

Predator-Prey for Three Species in the Open Ecological Space.

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Abstract: In this work a study of the coexistence in the open ecological space with three species, where a prey is born, a natural predator and a prey-predator; here the normal model is treated, the equilibrium positions are analyzed and conclusions are drawn from future behavior. If a generalization of the model is made, the system is simplified reducing it to quasi-normal form making use of the analytical theory of differential equations; this facilitated the application of the second method of Liapunov to give conclusions of the future development regarding the coexistence.

Key words: Ecology, Predator, Prey, Stability.

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I. Introdução

Ecology is the science that studies living beings and their interactions with the environment in which they live. This science is of utmost importance because the results of their studies provide data that reveal whether animals and ecosystems are in perfect harmony. At a time when deforestation and the extinction of several species are under way, the work of ecologists is of the utmost importance.

The problem of coexistence between different species in an open ecological space is addressed in [3], and [4], where the different types of coexistence and one can favor or hinder the development of the other.

Many types of biological systems have been modeled mathematically with the purpose of realizing a better study of the natural interaction that exists between different species; in particular the prey-predator model has a relevant position due to the applicability not only of biology where it practically governs the coexistence of different species in open space, but also because it can be applied in other areas including economics. Here, in addition to the highly publicized Lotka-Volterra models, we will analyze lesser known ones in addition to their qualitative study.

The model was discovered independently by Lotka and Volterra, and for this reason it is known as a model Lotka-Volterra or model predator-prey that describes the evolution of prey and predators very well when they are located in an isolated ecosystem. Nevertheless, we have to clarify that two distinct populations in the same environment have several ways of surviving, for example:

- Mutual competence, that is to say compete for the same food source, tend to cause the extinction of a population of them, and the other tends to take advantage of the maximum capacity of environmental resources.
- Interdependence, that is to say the two populations provide some food resources, live peacefully among them, and tend to a state of equilibrium.
- The law of the jungle, is to say a population survives depending on the abundance of natural resources, called prey; however, the other population lives depending on the populations of prey, called the predator. The two elements are composed by the prey-predator model.
- The parasitic life, where one species feed on the other without killing it, but which by all means shaves its quality of life.[9] refers to the mathematical modeling of several processes between them, dealing with the Prey-Predator model, which includes the possibility of system integration that simulates this interaction between two species.

In [10] the interaction of different species is treated in an open medium, indicating in particular a model for the coexistence between a prey and a predator. In addition, it draws a parallel in the economy coming to some conclusions of the process. The prey-predator model has been extensively treated using different techniques, here it may be included,[6].

Another focus on the Lotka-Volterra model is presented in [1]. In the master's dissertation [7] a very exhaustive study of the prey-predator model is made.

The treatment that we will make in this case corresponds with other models presented in the researches of diseases, especially the case of sickle cell anemia, quite treated and with a large number of already developed models; we will only mention some of these works, in [11] and [12], the qualitative study of different models in autonomous and non-autonomous form of the formation of polymers in the blood is treated.

Following these ideas from these previous works here is simulated the interaction between two species being simplified the referred system to arrive at conclusions of this process of coexistence in the open nature.

In nature the most frequent is the competition between different species in the struggle for survival, appearing here the prey-predator model developed by Lotka 1924; Volterra, 1926; Gause, 1934; Kostitzin, 1939. [2].

Drawing on the work of Lotka, the models that consider the population classified by age groups have been developed, in order to solve the limitations of the models that treat all the individuals of the population identically.

One of the most commonly used classical mathematical models is the dynamic system consisting of two elements (usually two species of animals) interacting in such a way that one (predator) species feeds on the other (prey). A typical example is the system consisting of foxes and rabbits, but it can be transferred without loss of generality to any other context, for example, that formed by sellers and buyers applicable to the Economy.

Foxes feed on rabbits and grass rabbits that we assume will never run out. When there are many rabbits, the population of foxes will increase since food is abundant, but there will come a time when the rabbit population will decline as foxes are abundant. By not having the foxes, enough food their population will decrease, which will again favor the rabbit population. That is to say, if they produce cycles of growth and decrease of both to the populations. Is there a mathematical model that explains this periodic behavior?

