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Study of Order Statistics of New Distribution

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Abstract: *Here the order statistics of new distribution [1] with the recurrence relation for single and product moments have been studied. Some statistical properties have been studied with Residual entropy and Past residual entropy.*

Keywords: *Order statistics, Single Moments, Joint Moments, Residual entropy, Past Residual entropy.*

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I. Introduction

New distribution [1] has the *pdf:*

$$f(x) = (1-p)\, x^{-p}\, \theta^{p-1} \qquad (1)$$

Here p is the shape parameter and θ is the scale parameter.
where, $0 < x < \theta;\ 0 < p < 1$

For lesser value of parameter *p* proposed distribution is a growth function and for higher value of *p* it is treated as a decay function.Therefore, it can be used for profit and loss both in financial data.

The *Cdf*of New distribution is defined as:

$$F(x) = \theta^{p-1} x^{1-p} \qquad (2)$$

where, $0 < x < \theta;\ 0 < p < 1$

Reliability function of the new distribution is

$$R(x) = 1 - F(x) = 1 - \theta^{p-1}\, x^{1-p} \qquad (3)$$

Where θ and *p* are scale and shape parameters respectively. At t = 0;eqn 3gives R(0) = 1 and as time t increases reliability decreases.

II. Distribution of Order Statistics

If $X_1, X_2 ...... X_n$ are n iid observations of size n,then the ordered arrangements of above size will be is $X_{1:n}, X_{2:n} ...... X_{n:n}$.
New power distribution has the probability distribution function of order statistics by[2]and [3] is,

$$f_{r:n}(x) = (1-p)C_{r:n}\Sigma_{i=0}^{n-r}\begin{bmatrix} n-r \\ i \end{bmatrix}(-1)^i\theta^{(r+i)(p-1)}x^{(r+i)(1-p)-1} \quad (4)$$

for $r = 1$ and $r = n$ order statistics for the New distribution are:

$$f_{i:n}(x) = \frac{n(1-p)(x^{p-1}\theta^{1-p}-1)^{n-1}}{\theta^{n(1-p)}x^{(n(p-1)-1)}} \quad (5)$$

$$f_{n:n}(x) = n(1-p)\,\theta^{n(p-1)}\,x^{n(1-p)-1} \quad (6)$$

Similarly if $1 \leq r < s \leq n$ $X_{r:n}$ is rth and $X_{s:n}$ is the sth order statistics. Hence the joint *pdf* of $X_{r:n}$ and $X_{s:n}$ by [2] is

$$f_{r:s:n} = (1-p)^2C_{r:s:n}\Sigma_{i=0}^{s-r-1}\Sigma_{i=0}^{n-s}(-1)^{i+j}[\frac{x^{(r+i)(1-p)-1}y^{(s-r-i+j)(1-p)-1}}{\theta^{(s+i)(1-p)}}] \quad (7)$$

shows the joint *pdf*.

For any $k = 1, 2...$ kth moment of order statistics is

$$\mu_{r:n}^k = \theta^k C_{r:n}B\left(\frac{k}{1-p}+r, n+1-r\right) \quad (8)$$

where n is the size of the sample.
Above equation depicts the single moments of order statistics of new distribution.
For$k = 1$

$$\mu_{r:n}^{(1)} = \theta C_{r:n}B\left(\frac{1}{1-p}+1; n+1-r\right) \quad (9)$$

For $k = 2$

$$\mu_{r:n}^{(1)} = \theta^2 C_{r:n}B\left(\frac{2}{1-p}+r; n+1-r\right) \quad (10)$$

$$V(X_{r:n}) = \mu_{r:n}^{(2)} - \left(\mu_{r:n}^{(1)}\right)^2 \quad (11)$$

$$= \theta^2 C_{r:n}\left[B\left(\frac{2}{1-p}+r; n+1-r\right) - C_{r:n}B^2\left(\frac{1}{1-p}+r; n+1-r\right)\right] \quad (12)$$

Here (9)and (12) are mean and the variance of order statistics respectively.
Similarly the further moment values $\mu_{r:n}^{(3)}, \mu_{r:n}^{(4)}...$ can be formed by putting $k = 3, 4.....$and so on.
Now

$$\mu_{1:n} = n\theta B\left(\frac{1}{1-p}+1; n\right) \quad (13)$$

$$\mu_{1:n}^2 = n\theta^2 B\left(\frac{2}{1-p}+1; n\right) \quad (14)$$

$$V\left(X_{r:n}\right)=\mu_{r:n}^{(2)}-\left(\mu_{r:n}^{(1)}\right)^{2} \tag{15}$$

$$=n\theta^{2}\left[B\left(\frac{2}{1-p}+1;n\right)-nB^{2}\left(\frac{1}{1-p}+1;n\right)\right] \tag{16}$$

