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Mathematical Study of Natural Frequencies of Human Thoracic Aorta

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Abstract:- *To investigate vibration of a simplified human thoracic aorta and its natural frequencies DMV(Donnell MushtariVlsasov) theory is used in present study. The theoretical analysis is presented to calculate natural frequencies and variation in length of thoracic aorta with respect to age of human being.The results show that the first natural frequency of thoracic aorta is near about 1.2 hertz, which corresponds to the natural heart beat rate(1.2 beat per second).*

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I. Introduction

Viscoelastic behavior of arteries is widely investigated by modern researchers because of increasing patients of cardiovascular disease. Many of them established correlations between viscoelastic properties and cardiovascular disease. Furthermore increased arterial stiffness is associated with many risk factors for vascular disease, including blood cholesterol, blood pressure and diabeties[2,7].Lin Wang studied the natural frequencies of arterial system and their relation to the heart rate.[9]The circumferential stress and strain are related to wall material properties.[4,5,11]. Vibration mode of artery has its own characteristic frequency depending on geometry, mass density,elasticity and tethering of arterial system. If the heart rate is near the fundamental natural frequency the system is in a resonance condition that helps in better blood circulation[9]. Recent investigations show that distensibility of an artery segment affects the arterial wall during the cardiac cycle[6].Galina Baltgaile developed some calculations of arterial elastic modulus, stiffness and intimamedia thickness[6]. V. Mohan studied pulsatile blood flow in arterioles[12]. I.S. Mackenzie discussed methods of measuring arterial stiffness in detail[10]. Charles T. Dotter studied that the thoracic aorta increases in length with age[3].

In this paper we investigate the natural frequency of anthoracic aorta segment using DMV theory. Mathematical and simulation results also show that natural frequency of aorta segment is close to heart beat rate (resonance condition). In future this research work can be helpful to develop specific models for better understanding of artery mechanics and cardiovascular system.

Mathematical model :- In the analysis a thoracic aorta is assumed as a simplified elastic circular cylindrical shell to simplify calculations. Therefore the analysis is based on following assumptions.

- The aorta wall is considered as an isotropic, incompressible, and linear elastic material.
- The aorta segment is assumed to be axisymmetric circular cylindrical.

DMV theory is used to calculate natural frequencies[7].The assumptions of DMV theory are as follows.

- The contribution of in-plane displacement u and v to the bending strain parameters is negligible.
- The effect of the shear stress on v is negligible.

We assume the aorta segment is supposed as a cylindrical shell of length “l” and simply supported on itsone edge and other edge is free.

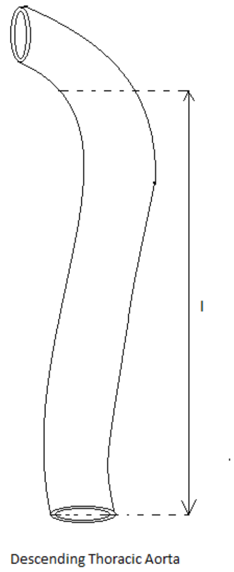
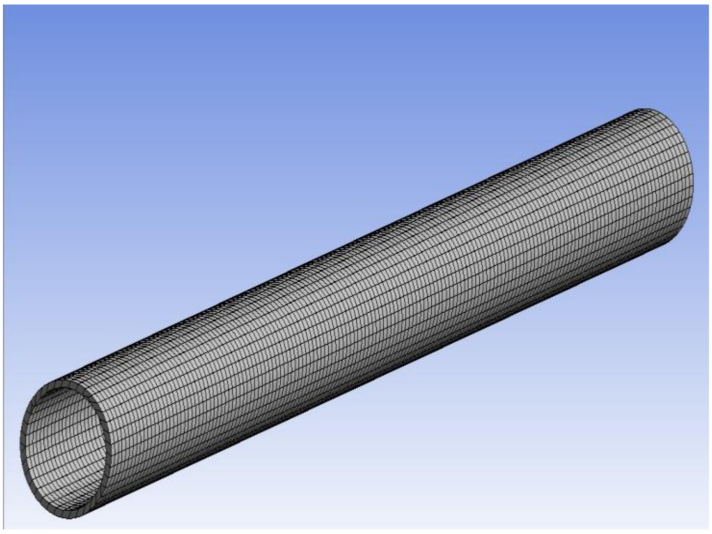