On the other hand, in the second decade of the 20th century the Italian biologist Umberto D'Ancona studied and compiled data on catches of fish of some types in the Mediterranean, on the one hand, seals (sharks, rachs, etc.), and other fish that were eaten by the previous ones (sardines, anchovies, etc.), in other words, one prey (the edible fish) and the other predator (seals).

One of the first reasons he thought was related to the First World War. In fact, at that time the first great war developed and this forced less boats to go fishing, and therefore, by reducing the intensity of fishing, this caused an increase in the number of predatory fish (seals). However, this argument had a problem and it was also that the number of edible fish had increased. In fact, if the intensity of fishing is small, then this fact benefits the predators more than the prey. The pertinent question was why?

Briefly, two questions were raised:

- How to explain the cyclical behavior of the evolution of two populations, where one species feed on the other?
- Why does a low catch intensity favor predator more than prey?

A detailed study of these types of systems is analyzed in the authors' work [5], which characterizes the behavior of the Lotka-Volterra systems under the hypothesis that the prey grows exponentially in the absence of predators and the predator disappears in absence of prey, studying the behavior of the trajectories in an environment of the equilibrium positions, one can perceive the existence of closed orbits due to the periodicity of the solutions.

Among the models of interaction between species the classic prey-predator model can be highlighted, whose mathematical formulation is composed of Malthusian models and the law of mass action. The analogy can be easily observed in epidemiological models. The prey-predator model also known as the Lotka-Volterra model has also been the starting point for the development of new techniques and mathematical theories.

Predation is a very fundamental type of interaction in nature, where predators catch prey for their food. We can imagine that this relationship is beneficial only to the predator, but from the ecological point of view this is important to regulate the population density of both prey and predator.

Predators remove individuals from the population, consuming them; the ease of catching the prey depends greatly on the size relationship between the prey population and the predator. The greater the population of prey, the greater the possibility of its capture.

Predation occurs when an organism kills and feeds on beings of another species; the animal that killed it is called a predator, which already fed on the prey. Predators are usually found in smaller quantities and have characteristics that favor prey capture; among these characteristics, we can mention the sharp claws, speed and agility.

II. Formulation Of The Model

In the study of the interaction of different species in nature in recent times has been very important mathematical modeling, since this has allowed to be able to make predictions regarding the future behavior of this coexistence process, determining if any species could enter at some time in danger of extinction; all this is important, because that way it is known when you can introduce some artificial element like fishing or house according to the case.

The Prey-Predator model simulates the interaction between species in an open ecological space, in which case we will treat the case of three species. Let's admit that during the process, the medium should not change. The case where there are only two species has already been dealt with in [5] and [8].

There are innumerable cases of interaction between three species where a prey and a natural predator appear, and a Prey-Prey, but it may be the situation that the born predator feeds or not the prey, in this model we present the continuation, the natural predator does not feed on the prey cream.

In the model we will consider that between the prey there is a good competition for space or the conquest of food; the prey-predator without the presence of the prey disappears, as much by the lack of food, as by the presence of the natural predator; the born predator without the presence of the Prey-predator will go extinct; here it is further assumed that there is no interaction with another species.

For the development of the model we will consider the following variables:

$x = x(t)$ - Concentration of prey cream at the moment t .

$y = y(t)$ - Prey-predator concentration at the moment t .