Above equations(13),(14) and (16)are mean,second order moments and Variance of $r=1$ statistics
Similar information for $r=n$:

$$\mu_{n:n}=n\theta B\left(\frac{1}{1-p}+n;1\right) \tag{17}$$

$$=\frac{n\theta(p-1)}{1+n(p-1)} \tag{18}$$

$$\mu_{n:n}^{(2)}=\frac{n\theta^{2}(p-1)}{2-n(p-1)} \tag{19}$$

$$V\left(X_{r:n}\right)=n(p-1)\theta^{2}\left[\frac{1}{2-n(p-1)}-\frac{n(p-1)}{[1+n(p-1)]^{2}}\right] \tag{20}$$

Note: computability of the moments can be checked in appendices
From (8)it can be seen that

$$\mu_{r:n}^{k} = =C_{r:n}\theta^{k}B\left(\frac{k}{1-p}+r;n-r+1\right) \tag{21}$$

$$\mu_{r+1:n}^{k} = =C_{r+1:n}\theta^{k}B\left(\frac{k}{p}+1;n+1-r\right) \tag{22}$$

from equation (21) and (22)it can be shown that

$$\mu_{r:n}^{k}=\frac{pr}{k+pr}\mu_{r+1:n}^{k} \tag{23}$$

similarly we have product moments of the new distribution can be shown as

$$\frac{k_{2}+p(n-s+1)}{k_{2}}\mu_{r;s:n}^{k_{1},k_{2}}=\frac{np}{k_{2}}\left(\mu_{r;s:n-1}^{(k_{1},k_{2})}-\mu_{r;s-1:n-1}^{(k_{1},k_{2})}\right)+\frac{p(n-s-1)}{k_{2}}\mu_{r;s-1:n}^{(k_{1},k_{2})} \tag{24}$$

for $1\leq r\leq s\leq n$ and $n\epsilon N$

III. Entropy of Order statistics

Order statistics entropy can be shown by using the transformation $V_r = F(Y_r)$ from [3] and [4], as below

$$H(Y_r) = H_n(V_r) - E_{z_r}[log f(F^{-1}(V_r))] \tag{25}$$

Here $H_n(V_r)$ is the entropy of the Beta distribution

$$\begin{aligned} H_n(V_r) = log B(r, n-r+1) - (r-1)[\psi(r) - \psi(n+1)] \\ -(n-r)[\psi(n-r+1) - \psi(n+1)], \end{aligned} \tag{26}$$

here $\psi(\theta) = (d/d\theta) \log \Gamma(\theta)$ is the diagamma function.

$$H_n(V_1) = 1 - log(n) - \frac{1}{n}$$

shows the smallest order statistics ($r = 1$)

$$H_n(V_n) = 1 - log(n) - \frac{1}{n}$$

shows the largest order statistics ($r = n$)

Here, $\psi(n+1) - \psi(n) = \frac{1}{n}$

Using the probability integral transformation $V_r = F(Y_r)$ OR $Y_r = F^{-1}(V_r)$ we can find that

$$F^{-1}(V_r) = \theta(V_r)^{1/1-p} \tag{27}$$

combining above result along with (26)and applying in (25) we get the desired result. Entropy of the rth order statistics for the new distribution is

$$\begin{aligned} H(Y_r) = log B(r, n-r+1) - (r-1)[\psi(r) - \psi(n+1)] \\ -(n-r)[\psi(n-r+1) - \psi(n+1)] \\ -log(1-p)/\theta - p/(1-p)[\psi(i) - \psi(n+1)] \end{aligned} \tag{28}$$

The smallest order statistics($i = 1$)

$$\begin{aligned} H_n(V_1) &= 1 - log(n(1-p)/\theta) - \frac{1}{n} - p/(1-p)[\psi(n+1) - \psi(1)] \\ &= 1 - log(n(1-p)/\theta) - \frac{1}{n} - p/(1-p)[\psi(n+1) + \gamma] \end{aligned}$$

Here the Eulers constant $\gamma = 0.5772$ replaces $-\psi(1)$.

