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Real Interpolationof Operatorsin Banach-Saks and Invariant Spaces WithApplications

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Abstract: *LinearOperators on invariant spaces and between Banach spaces we define a semi norm vanishing on the subspace of operators having the alternate signs Banach-Saks property. In particular, the estimates show that the alternate signsinvariant spaces and Banach-Saks property are inherited from a space of an interpolation pair* (A_0, A_1) *tothe real interpolation spaces* $A_{\theta,p}$. *Finally, examples are given to support our results.*

Keywords: *invariant spaces ,Banach-Saks , Lions-Peetre*

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I. Introduction

A linear transformation $T : V \rightarrow V$and $\leq V$. T is invariant under T if TW ⊂ W and a bounded linear operator $T : V \rightarrow W$acting between Banach spaces is said to have the Banach-Saks (BS) property if every bounded sequence (v_n) in Vcontains a subsequence (v'_n) such that the Cesáro means of (Tv'_n) converge in Y. If we restrict this definition to all weakly null sequences (v_n) in X, we say that T has the weak Banach-Saks (WBS) property or the Banach-Saks-Rosenthal property. We say that T has the alternate signs Banach-Saks (ABS) property if every bounded sequence (v_n) in Vcontains a subsequence (v'_n) such that the Cesáro means of$((-1)^n Tv'_n)$ converge in Y .

A Banach space $V$is called to have the BS, WBS or ABS property if the corresponding property is possessed by the identity operator $I : V \rightarrow V$. For a detailed study of these properties we refer the reader to [11].

A natural question is the behavior ofinvariant spaces andBanach-Saks properties under interpolation. Beauzamy [11] proved that if (A_0, A_1) is an interpolation pair such that A_0 is continuously embedded inA_1 and the embedding has the ABS property, then the real interpolation spaces $A_{\theta,p}$with respect to (A_0, A_1) have the ABS property for all $0 < \theta < 1$ and $1 < p < \infty$. This in turn served to show that every operator withthe BS or ABS property factors through a space with the same property (see also [13]). Heinrich [3] proved that if the embedding $I : A_0 \cap A_1 \rightarrow A_0 + A_1$ has the BS property, then so has $A_{\theta,p}$with respect to (A_0, A_1) for all $0 < \theta < 1$ and $1 < p < \infty$ (see also [1,12]).We find a measure of deviation from the ABS property with good interpolation properties.

Our work is motivated by[2 ,9 , 11 ,14], where similar results for a measure of weak noncompactness were obtained.

II. Invariant spaces and Banach-Saks property and spreading models

One of the basic results on invariant spacesBanach-Saks properties is the following one of Rosenthal [8]: if a Banach space X does not have the WBS property, then there exist a number $\delta > 0$ and a bounded double sequence $(v\ _n^m)$ in Vsuch that for all $k \in \mathbb{N}$, all subsets $A \subset \mathbb{N}$ with $|A| = 2^k$and $k \leq minA,$ and all sequences of scalars(c_n) , we have

$$\left\| \sum_{m,n \in A} c_n v\ _n^m \right\| \geq \delta \sum_{n \in A} |c_n| .$$

Definition 1. Let $(v\ _n^m)$ be a bounded sequence in a Banach space V. Define

$$\phi_{vsm}(v\ _n^m) = \inf \left\| |A|^{-1} \sum_{m,n \in A} \epsilon_n v\ _n^m \right\|, \phi_{am}(v\ _n^m) = \inf \left\| |A|^{-1} \sum_{m,n \in A} v\ _n^m \right\|,$$

the infimum for $\phi_{vsm}(v\ {}^{m}_{n})$being taken over all finite subsets $A \subset \mathbb{N}$ and all sequences of signs(ϵ_n), the infimum for φambeing taken over all finite subsets $A \subset \mathbb{N}$. If$(w\ {}^{m}_{n})$ is a sequence of svsm for $(v\ {}^{m}_{n})$, in particular, $(w\ {}^{m}_{n})$is double a subsequence of $(v\ {}^{m}_{n})$ or $(w\ {}^{m}_{n})$ is double a sequence of sam for (v_n) ,then $\phi_{vsm}(v\ {}^{m}_{n}) \leq \phi_{vsm}(w\ {}^{m}_{n})$.

Definition 2 $T : V \to V$ and $W \leq V$.*T is* ***invariant under*** *T if* TW ⊂ W.

Note that $g(T)W \subset W$ for any polynomial g.

