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On Non-invariant Hypersurfaces of a Nearly Kenmotsu Manifold

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Abstract: *The object of this paper is to study non-invariant hypersurfaces of a nearly Kenmotsu manifold equipped with (f, g, u, v, λ)- structure. Some properties obeyed by this structure are obtained. The necessary and sufficient conditions also have been obtained for totally umbilical non-invariant hypersurfaces with (f, g, u, v, λ)- structure of nearly Kenmotsu manifold to be totally geodesic. The second fundamental form of a non-invariant hypersurfaces of a nearly Kenmotsu manifold with (f, g, u, v, λ)- structure has been traced under the condition when f is parallel.*

Keywords and Phrases: *Nearly Kenmotsu Manifold, Totally Geodesic, Totally Umbilical.*
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I. Introduction

In 1970, *S.I.Goldberg et. al*[1]introduced the notion of a non-invariant hypersurfaces of an almost contact manifold in which the transform of a tangent vector of the hypersurface by the (1,1) structure tensor field *f* defining the almost contact structure is never tangent to the hypersurface. The notion of (*f, g, u, v, λ*) - structure was given by *K.Yano*[2]. It is well *known*[3,4]that hypersurfaces of an almost contact metric manifold always admits a (*f,g,u,v,λ*) - structure. *Authors*[1] proved that there always exists a (*f, g, u, v, λ*) - structure on a non-invariant hypersurface of an almost contact metric manifold. They also proved that there does not exist invariant hypersurface of a contact manifold. *R. Prasad* [5,7]studied the non-invariant hypersurfaces of trans-Sasakian manifolds and nearly Sasakian manifolds. In the present paper, we study the non-invariant hypersurfaces of nearly Kenmotsu manifolds.

The paper is organized as follows. In section 2, we give a brief description of nearly Kenmotsu manifold. In section 3, introduce the non-invariant hypersurfaces and induced (*f, g, u, v, λ*)- structure on non-invariant hypersurface *M* getting some equation. Some results of non-invariant hypersurfaces with (*f,g,u,v,λ*)-structure of nearly Kenmotsu manifold. The necessary and sufficient conditions also have been obtained for totally umbilical non-invariant hypersurfaces with (*f,g,u,v,λ*)- structure of nearly Kenmotsu manifold to be totally geodesic. The second fundamental form of a non-invariant hypersurfaces of a nearly Kenmotsu manifold with (*f, g, u, v, λ*)-structure has been traced under the condition when *f* is parallel.

II. Preliminaries

Let $\bar{M}$ be an $(2n+1)$-dimensional almost contact metric manifold with a metric$g$, tensor field$\emptyset$of type$(1,1)$, a vector fieldξ, a dual 1-form η which are satisfying the following

$$\emptyset^2X = -X + \eta(X)\xi,\ \eta(\xi) = 1,\ \emptyset(\xi) = 0,\ \eta o\emptyset = 0 \quad (2.1)$$

$$g(\emptyset X, \emptyset Y) = g(X,Y) - \eta(X)\eta(Y) \quad (2.2)$$

for any X, Y tangent to $\bar{M}$.

If addition to the above condition we have

$$g(\emptyset X, Y) = -g(\emptyset Y, X), \quad g(X,\xi) = \eta(X) \quad (2.3)$$

The structure is said to be contact metric structure.

An almost contact metric Manifold $\bar{M}$ is called nearly Kenmotsu manifold if it satisfy the condition [6]

$$(\bar{\nabla}_X\emptyset)Y + (\bar{\nabla}_Y\emptyset)X = -\eta(Y)\emptyset X - \eta(X)\emptyset Y \quad (2.4)$$

Where$\bar{\nabla}$ denote the Riemannian connection with respect to g, if moreover M sathisfies

$$(\bar{\nabla}_X\emptyset)Y = g(\emptyset X, Y)\xi - \eta(Y)\emptyset X$$

Then it is called Kenmotsu manifold [6].

Obliviously, a Kenmotsu manifold is also a nearly Kenmotsu manifold.

From (2.4), we have

$$\bar{\nabla}_X\xi = X - \eta(X)\xi - \emptyset((\bar{\nabla}_\xi\emptyset)X). \quad (2.5)$$

A hypersurface of an almost contact metric manifold $\bar{M}$ on $(\emptyset, \xi, \eta, g)$ is calleda non-invariant hypersurfaces, if the transform of a tangent vector of thehypersurface under the action of (1,1) tensor field $\emptyset$ defining the contactstructure is never tangent to the hypersurface.