The largest order statistics($i = n$)

$$H_n(V_1) = 1 - \frac{(1-2p)}{n(1-p)} - log(n(1-p)/\theta)$$

3.1 Residual entropy

$$H(X_{r:n},t) = -\int_0^\infty \frac{f_{r:n}(x)}{\overline{F}_{r:n}(t)} \, log \, \frac{f_{r:n}(x)}{\overline{F}_{r:n}(t)} \, dx \tag{29}$$

by above residual entropy equation of order statistics and by the Integral transformations $U = F(x)$ and $f(F^{-1}(u)) = \dfrac{(1-p)u^{-p/1-p}}{\theta}$ .Residual entropy of new distribution for $r = 1$ is

$$H(X_{1:n},t) = \frac{n-1}{n} log(n) + log(\overline{F}(t)) - \frac{n}{\overline{F}^n(t)} \int_{F(t)}^1 (1-u)^{n-1} log[f(F^{-1}(u))]du. \tag{30}$$

$$\begin{aligned} H(X_{1:n},t) = \frac{n-1}{n} & - -log\frac{n(1-p)}{\theta} + log(\overline{F}(t)) \\ & - \frac{p}{(1-p)} logF(t) + (1 - F^n(t) \\ & \times \Sigma_{i=1}^n (-1)^i (1/i)(1 - F^i(t)) \end{aligned} \tag{31}$$

where $\overline{F}(t)$ is survival function of new distribution For the higher order $r = n$ we follows the similar way.

3.2 Past Residual Entropy

Similarly past residual entropy of new distribution by [6]is

$$H^0(X_{r:n},t) = -\int_0^t \frac{f_{r:n}(x)}{F_{r:n}(t)} \, log \, \frac{f_{r:n}(x)}{F_{r:n}(t)} \, dx \tag{32}$$

By above past residual entropy equation and integral transformation $U = F(x)$ $f(F^{-1}(u)) = \dfrac{(1-p)u^{-p/1-p}}{\theta}$ and putting $r = n$ past residual entropy of the new distribution obtained as.

$$\begin{aligned} H^0(X_{r:n},t) = \frac{n-1}{n} & - log\,(n) + logF(t)) - \\ \frac{n}{F^n(t)} \int_0^{F(t)} & u^{n-1} log\,[f(F^{-1}(u))]du. \end{aligned} \tag{33}$$

$$H^0(X_{r:n},t) = \frac{n-2}{n} - log\frac{n(1-p)}{\theta} + \frac{1}{1-p} logF(t) \tag{34}$$

In a similar way, we can find the smallest order statistics for $r = 1$.

IV. Conclusion

Here the order statistics of new distribution(2019)with the recurrence relationfor single and product moments have been studied.Some statistical propertieshave been studied with entropy.
Appendices shows the mean and variance for different values of shape and scale parameters with different *r*.

V. References

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VI. Appendices

Table 1: for θ = 0.5, r = 1, 2, 3... and p=0.5

Θ	r	n	$E(X_{r:n})$	$E(X^2_{r:n})$	$V(X_{r:n})$
0.5	1	1	0.166667	0.05000	0.022222222
		2	0.083333	0.016667	0.009722222
		3	0.050000	0.007143	0.004642857
		4	0.033333	0.003571	0.002460317
		5	0.023810	0.001984	0.001417234
		6	0.017857	0.001190	0.000871599
		7	0.013889	0.000758	0.000564675
		8	0.011111	0.000505	0.000381594
		9	0.009091	0.000350	0.000267006
	2	2	0.250000	0.083333	0.020833333
		3	0.150000	0.035714	0.013214286
		4	0.100000	0.017857	0.007857143
		5	0.071429	0.009921	0.004818594
		6	0.053571	0.005952	0.003082483
		7	0.041667	0.003788	0.002051768
		8	0.033333	0.002525	0.001414141
		9	0.027273	0.001748	0.00100445
	3	3	0.300000	0.107143	0.017142857
		4	0.200000	0.053571	0.013571429
		5	0.142857	0.029762	0.009353741
		6	0.107143	0.017857	0.006377551
		7	0.083333	0.011364	0.004419192
		8	0.066667	0.007576	0.003131313
		9	0.054545	0.005245	0.002269549
	4	4	0.333333	0.125000	0.013888889
		5	0.238095	0.069444	0.012755102
		6	0.178571	0.041667	0.009778912
		7	0.138889	0.026515	0.007225028
		8	0.111111	0.017677	0.005331089
		9	0.090909	0.012238	0.003973299
	5	5	0.357143	0.138889	0.011337868
		6	0.267857	0.083333	0.011585884
		7	0.208333	0.053030	0.009627525
		8	0.166667	0.035354	0.007575758
		9	0.136364	0.024476	0.005880483
	6	6	0.375000	0.150000	0.00937500
		7	0.291667	0.095455	0.010385101
		8	0.233333	0.063636	0.009191919
		9	0.190909	0.044056	0.007609663
	7	7	0.388889	0.159091	0.007856341
		8	0.311111	0.106061	0.009270483
		9	0.254545	0.073427	0.008633185
	8	8	0.400000	0.166667	0.006666667
		9	0.327273	0.115385	0.008277177
	9	9	0.409091	0.173077	0.005721551