$z = z(t)$ - Concentration of predator born in time t .

$$\begin{cases} \frac{dx}{dt} = a_1 x - a_2 xy - a_3 x^2 \\ \frac{dy}{dt} = -b_1 y + b_2 xy - b_3 yz \\ \frac{dz}{dt} = -c_1 z + c_2 yz \end{cases} \quad (1)$$

With the initial conditions $x(0) = x_0$, $y(0) = y_0$, $z(0) = z_0$; in the model we are considering that the coefficients are all positive. This system has three equilibrium positions which are as follows:

$$(0,0,0), \left(\frac{b_1}{b_2}, \frac{a_1 b_2 - a_3 b_1}{a_2 b_2}, 0 \right), e \left(\frac{a_1 c_2 - a_2 c_1}{a_3 c_2}, \frac{c_1}{c_2}, \frac{b_2 (a_1 c_2 - a_2 c_1) - a_3 b_1 c_2}{a_3 b_3 c_2} \right)$$

In order for these equilibrium positions to be in the first position, it is necessary and sufficient that the following conditions are met:

- 1) $a_1 b_2 > a_3 b_1$,
- 2) $a_1 c_2 > a_2 c_1$,
- 3) $a_1 b_2 c_2 > a_3 b_1 c_2 + a_2 b_2 c_1$.

It is clear that the equilibrium position $(0,0,0)$, is unstable because one can directly apply the first approximation method, where there is a positive own value; which indicates that the species will not have near zero concentrations; as for the second position of equilibrium, to have a null component, would cause an ecological imbalance, which would not be very interesting study.

For the study of the third position of equilibrium it is necessary to carry out a coordinate transformation to bring the origin to this position, thus one would have to do the transformation of coordinates:

$$\begin{cases} x = x_1 + h \\ y = y_1 + k \\ z = z_1 + l \end{cases} \quad (2)$$

At where

$$(h, k, l) = \left(\frac{a_1c_2 - a_2c_1}{a_3c_2}, \frac{c_1}{c_2}, \frac{b_2(a_1c_2 - a_2c_1) - a_3b_1c_2}{a_3b_3c_2} \right).$$

By deriving the transformation (14) and taking into account the system (13), the system is obtained,

$$\begin{cases} \frac{dx_1}{dt} = \frac{a_2c_1 - a_1c_2}{c_2} x_1 - \frac{a_2(a_1c_2 - a_2c_1)}{a_3c_2} y_1 - a_2xy - a_3x^2 \\ \frac{dy_1}{dt} = \frac{b_2c_1}{c_2} x_1 - \frac{b_3c_1}{c_2} z_1 + b_2x_1y_1 - b_3y_1z_1 \\ \frac{dz_1}{dt} = \frac{b_2a_1c_2 - b_2a_2c_1 - a_3b_1c_2}{a_3b_3} y_1 + c_2y_1z_1 \end{cases} \quad (3)$$

The characteristic equation of the system 15 has the form,

$$\lambda^3 + n_1\lambda^2 + n_2\lambda + n_3 = 0 ,$$

At where

$$n_1 = \frac{a_1c_2 - a_2c_1}{c_2} ,$$

$$n_2 = \frac{a_1b_2c_1c_2 - a_2b_2c_1^2 - a_3b_1c_1c_2 + a_1a_2b_2c_1c_2 - a_2^2b_2c_1^2}{a_3c_2^2} ,$$

$$n_3 = \frac{1}{a_3c_2^2} [a_2^2b_2c_1^3 + a_2a_3b_1c_1^2c_2 + a_1^2b_2c_1c_2^2 - 2a_1a_2b_2c_1^2c_2 - a_1a_3b_1c_1c_2^2]$$

For all eigenvalues to have negative real part it is necessary and sufficient that all the smaller diagonals of the following matrix (H) of Hurwitz are positive.

$$H = \begin{bmatrix} n_1 & 1 & 0 \\ n_3 & n_2 & n_1 \\ 0 & 0 & n_3 \end{bmatrix}$$