Figure:-1



Geometry of thoracic aorta
Figure-2

Governing equations:-

The equations of motion in cylindrical coordinate system corresponding to DMV theory and above assumptions can be expressed as follows:

$$\frac{\partial^2 u}{\partial x^2}+\frac{1-\nu}{2R^2}\frac{\partial^2 u}{\partial \theta^2}+\frac{\nu}{R}\frac{\partial w}{\partial x}+\frac{1+\nu}{2R}\frac{\partial^2 v}{\partial x\partial \theta}=\frac{(1-\nu^2)\rho}{E}\frac{\partial^2 u}{\partial t^2}----(1)$$

$$\frac{1-\nu}{2}\frac{\partial^2 v}{\partial x^2}+\frac{1}{R^2}\frac{\partial^2 v}{\partial \theta^2}+\frac{1}{R^2}\frac{\partial w}{\partial \theta}+\frac{1+\nu}{2R}\frac{\partial^2 u}{\partial x\partial \theta}=\frac{(1-\nu^2)\rho}{E}\frac{\partial^2 v}{\partial t^2}--(2)$$

$$-\left(\frac{\nu}{R}\frac{\partial u}{\partial x}+\frac{1}{R^2}\frac{\partial v}{\partial \theta}+\frac{w}{R^2}\right)-\frac{h^2}{12}\left(\frac{\partial^4 w}{\partial x^4}+\frac{2}{R^2}\frac{\partial^4 w}{\partial x^2\partial \theta^2}+\frac{1}{R^4}\frac{\partial^4 w}{\partial \theta^4}\right)=\frac{(1-\nu^2)\rho}{E}\frac{\partial^2 w}{\partial t^2}----(3)$$

Where
X = axial coordinate
θ = angular coordinate
t = time
u =u(x,θ) axial displacement
v= v(x,θ) tangential displacement
w = w(x,θ) radial displacement
E = Young's modulus(psi)
ρ = mass/volume
ν= Poisson ratio
h = wall thickness(inches)
R = radius(inches)
l = length of the descending thoracic aorta

ω= natural frequency(hertz)
m = axial mode number (the number of half waves of displacement in the length of artery.
n = circumferential mode number.(the number of half waves of displacement in the circumference of artery.
p = Age of human being (years)

Boundary conditions are:-

v(0,θ,t)=0 ------(4)
w(0,θ,t)=0 -----(5)

force and moment resultants

$$N_{xx} = C\left(\frac{\partial u}{\partial x} + \frac{v}{R}\frac{\partial v}{\partial \theta} + \frac{v}{R}w\right)(0,\theta,\mathrm{t}) = 0 \;-----(6)$$

Where C is middle surface stiffness of the artery given by

$C = \dfrac{Eh}{1-\nu^2}$

$$M_{xx}(0,\theta,t) = D\left(-\frac{\partial^2 w}{\partial x^2} + \frac{v}{R^2}\frac{\partial v}{\partial \theta} - \frac{v}{R^2}\frac{\partial^2 w}{\partial \theta^2}\right)(0,\theta,\mathrm{t}) = 0 --- (7)$$

Where D the bending stiffness of the artery is given by

$$D = \frac{Eh^3}{12(1-\nu^2)}$$

v(l,θ,t)=0 -----------(8)
w(l,θ,t)=0 ------(9)

$$N_{xx} = C\left(\frac{\partial u}{\partial x} + \frac{v}{R}\frac{\partial v}{\partial \theta} + \frac{v}{R}w\right)(\mathrm{l},\theta,\mathrm{t}) = 0 ----- (10)$$

$$M_{xx}(0,\theta,t) = D\left(-\frac{\partial^2 w}{\partial x^2} + \frac{v}{R^2}\frac{\partial v}{\partial \theta} - \frac{v}{R^2}\frac{\partial^2 w}{\partial \theta^2}\right)(\mathrm{l},\theta,\mathrm{t}) = 0 \;----(11)$$

Where $N_{xx}\, and M_{xx}\, are$force and moment resultants respectively.