Table 2: for θ = 1, r = 1, 2, 3, ... and p=0.5

θ	r	n	$E(X_{r:n})$	$E(X_{r:n}^2)$	$V(X_{r:n})$
1	1	1	0.33333333	0.2000000	0.08888889
		2	0.16666667	0.06666667	0.03888889
		3	0.1000000	0.02857143	0.01857143
		4	0.06666667	0.01428571	0.00984127
		5	0.04761905	0.00793651	0.00566893
		6	0.03571429	0.00476190	0.00348639
		7	0.02777778	0.00303030	0.00225870
		8	0.02222222	0.00202020	0.00152637
		9	0.01818182	0.00139860	0.00106802
	2	2	0.5000000	0.33333333	0.08333333
		3	0.3000000	0.14285714	0.05285714
		4	0.2000000	0.07142857	0.03142857
		5	0.14285714	0.03968254	0.01927438
		6	0.10714286	0.02380952	0.01232993
		7	0.08333333	0.01515152	0.00820707
		8	0.06666667	0.01010101	0.00565657
		9	0.05454545	0.00699301	0.0040178
	3	3	0.6000000	0.42857143	0.06857143
		4	0.4000000	0.21428571	0.05428571
		5	0.28571429	0.11904762	0.03741497
		6	0.21428571	0.07142857	0.0255102
		7	0.16666667	0.04545455	0.01767677
		8	0.13333333	0.03030303	0.01252525
		9	0.10909091	0.02097902	0.00907819
	4	4	0.66666667	0.5000000	0.05555556
		5	0.47619048	0.27777778	0.05102041
		6	0.35714286	0.16666667	0.03911565
		7	0.27777778	0.10606061	0.02890011
		8	0.22222222	0.07070707	0.02132435
		9	0.18181818	0.04895105	0.0158932
	5	5	0.71428571	0.55555556	0.04535147
		6	0.53571429	0.33333333	0.04634354
		7	0.41666667	0.21212121	0.0385101
		8	0.33333333	0.14141414	0.03030303
		9	0.27272727	0.09790210	0.02352193
	6	6	0.7500000	0.6000000	0.0375000
		7	0.58333333	0.38181818	0.0415404
		8	0.46666667	0.25454545	0.03676768
		9	0.38181818	0.17622378	0.03043865
	7	7	0.77777778	0.63636364	0.03142536
		8	0.62222222	0.42424242	0.03708193
		9	0.50909091	0.29370629	0.03453274
	8	8	0.8000000	0.66666667	0.02666667
		9	0.65454545	0.46153846	0.03310871
	9	9	0.81818182	0.69230769	0.02288620