Let Xbe a tangent vector ona non-invariant hypersurfaces of an almost contact metric manifold $\bar{M}$, then$\emptyset X$is never tangent to the hypersurface.

Let M be a non-invariant hypersurfaces of an almost contact metric manifold. Now, if we define the following,

$\emptyset X = fX + u(X)\bar{N}$ (2.6)

$\emptyset \bar{N} = -U$ (2.7)

$\xi = V + \lambda\bar{N} \quad , \lambda = \eta(\bar{N})$ (2.8)

$\eta(X) = v(X)$ (2.9)

where f is (1,1) tensor field, u & v are 1-form, $\bar{N}$ is a unit normal to the hypersurfaces , $X \in TM$ & $u(X) \neq 0$, then we get an induced (f, g, u, v, λ) - structure on M satisfying the condition.[2,3]

Making use of (2.6), (2.7), (2.8) & (2.9) through equations (2.1) to (2.4), we have

$f^2 = -I + u \otimes \mathrm{U} - v \otimes V$ (2.10)

$uof = \lambda v, \qquad vof = -\lambda u$ (2.11)

$v(V) = 1 - \lambda^2,\ u(V) = v(U) = 0 \ \ , u(U) = 1 - \lambda^2$ (2.12)

$fV = \lambda U, fU = -\lambda V$ (2.13)

$g(X, U) = u(X),\ g(X, V) = v(X)$ (2.14)

$g(fX, fY) = g(X, Y) - u(X)u(Y) - v(X)v(Y)$ (2.15)

$g(fX, Y) = -g(fY, X)$ (2.16)

for all $X, Y \in TM$ & $\lambda = \eta(\bar{N})$.

The Gauss & Weingarten formula are given by

$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y)\bar{N}$ (2.17)

$\bar{\nabla}_X \bar{N} = -A_{\bar{N}} X(2.18)$ (2.18)

for all $X, Y \in TM$, where ∇ &$\bar{\nabla}$ are the Riemannian & induced connection on $\bar{M}$&M respectively &$\bar{N}$ is the unit normal vector in the normal bundle $T^{\perp}M$.

In this formula h is the second fundamental form on M related to $A_{\bar{N}}$ by

$h(X, Y) = g(A_{\bar{N}} X, Y)$ (2.19)

for all $X, Y \in T$

III. Non-invariant Hypersurfaces

Lemma 3.1. If M be a non-invariant hypersurfaces with (f, g, u, v, λ) – structure of a nearly Kenmotsu manifold$\bar{M}$, then

$(\bar{\nabla}_X \eta)Y + (\bar{\nabla}_Y \eta)X = (\nabla_X v)Y + (\nabla_Y v)X - 2\lambda h(X, Y)$ (3.1)

$\bar{\nabla}_X \xi = \nabla_X V - \lambda A_{\bar{N}} X + (h(X, \xi) + X\lambda)\bar{N}$ (3.2)

for all $X, Y \in TM$.

Proof. After computations similar to lemma 3.1 in [7], lemma follows.

Theorem 3.2. If M be a non-invariant hypersurface with (f, g, u, v, λ) – structure of a nearly Kenmotsu manifold$\bar{M}$, then

$(\nabla_X f)Y + (\nabla_Y f)X = u(X)A_{\bar{N}} Y + u(Y)A_{\bar{N}} X - 2h(X, Y)U - v(X)fY - v(Y)fX$ (3.3)

$(\nabla_X u)Y + (\nabla_Y u)X = -u(X)v(Y) - u(Y)v(X) - h(X, fY) - h(Y, fX)$ (3.4)

for all $X, Y \in TM$.