Analytical solution:-

The solution of the equations (1)-(3) is assumed to be in following form

$$u(x,\theta) = \sum_m\sum_n U_{mn}\, cos\frac{m\pi x}{l} cosn\theta cos\omega t ----- (12)$$

$$v(x,\theta) = \sum_m\sum_n V_{mn}\, sin\frac{m\pi x}{l} sinn\theta cos\omega t ----- (13)$$

$$w(x,\theta) = \sum_m\sum_n W_{mn}\, sin\frac{m\pi x}{l} cosn\theta cos\omega t ----- (14)$$

Where U_{mn}, V_{mn}, W_{mn} are constant and the boundary conditions are followed by assumed solutions.
Using eq (12) ,(13) and (14) in eqs. (1), (2),(3) we have

$$(-\eta^2 - a_1 n^2 + \Gamma)\mathrm{U_{mn}} + (a_2\eta n)V_{mn} + (\nu\eta)W_{mn} = 0 \;--- (14)$$

$$(a_2\eta n)U_{mn} + (-a_1\eta^2 - n^2 + \Gamma)\mathrm{V_{mn}} + (-\mathrm{n})\mathrm{W_{mn}} = 0 --- (15)$$

$$(\nu\eta)U_{mn} + (-n)V_{mn} + (-1 - \eta^4 - 2\eta^2 n^2\gamma - n^2\gamma + \Gamma)W_{mn} = 0 --(16)$$

Where $\eta = \frac{m\pi R}{l}$, $a_1 = \frac{1-\upsilon}{2}, a_2 = \frac{1+\upsilon}{2}$, $\Gamma = \frac{(1-\nu^2)R^2\rho}{E}\omega, \gamma = \frac{h^2}{12R^2}$

Equations (14) – (16) can be assembled into matrix form as follows

$$\begin{bmatrix} (-\eta^2 - a_1 n^2 + \Gamma) & (a_2\eta n) & (\nu\eta) \\ (a_2\eta n) & (-a_1\eta^2 - n^2 + \Gamma) & (-\mathrm{n}) \\ (\nu\eta) & (-n) & (-1 - \eta^4 - 2\eta^2 n^2\gamma - n^2\gamma + \Gamma) \end{bmatrix} \begin{bmatrix} U_{mn} \\ V_{mn} \\ W_{mn} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

For a nontrivial solution of U_{mn}, V_{mn}, W_{mn} the determinant of the coefficient matrix in eq. (14)-(16) must be zero.
And hence

$$\begin{vmatrix} (-\eta^2 - a_1 n^2 + \Gamma) & (a_2 \eta n) & (\nu\eta) \\ (a_2 \eta n) & (-a_1 \eta^2 - n^2 + \Gamma) & (-\mathrm{n}) \\ (\nu\eta) & (-n) & (-1 - \eta^4 - 2\eta^2 n^2 \gamma - n^2 \gamma + \Gamma) \end{vmatrix} = 0$$

$$\Gamma^3 + L_1 \Gamma^2 + L_2 \Gamma + L_3 = 0 ---------------(17)$$

Where

$$L_1 = -\eta^2(1 + a_1 + 2n^2\gamma) - n^2(1 + a_1) - \eta^4\gamma - n^4\gamma - 1 \quad ----(18)$$

$$L_2 = -\eta^6\gamma(1 + a_1) + \eta^4(a_1 + 3n^2\gamma + 3a_1 n^2\gamma) + \eta^2(1 + n^2 + a_1^2 n^2 - a_2^2 n^2 - \nu^2 + 3a_1 n^4\gamma + 3n^4\gamma) + n^6\gamma(1 + a_1) + a_1 n^4 + n^2 a_1 \quad ------------------(19)$$

$$L_3 = -\eta^8 a_1\gamma - \eta^6 n^2\gamma(1 + 2a_1 - a_2^2 + a_1^2) - \eta^4(a_1 + 2a_1 n^4\gamma + 2a_1^2 n^4\gamma + 2n^4\gamma - 2a_2^2 n^4\gamma - a_2\nu^4 - \eta^2[n^6\gamma(1 + a_1^2 + 2a_1 - a_2^2) + n^2(-a_2^2 + a_1^2 + 2a_2\nu - \nu^2)] \; - n^8 a_1\gamma \quad ------------------------------(20)$$

Natural frequencies of transverse vibrations for small values of Γ for different combinations of n and m are obtained by neglecting square and cubic terms of in eq (17).

$$\Gamma \approx -\frac{L_3}{L_2} ----------------------(18)$$

Using value of Γ in eq(18) in we have

$$\omega^2 \approx -\frac{EL_3}{(1-\nu^2)R^2\rho L_2} ----- (19)$$

Length(cm) of thoracic aorta is calculated by

$$l = 6.76 + 15.86 log_{10}(p) ------------- (20)$$

Numerical Calculation:-

Numerical calculations are done for a healthy thoracic aorta of a human being. As length of aorta increases with age and natural frequencies depend on geometry of aorta. On the basis of past investigations the inner radius is 0.43(inches), thickness is 0.06(inches), Young's modulus is 156 psi and mass density is 0.036(lb per inch^3), m is 1 and n is 1.

