollary3.1: Let (X,d) be a complete C^*-algebra valued metric space and f has the property that is for all $0 \ne \varepsilon \in P$ there exist $\delta >> 0$ such that

$d(x, y) < \varepsilon + \delta$ implies $d(fx, fy) < \varepsilon$ for all x,y $\in$ X. Then f has unique fixed point.

Corollary3.2: Let (X,d) be a complete C^*-algebra valued metric space and f be a mapping on X.Suppose that there exist a function θ from P into itself satisfying the following:

1. $\theta(0) = 0$ and $\theta(t) >> 0$ for all t>>0

2. θ is non-decreasing and continuous function.Moreover its inverse is continuous

3. For all $0 \ne \varepsilon \in P$ there exists $\delta >> 0$ for all x,y $\in$ X

   $\theta(d(x, y))\varepsilon + \delta$ implies $\theta(d(fx, fy)) < \varepsilon$

4. For all x,y $\in$ X $\theta(x + y) \le \theta(x) + \theta(y)$. Then f has unique fixed point.

Remark3.3: This theorem is the generalization of the C^*-algebra valued contractive mapping, by setting $\psi(z) = 1$

$$\int_0^{d(f(x), f(y))} \psi(z) dz = d(f(x), f(y)) \le c^* d(x, y) c = \int_0^{d(x,y)} \psi(z) dz$$

Example3.4: Let X=[0, 1] be any non empty set and d be metric space defined as

$d(x, y) = \| x - y \| I$, and define $h : X \to X, \psi : [0,\infty) \to [0,\infty)$ by

$$h(z) = \begin{Bmatrix} \dfrac{z}{1+qz} if z = \dfrac{1}{m} \\ 0 if z \ne \dfrac{1}{m} \end{Bmatrix} \text{ and } \tag{3.4}$$

$$\phi(t) = \begin{Bmatrix} t^{\frac{1}{t}-2}(1 - \log t) if, t \succ 0 \\ 0, if t = 0 \end{Bmatrix} \tag{3.5}$$

For all m $\in \mathbb{N}$ and q be any positive integer. As we know that (A) is equivalent to

$$\| d(fx,fy)\|^{\frac{1}{\|d(fx,fy)\|}} \le \|c\| \| d(x,y)\|^{\frac{1}{\|d(x,y)\|}} \quad \forall x,y\in X \tag{3.6}$$

Now our next target is to show that (3.6) is satisfied for $c=\|c\|=\frac{1}{\sqrt{2}}\prec 1.$ For this let us consider

$x=\frac{1}{m+1}$ & $y=\frac{1}{m}$ for $m\in\mathbb{N}$, then we have

$$\| d(fx,fy)\|^{\frac{1}{\|d(fx,fy)\|}} = \left\| \frac{1}{m+1+p}-\frac{1}{m+p}\right\|^{\frac{1}{\left\|\frac{1}{m+1+p}-\frac{1}{m+p}\right\|}}$$

$$=\left[\frac{1}{(m+1+p)(m+p)}\right]^{(m+1+p)(m+p)} \tag{3.7}$$

Now R.H.S of (3.6) implies that,

$$\| d(x,y)\|^{\frac{1}{\|d(x,y)\|}} = \left\| \frac{1}{m}-\frac{1}{m+1}\right\|^{\frac{1}{\left\|\frac{1}{m}-\frac{1}{m+1}\right\|}}$$

$$=\left[\frac{1}{m(m+1)}\right]^{m(m+1)} \tag{3.8}$$

Putting value of (3.7), (3.8) in (3.6) then we get

$$\left[\frac{1}{(m+1+p)(m+p)}\right]^{(m+1+p)(m+p)} \le \|c\| \left[\frac{1}{m(m+1)}\right]^{m(m+1)} \tag{3.9}$$

Therefore (3.9) is true for $\|c\|=\frac{1}{\sqrt{2}}\prec 1,$ so h is an integral C^*-valued contraction with contraction constant

$\|c\|=\frac{1}{\sqrt{2}}\prec 1,$ thus all the condition of Main theorem is satisfied and

f has a unique fixed point 0.

III. Conclusion

The idea of an integral type C^*- algebra valued contraction is not only the extension of C^*- algebra valued contraction, but it develops the inequality (3.1). Where as the notion of Additive of C^*-algebra and sub additive C^*- algebra valued contraction extends the idea of C^*- algebra valued contraction but it slightly generalizes the inequality (3.1)

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