Proof. By covariant differentiation, we know that

$(\bar{\nabla}_X \emptyset)Y = \bar{\nabla}_X \emptyset Y - \emptyset(\bar{\nabla}_X Y)$

Using equation (2.6), (2.7) and (2.17) in above equation, we have

$(\bar{\nabla}_X \emptyset)Y = \bar{\nabla}_X (fY + u(Y)\bar{N}) - \emptyset(\nabla_X Y + h(X, Y)\bar{N})$

$(\bar{\nabla}_X \emptyset)Y = (\nabla_X f)Y - u(Y)A_{\bar{N}} X + h(X, Y)U + \big((\nabla_X u)Y\big)\bar{N} + h(X, fY)\bar{N}$ (3.5)

Similarly, we have

$(\bar{\nabla}_Y \emptyset)X = (\nabla_Y f)X - u(X)A_{\bar{N}} Y + h(X, Y)U + \big((\nabla_Y u)X\big)\bar{N} + h(Y, fX)\bar{N}$ (3.6)

Adding equation (3.5) & (3.6), we have

$$(\bar{\nabla}_X \emptyset)Y + (\bar{\nabla}_Y \emptyset)X = (\nabla_X f)Y + (\nabla_Y f)X + 2h(X, Y)U - u(Y)A_{\bar{N}} X - u(X)A_{\bar{N}} Y \\ +\big((\nabla_X u)Y\big) + \big((\nabla_Y u)X\big) + h(X, fY) + h(Y, fX))\bar{N} \quad (3.7)$$

Now using equation (2.6) and (2.9) in (2.4), we have

$(\bar{\nabla}_X \emptyset)Y + (\bar{\nabla}_Y \emptyset)X = -v(X)(fY) - v(X)u(Y)\bar{N} - v(Y)(fX) - v(Y)u(X)\bar{N})$ (3.8)

From equation (3.7) and (3.8), comparing tangential and normal part, we have the desired results.

Theorem 3.3. If M be a non-invariant hypersurfaces with (f,g,u,v,λ) – structure of a nearly Kenmotsu manifold $\bar{M}$, then

$$h(X,\xi)U = f(\nabla_X \xi) - fX + f^2((\bar{\nabla}_\xi \emptyset)X) - u((\bar{\nabla}_\xi \emptyset)X)U \tag{3.9}$$

$$u(\nabla_X \xi) = u(X) - u(f((\bar{\nabla}_\xi \emptyset)X)) \tag{3.10}$$

for all $X, Y \in TM$.

Proof. Let us consider

$$(\bar{\nabla}_X \emptyset)\xi = \bar{\nabla}_X \emptyset\xi - \emptyset(\bar{\nabla}_X \xi) \tag{3.11}$$

Using equation (2.1), (2.5) & (2.6), we have

$$(\bar{\nabla}_X \emptyset)\xi = -\emptyset(X - \eta(X)\xi - \emptyset((\bar{\nabla}_\xi \emptyset)X))$$

$$(\bar{\nabla}_X \emptyset)\xi = -fX + f^2((\bar{\nabla}_\xi \emptyset)X) - u((\bar{\nabla}_\xi \emptyset)X)U - u(X)\bar{N} + u(f((\bar{\nabla}_\xi \emptyset)X))\bar{N} \tag{3.12}$$

Since we know that

$$(\bar{\nabla}_X \emptyset)\xi = \bar{\nabla}_X \emptyset\xi - \emptyset(\bar{\nabla}_X \xi)$$

Using (2.1),(2.6)& (2.17), we have

$$(\bar{\nabla}_X \emptyset)\xi = -f(\nabla_X \xi) - u(\nabla_X \xi)\bar{N} + h(X,\xi)U \tag{3.13}$$

Using (3.12), (3.13) and comparing tangential & normal part, we get the desired results.

Theorem 3.4. If M be a non-invariant hypersurfaces with (f,g,u,v,λ) – structure of a nearly Kenmotsu manifold $\bar{M}$, then

$$(\bar{\nabla}_X \emptyset)Y + (\bar{\nabla}_Y \emptyset)X = -v(X)fY - v(Y)fX - (u(X)v(Y) + u(Y)v(X))\bar{N} \tag{3.14}$$

for all $X, Y \in TM$.

Proof. Using (3.3), (3.4) & (3.7), we have

$$(\bar{\nabla}_X \emptyset)Y + (\bar{\nabla}_Y \emptyset)X = u(X)A_{\bar{N}}Y + u(Y)A_{\bar{N}}X - u(X)A_{\bar{N}}Y + 2h(X,Y)U - 2h(X,Y)U - u(Y)A_{\bar{N}}X$$
$$-u(X)A_{\bar{N}}Y + 2h(X,Y)U + (-u(X)v(Y) - u(Y)v(X) - h(X,fY) - h(Y,fX) + h(X,fY) + h(Y,fX))\bar{N}$$

$$(\bar{\nabla}_X \emptyset)Y + (\bar{\nabla}_Y \emptyset)X = -v(X)fY - v(Y)fX - \big(u(X)v(Y) + u(Y)v(X)\big)\bar{N}.$$

for all $X, Y \in TM$.