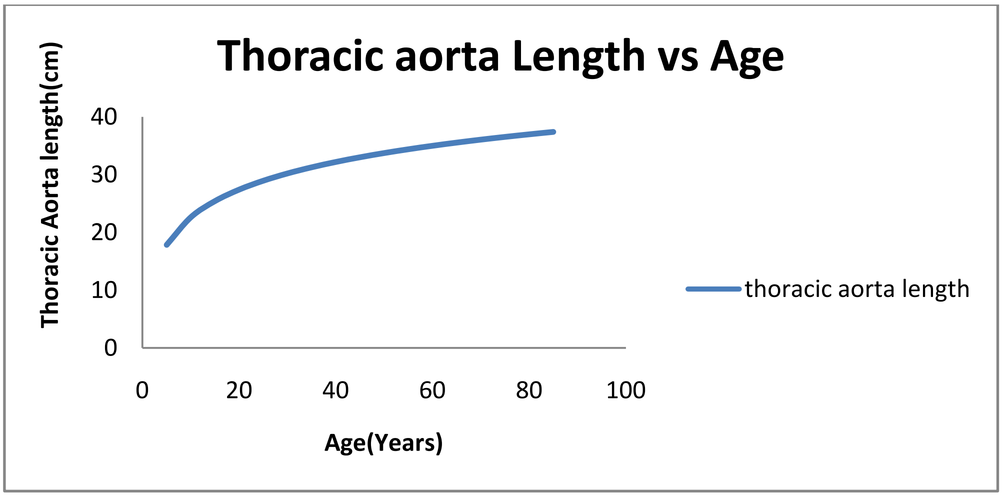


Figure:-3

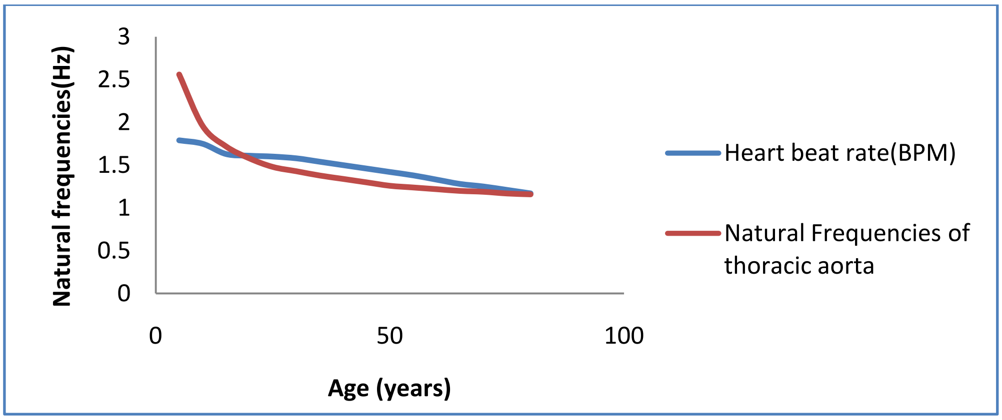


Figure :-4

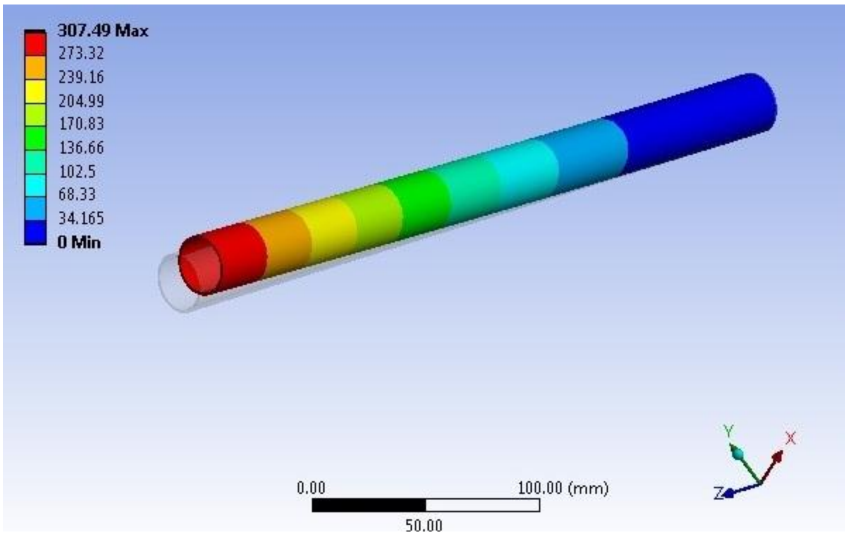


Figure-5 (For m=1, n=1)