It follows that,

$$n_1 > 0, n_2 > 0, n_3 > 0 \text{ e } n_1n_2 > n_3.$$

So that all the roots of the characteristic equation of the system (3) have real negative parts it is necessary and sufficient that the following conditions are satisfied:

$$1) \quad a_1c_2 > a_2c_1,$$

- 2) $a_1 b_2 c_1 c_2 + a_1 a_2 b_2 c_1 c_2 > a_2 b_2 c_1^2 + a_3 b_1 c_1 c_2 + a_2^2 b_2 c_1^2$,
- 3) $a_2^2 b_2 c_1^3 + a_2 a_3 b_1 c_1^2 c_2 + a_1^2 b_2 c_1 c_2^2 > 2a_1 a_2 b_2 c_1^2 c_2$,
- 4) $2a_1 a_2 b_2 c_1^2 c_2 + 2a_1 a_2^2 b_2 c_1^2 c_2 + a_1 a_3 b_1 c_1 c_2^2 + 2a_1 a_2 b_2 c_1 c_2^2 + a_1 a_3 b_1 c_1 c_2^3 > 2a_2^2 b_2 c_1^3 + a_2 a_3 b_1 c_1^2 c_2 + a_1^2 b_2 c_1 c_2^2 + a_1^2 a_2 b_2 c_1 c_2^2 + a_2^2 b_2 c_1^3 c_2 + a_2 a_3 b_1 c_1^2 c_2^2 + a_1 b_2 c_1 c_2^3$.

Theorem 1: If conditions (1), (2), (3), and (4) are satisfied, then solutions of system (3) converge to (0,0,0) and therefore the solutions of the system (1) converge to the equilibrium position (h, k, l).

III. Quasi-Normal Form.

Because if the point (x, y, z) in the system (1) is sufficiently close to the origin of the coordinate it would represent that the concentrations of the species is very small, so that the encounters between prey and predators would decrease, this causes that the nonlinear part of the system would represent an infinitesimal of degree superior to the second; so the system could be written in a more general way as the continuation appears,

$$\begin{cases} x' = ax - xf_1(x, y) \\ y' = -by + yf_2(x, y, z) \\ z' = -cz + zf_3(y, z) \end{cases} \quad (4)$$

Nesse caso é evidente que existe a posição de equilíbrio (0,0,0), se além disso, como no sistema (1) existe no primeiro quadrante o ponto (h, k, l), que seja também uma posição de equilíbrio, é dizer, tal que, $f_1(h, k) = a$, $f_2(h, k, l) = b$, $f_2(k, l) = c$, isso é possível, pois se f_1 , f_2 , e f_3 , tem a forma em que aparece no sistema um isso cumpre-se. É evidente que o ponto (0,0,0) é instável, pois existe um valor próprio positivo, então é de interesse analisar o comportamento nesse segundo ponto (h, k, l).

If in the characteristic equation one has to, $n_3 = 0$, but $n_i > 0$, $i = 1, 2$, then the matrix of the linear part of the system has a unique null value and two with a real negative part, that is to say we are in the presence of a critical case, for which it is necessary to simplify the system and apply the Qualitative Theory of Differential Equations.

Here is the case where, $\lambda_1 = 0$ and $\text{Re } \lambda_i < 0$ for $i = 2, 3$. By means of a non-degenerate linear transformation $X = QY$, the system (4) can be reduced to the form,

$$\begin{cases} y_1' = Y_1(y_1, y_2, y_3) \\ y_2' = \lambda_2 y_2 + Y_2(y_1, y_2, y_3) \\ y_3' = \lambda_3 y_3 + Y_3(y_1, y_2, y_3) \end{cases} \quad (5)$$

Theorem 2: The exchange of variables,

$$\begin{cases} y_1 = z_1 + h_1(z_1) + h^0(z_1, z_2, z_3) \\ y_2 = z_2 + h_2(z_1) \\ y_3 = z_3 + h_3(z_1) \end{cases} \quad (6)$$

transforms the system (5) in the following quasi-normal form,

$$\begin{cases} z_1' = Z_1(z_1) \\ z_2' = \lambda_2 z_2 + Z_2(z_1, z_2, z_3) \\ z_3' = \lambda_3 z_3 + Z_3(z_1, z_2, z_3) \end{cases} \quad (7)$$

At where h^0, Z_2, Z_3 annul for $z_2 = z_3 = 0$.