Theorem 3.5. If M be a totally umbilical non-invariant hypersurfaces with (f,g,u,v,λ) – structure of a nearly Kenmotsu manifold $\bar{M}$, then it is totally geodesic if and only if

$$\lambda v(X) + u((\bar{\nabla}_\xi \emptyset)X) + X\lambda = 0 \tag{3.15}$$

and in particular, if nearly Kenmotsu manifold admits a contact structure then (3.15) can be expressed as

$$v + d(\log \lambda) = 0 \tag{3.16}$$

for all $X, Y \in TM$.

Proof. By Gauss formula, we have

$$\bar{\nabla}_X \xi = \nabla_X \xi + h(X,\xi)\bar{N}$$

Using equation (2.8), we have

$$\bar{\nabla}_X \xi = \nabla_X V - \lambda A_{\bar{N}}X + (h(X,V) + X\lambda)\bar{N} \tag{3.17}$$

Using (2.6), (2.8) & (2.9) in (2.5), we have

$$\bar{\nabla}_X \xi = X - v(X)V - \lambda v(X)\bar{N} - f((\bar{\nabla}_\xi \emptyset)X) - u((\bar{\nabla}_\xi \emptyset)X)\bar{N} \tag{3.18}$$

From (3.17) & (3.18), we have

$$\nabla_X V - \lambda A_{\bar{N}}X + (h(X,V) + X\lambda)\bar{N} = X - v(X)V - \lambda v(X)\bar{N} - f((\bar{\nabla}_\xi \emptyset)X) - u((\bar{\nabla}_\xi \emptyset)X)\bar{N}$$

Equating normal part, we have

$$h(X,V) = -\lambda v(X) - u((\bar{\nabla}_\xi \emptyset)X) - X\lambda \tag{3.19}$$

Now if M is totally umbilical then$A_{\bar{N}} = \zeta I$, where ζ is Kahlerian metric and (2.19) reduce into

$$h(X,Y) = g(A_{\bar{N}}X,Y) = g(\zeta X,Y) = \zeta g(X,Y)$$

therefore, $h(X,Y) = \zeta g(X,Y) = \zeta g(X,\xi) = \zeta\eta(X)$

$$\Longrightarrow h(X,\xi) = \zeta v(X)$$

So (3.19) reduce as

$$\lambda v(X) + u((\bar{\nabla}_\xi \emptyset)X) + X\lambda + \zeta v(X) = 0$$

if$M$ is totally umbilical i.e. $\zeta = 0$, then above becomes

$$\lambda v(X) + u((\bar{\nabla}_\xi \emptyset)X) + X\lambda = 0$$

Now if nearly Kenmotsu manifold is equipped with contact structure, then above can be written as

$$\lambda v(X) + X\lambda = 0$$

$$\Longrightarrow v + d(\log \lambda) = 0.$$

for all $X, Y \in TM$.

Theorem 3.6. If M be a non-invariant hypersurfaces with (f,g,u,v,λ) – structure of a nearly Kenmotsu manifold $\bar{M}$, if U is parallel then we have

$$\lambda fX + f(A_{\bar{N}}X) + (\nabla_{\bar{N}}f)X - v(X)U = 0 \quad (3.20)$$

for all $X, Y \in TM$.