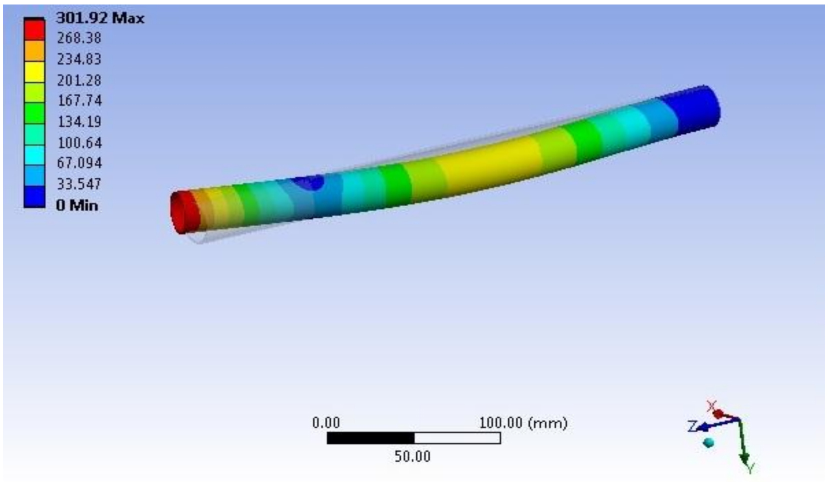


Figure-6 (For m=2, n=1)

m	n	Natural frequency(calculated)	Natural frequency(simulated)
1	1	1.2613	1.3474
2	1	5.8213	7.9942

Table-1

II. Results

From figure-3 we observe that thoracic aorta length increases with age with high rate till 35 year and after 35 year rate of increment in length is comparatively slow. From figure -2 it can be conclude that the first natural frequencies (for m=1 and n=1) of thoracic wall is near to the natural heart beat rate. Which theoretically supports a better blood circulation in human beings[3].Table -1 shows the comparison of natural frequencies of analytical solution and simulation results obtained by ANSYS R18.0. Strong correlations exist between arterial elastic properties and cardiac disorders. This study will help in better understanding of cardiovascular mechanism and human blood circulation system.

References

[1]. Anooshiravan F., Pouria O. Free Vibration Analysis of Circular Cylindrical Shells: Comparison of Different Shell Theories. International Journal of Mechanics and Applications ,2012; 2(5): 74-80.

[2]. Arnett DK, Evans GW, Riley WA. Arterial stiffness: a new cardiovascular risk factor? Am J Epidemiol 1994; 140: 669–682.

[3]. Charles T. Dotter, Aortic Length: Angiocardiographic Measurements, American Heart Association, 1950;2:915-920.

[4]. Choung C J, Fung Y C on residual stress in arteries, Journal of Biomechanical engineering, 1986,108:189-192.

[5]. Delfino A, StergiopulosN,Moore J E, Meister JJ. Residual strain effects on the stress field in a thick wall finite element model of the human carotid bifurcation. Journal of Biomechanics, 1997, 30(8):777-786.

[6]. Galina Baltgaile, Arterial wall dynamics. Perspectives in Medicine (2012) 1, 146—151.

[7]. Gamble G, Zorn J, Sanders G, MacMahon S, Sharpe N Estimation of arterial stiffness, compliance, and distensibility from M-mode ultrasound measurements of the common carotid artery. Stroke 1994; 25: 11–16.

[8]. HuiA N, Fan HE, Lingxia Xing, Xiaoyang LI, Oscillation frequency of simplified arterial tubes. Front Mech. Eng. China 2009,4(3): 300-304.

[9]. Lin Wang YY1, Jan MY, Shyu CS, Chiang CA, Wang WK.The natural frequencies of the arterial system and their relation to the heart rate. IEEE Trans Biomed Eng. 2004 Jan;51(1):193-205.

[10]. S. I. Mackenzie, Assessment of arterial stiffness in clinical practice, Q J Med 2002; 95:67–74.

[11]. Takamizawa K, Haysahi K. Uniform strain hypothesis and thinwalled theory in arterial mechanics. Biorheology,1988,25(3):555-565.

[12]. V. Mohan, V. Prasad, J singh , Blood flow in artererioles, International Journal of Mathematical archive-3(1) 2012: 1-5.

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