Table 3: for *θ = 0.5, r = 1, 2, 3... p=0.95*

θ	r	n	$E(X_{r:n})$	$E(X^2_{r:n})$	$V(X_{r:n})$
0.5	1	1	0.02381	0.006098	0.005530668
		2	0.002165	0.00029	0.000285675
		3	0.000282	2.03E-05	2.0178E-05
		4	4.71E-05	1.84E-06	1.83939E-06
		5	9.41E-06	2.05E-07	2.04534E-07
		6	2.17E-06	2.67E-08	2.66852E-08
		7	5.63E-07	3.98E-09	3.97478E-09
		8	1.61E-07	6.63E-10	6.62491E-10
		9	4.99E-08	1.22E-10	1.21684E-10
	2	2	0.045455	0.011905	0.009838646
		3	0.005929	0.000831	0.000795413
		4	0.000988	7.55E-05	7.45295E-05
		5	0.000198	8.39E-06	8.35049E-06
		6	4.56E-05	1.09E-06	1.09221E-06
		7	1.18E-05	1.63E-07	1.62839E-07
		8	3.38E-06	2.72E-08	2.71518E-08
		9	1.05E-06	4.99E-09	4.98806E-09
	3	3	0.065217	0.017442	0.013188552
		4	0.01087	0.001586	0.001467476
		5	0.002174	0.000176	0.000171455
		6	0.000502	2.3E-05	2.27284E-05
		7	0.00013	3.42E-06	3.40564E-06
		8	3.72E-05	5.7E-07	5.69046E-07
		9	1.15E-05	1.05E-07	1.04639E-07
		4	0.083333	0.022727	0.015782828
		5	0.016667	0.002525	0.002247475
		6	0.003846	0.000329	0.000314588
		7	0.000997	4.91E-05	4.80624E-05
		8	0.000285	8.18E-06	8.09495E-06
		9	8.84E-05	1.5E-06	1.49392E-06
	5	5	0.10000	0.027778	0.017777778
		6	0.023077	0.003623	0.003090644
		7	0.005983	0.00054	0.000503829
		8	0.001709	8.99E-05	8.70152E-05
		9	0.000531	1.65E-05	1.62377E-05
	6	6	0.115385	0.032609	0.019295086
		7	0.029915	0.004857	0.003961735
		8	0.008547	0.000809	0.000736384
		9	0.002653	0.000149	0.000141636
	7	7	0.12963	0.037234	0.020430202
		8	0.037037	0.006206	0.004833932
		9	0.011494	0.00114	0.0010077
	8	8	0.142857	0.041667	0.021258503
		9	0.044335	0.007653	0.005687471
	9	9	0.155172	0.045918	0.021839889

Table 4: for *θ = 1, r = 1, 2, 3... and p=0.95*

θ	r	n	$E(X_{r:n})$	$E(X_{r:n}^2)$	$V(X_{r:n})$
1	1	1	0.04761905	0.02439024	0.02212267
		2	0.004329	0.00116144	0.0011427
		3	0.00056465	8.1031E-05	8.0712E-05
		4	9.4109E-05	7.3664E-06	7.3576E-06
		5	1.8822E-05	8.1849E-07	8.1814E-07
		6	4.3435E-06	1.0676E-07	1.0674E-07
		7	1.1261E-06	1.59E-08	1.5899E-08
		8	3.2174E-07	2.6501E-09	2.65E-09
		9	9.985E-08	4.8675E-10	4.8674E-10
	2	2	0.09090909	0.04761905	0.03935458
		3	0.01185771	0.00332226	0.00318165
		4	0.00197628	0.00030202	0.00029812
		5	0.00039526	3.3558E-05	3.3402E-05
		6	9.1213E-05	4.3772E-06	4.3688E-06
		7	2.3648E-05	6.5192E-07	6.5136E-07
		8	6.7565E-06	1.0865E-07	1.0861E-07
		9	2.0969E-06	1.9957E-08	1.9952E-08
	3	3	0.13043478	0.06976744	0.05275421
		4	0.02173913	0.00634249	0.0058699
		5	0.00434783	0.00070472	0.00068582
		6	0.00100334	9.192E-05	9.0914E-05
		7	0.00026013	1.369E-05	1.3623E-05
		8	7.4322E-05	2.2817E-06	2.2762E-06
		9	2.3065E-05	4.1909E-07	4.1856E-07
	4	4	0.16666667	0.09090909	0.06313131
		5	0.03333333	0.01010101	0.0089899
		6	0.00769231	0.00131752	0.00125835
		7	0.0019943	0.00019623	0.00019225
		8	0.0005698	3.2704E-05	3.238E-05
		9	0.00017683	6.0069E-06	5.9757E-06
	5	5	0.200000	0.11111111	0.07111111
		6	0.04615385	0.01449275	0.01236258
		7	0.01196581	0.0021585	0.00201531
		8	0.0034188	0.00035975	0.00034806
		9	0.00106101	6.6076E-05	6.4951E-05
	6	6	0.23076923	0.13043478	0.07718034
		7	0.05982906	0.01942646	0.01584694
		8	0.01709402	0.00323774	0.00294554
		9	0.00530504	0.00059469	0.00056654
	7	7	0.25925926	0.14893617	0.08172081
		8	0.07407407	0.0248227	0.01933573
		9	0.02298851	0.00455927	0.0040308
	8	8	0.28571429	0.16666667	0.08503401
		9	0.08866995	0.03061224	0.02274988
	9	9	0.31034483	0.18367347	0.08735956