Proof. Consider

$$(\bar{\nabla}_X\emptyset)\bar{N} = \bar{\nabla}_X\emptyset\bar{N} - \emptyset(\bar{\nabla}_X\bar{N})$$

$$(\bar{\nabla}_X\emptyset)\bar{N} = -\nabla_X U + f(A_{\bar{N}}X) \quad (3.21)$$

Now replacing Y by $\bar{N}$ in (2.4), we have

$$(\bar{\nabla}_X\emptyset)\bar{N} + (\bar{\nabla}_{\bar{N}}\emptyset)X = -\eta(X)\emptyset\bar{N} - \eta(\bar{N})\emptyset X$$

Using equation (2.6), (2.7), (2.8) & (2.9) in above, we have

$$(\bar{\nabla}_X\emptyset)\bar{N} + (\bar{\nabla}_{\bar{N}}\emptyset)X = v(X)U - \lambda fX - \lambda u(X)\bar{N}) \quad (3.22)$$

Also we have

$$(\bar{\nabla}_{\bar{N}}\emptyset)X = (\nabla_{\bar{N}}f)X - u(X)A_{\bar{N}}\bar{N} + \{(\nabla_{\bar{N}}u)X + h(\bar{N}, fX)\}\bar{N} \quad (3.23)$$

Using equation (3.21) & (3.23) in (3.22), we have

$$-\nabla_X U + f(A_{\bar{N}}X) + (\nabla_{\bar{N}}f)X - u(X)A_{\bar{N}}\bar{N} + \{(\nabla_{\bar{N}}u)X + h(\bar{N}, fX)\}\bar{N} = v(X)U - \lambda fX - \lambda u(X)\bar{N})$$

Equating tangential part, we have

$$\nabla_X U = \lambda fX + f(A_{\bar{N}}X) + (\nabla_{\bar{N}}f)X - v(X)U$$

If U is parallel then we have

$$\lambda fX + f(A_{\bar{N}}X) + (\nabla_{\bar{N}}f)X - v(X)U = 0.$$

for all $X, Y \in TM$.

Theorem 3.7. If M be a non-invariant hypersurfaces with (f,g,u,v,λ) – structure of a nearly Kenmotsu manifold $\bar{M}$, if f is parallel then we have

$$h(X,Y) = \frac{\mu}{1-\lambda^2}u(X)u(Y) - \frac{1}{2(1+\lambda^2)}(v(X)u(fY) + v(Y)u(fX)) \quad (3.24)$$

Where$\mu = h(U,U) = g(A_{\bar{N}}U, U)$,Also$M$ is totally geodesic if and only if

$$\frac{1}{2}\lambda v(X) + u((\bar{\nabla}_\xi\emptyset)X) + X\lambda - \frac{1}{2}u(fX) = 0 \quad (3.25)$$

for all $X, Y \in TM$.

Proof: As f is parallel then from equation (3.5), we have

$$2h(X,Y)U = u(X)A_{\bar{N}}Y + u(Y)A_{\bar{N}}X - v(X)fY - v(Y)fX$$

Appling u both sides & using (2.12), we have

$$2(1-\lambda^2)h(X,Y) = u(X)u(A_{\bar{N}}Y) + u(Y)u(A_{\bar{N}}X) - v(X)u(fY) - v(Y)u(fX) \quad (3.26)$$

Replacing $Y\ by\ U$ both sides & using (2.12), (2.13) in (3.26), we have

$$2(1-\lambda^2)h(X,U) = u(X)u(A_{\bar{N}}U) + (1-\lambda^2)u(A_{\bar{N}}X) \quad (3.27)$$

As$h(X,Y) = g(A_{\bar{N}}X, Y)$

$$h(X,U) = g(A_{\bar{N}}X, U) = u(A_{\bar{N}}X)$$

So, equations (3.27)reduces as

$$u(A_{\bar{N}}X) = \frac{\mu}{1-\lambda^2}u(X) \quad (3.28)$$

Similarly, we have

$$u(A_{\bar{N}}Y) = \frac{\mu}{1-\lambda^2}u(Y) \quad (3.29)$$

Where$\mu = h(U,U) = u(A_{\bar{N}}U)$

Using (3.28), (3.29) in (3.26), we have

$$h(X,Y) = \frac{\mu}{(1-\lambda^2)^2}u(X)u(Y) - \frac{1}{2(1-\lambda^2)}(v(X)u(fY) + v(Y)u(fX)) \quad (3.30)$$

Now putting $Y = V$ and using (2.12), (2.13) in (3.30), we have

$$h(X,V) = -\frac{1}{2}\lambda v(X) - \frac{1}{2}u(fX) \quad (3.31)$$

Now from (3.30) and (3.31), we have

$$\frac{1}{2}\lambda v(X) + u((\bar{\nabla}_\xi\emptyset)X) + X\lambda - \frac{1}{2}u(fX) = 0.$$

for all $X, Y \in TM$.

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