Demonstration: By deriving the transformation (6) along the trajectories of systems (5) and (7) we obtain the system of equations,

$$\begin{cases} (p_2\lambda_2 + p_3\lambda_3)h^0 + Z_1(z_1) = Y_1 - \frac{dh_1}{dz_1}Z_1 - \frac{\partial h^0}{\partial z_1}Z_1 - \frac{\partial h^0}{\partial z_2}Z_2 - \frac{\partial h^0}{\partial z_3}Z_3 \\ \lambda_2 h_2 + Z_2 = Y_2 - \frac{dh_2}{dz_1}Z_1 \\ \lambda_3 h_3 + Z_3 = Y_3 - \frac{dh_3}{dz_1}Z_1 \end{cases} \quad (8)$$

To determine the series that intervene in the systems and the transformation, we will separate the coefficients of the power of degree $p = (p_1, p_2, p_3)$ in the following two cases:

Case I: Making in the system (8) $z_2 = z_3 = 0$, is to say to the vector $p = (p_1, 0, 0)$ results the system,

$$\begin{cases} Z_1(z_1) = Y_1(z_1 + h_1, h_2, h_3) - \frac{dh_1}{dz_1}Z_1 \\ \lambda_2 h_2 = Y_2 - \frac{dh_2}{dz_1}Z_1 \\ \lambda_3 h_3 = Y_3 - \frac{dh_3}{dz_1}Z_1 \end{cases} \quad (9)$$

The system (9) allows to determine the coefficients of the series, Z_1, h_1, h_2 and h_3 , where for being the resonant case $h_1 = 0$, and the remaining series are determined uniquely.

Case II: In the case when $z_2 \neq 0$ and $z_3 \neq 0$ of the system (8) it follows that,

$$\begin{cases} (p_2\lambda_2 + p_3\lambda_3)h^0 = Y_1 - \frac{\partial h^0}{\partial z_1}Z_1 - \frac{\partial h^0}{\partial z_2}Z_2 - \frac{\partial h^0}{\partial z_3}Z_3 \\ Z_2 = Y_2(z_1 + h_1 + h^0, z_2 + h_2, z_3 + h_3) \\ Z_3 = Y_3(z_1 + h_1 + h^0, z_2 + h_2, z_3 + h_3) \end{cases} \quad (10)$$

Because the series of the system (7) are known expressions, the system (10) allows to calculate the series h^0, Z_2 and Z_3 . This proves the existence of the exchange of variables.

In the system (7) the function Z_1 admits the following development in series of powers:

$$Z_1(z_1) = \alpha z_1^s + \dots$$

At where α is the first non-zero coefficient and s is the corresponding power.

Theorem3: If $\alpha < 0$ and s is odd, then the trajectories of the system (7) are asymptotically stable, otherwise they are unstable.

Demonstration: Consider the positive defined Lyapunov function,

$$V(z_1, z_2, z_3) = \frac{1}{2}(z_1^2 + z_2^2 + z_3^2)$$

The function V is such that its derivative along the trajectories of the system (7) has the following expression,

$$V'(z_1, z_2, z_3) = \alpha z_1^2 + \lambda_1 z_2^2 + \lambda_2 z_3^2 + R(z_1, z_2, z_3)$$

This function is defined as negative because in R potencies of degrees higher than those indicated in the initial part of the expression of the derivative of V , this allows us to state that the equilibrium position (h, k, l) is asymptotically stable.

IV. Conclusion

1. Due to the characteristics of the problem considered, it is natural that the critical case analyzed appears.
2. The almost normal form combined allows great difficulties to make a qualitative studio of the trajectories of the system.
3. Theorem one gives conditions that guarantee a balance between the three species participating in the process.
4. Theorem two gives the following methodology to simplify the original system, transforming it into almost normal form, in order to find a better treatment to the studied process.
5. If e is odd, the concentrations of the species will remain in equilibrium converging to the equilibrium position (h, k, l) .
6. If the above conditions are not met, no equilibrium can be guaranteed as to the concentrations of the three participating species, which could lead to excessive growth in one of them and extinction of one of the others.

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