Proposition 3. suppose$(v\ {}^{m}_{n})$ be double a bounded sequence in a Banach space X. There exist double a subsequence$(v\ {}'^{m}_{n})$ of$(v\ {}^{m}_{n})$ and a seminorm L in the set S of all finite sequences of scalars (real or complex), with the following property: for every$\epsilon > 0$*and every* $a = (a_1, \ldots, a_m) \in S$ *there exists* $v \in N$*such that, if* $v \leq n_1 < \ldots < n_m$, *then*

$$\left| \left\| \sum_{i=1}^{m} a_i v\ {}'^{m}_{ni} \right\| - L(a) \right| < \varepsilon.$$

If $(v\ {}^{m}_{n})$ has no Cauchy subsequence, the formula

$$\| a_1 v_1'^1 + \ldots + a_m v_m'^m \|_E = L(a), \; a = (a_1, \ldots, a_m),$$

defines a norm in the space spanned by vectors $v\ {}'^{m}_{n}$. Let Ebethe completion of span $\{v\ {}'^{m}_{n}\}$under this norm. The space $E$is called the spreadingmodel of Vbuilt on $(v\ {}^{m}_{n})$. The sequence $(v\ {}^{m}_{n})$ is called the fundamental sequence of$E$.The norm of $E$is invariant under spreading; that is$\| a_1 v_1'^1 + \ldots + a_m v_m'^m \|_E = \| a_1 v_{n_1}'^1 + \ldots +$ $amvnm'mE$for all

$n_1 < \ldots < n_m$.

The next proposition will play a key role in our considerations. Its assertion is related to property (P_1') of [11 ,15]. In the proof, we follow the main line of the proof of Theorem II.2 of [11].

Proposition 4. Let $(v\ {}^{m}_{n})$ be double a bounded sequence in a Banach space X. Then for every $\epsilon > 0$ there exist a sequence$(w\ {}^{m}_{n})$ of svsm for $(v\ {}^{m}_{n})$and a sequence$(v\ {}^{m}_{n})$of sam for$(v\ {}^{m}_{n})$such that for all finite subsets$A \subset \mathbb{N}$ and all sequences of signs(ϵ_n),

$$\left\| |A|^{-1} \sum_{m,n \in A} \epsilon_n w\ {}^{m}_{n} \right\| \leq \phi_{vsm}(w\ {}^{m}_{n}) + \epsilon \, , \left\| |A|^{-1} \sum_{m,n \in A} v\ {}^{m}_{n} \right\| \leq \phi_{am}(v\ {}^{m}_{n}) + \epsilon$$

Proof. We prove the assertion for the relation svsm. The proof for the relation sam is almost the same. Fix $\varepsilon > 0$. First assume that (v_n) contains a Cauchy subsequence $(v_n'^m)$.Let$w\ {}^{m}_{n} = \frac{v_{2n}'^1 - v_{2n-1}'^m}{2}$. Ignoring a finite number of terms of $(w\ {}^{m}_{n})$,we see that $(w\ {}^{m}_{n})$satisfies the assertion.Now assume that $(v\ {}^{m}_{n})$has no Cauchy subsequence. Let a double subsequence $(v_n'^m)$ of$(v\ {}^{m}_{n})$be the fundamental sequence of the spreading model Ebuilt on $(v\ {}^{m}_{n})$, givenby Proposition 3. Taking $(v_n'^m)$ in the norm $\|.\|_E$,we put $K = \phi_{vsm}(v_n')$. Thereexists $u = m^{-1} \sum_{i=1}^{m} \epsilon_i' v_{n_i}'^m$, where $n_1 < \ldots < n_m$and $\epsilon_1', \ldots, \epsilon_m'$is a finite sequenceof signs, such that $K \leq \|u\|_E \leq K + \frac{\varepsilon}{4}$.Let$u\ {}^{m}_{n} = m^{-1} \sum_{i=1}^{m} \epsilon_i' v_{(n-1)m+i}'^m$ for every $n \in \mathbb{N}$.Since$\|.\|_E$ is invariant under spreading ,

$K \leq \| u\ {}^{m}_{n} \|_E \leq K + \varepsilon/4$. Clearly,

for all finite subsets $A \subset \mathbb{N}$ and all sequences of signs (ϵ_n),

$$K \leq \left\| |A|^{-1} \sum_{n \in A} \epsilon_n u\ {}^{m}_{n} \right\|_E \leq K + \varepsilon/4.$$

Let $k \in N$.By Proposition 3, we get $n_k$such that if $B \subset \mathbb{N}$ with $|B| \leq 2^k$and $n_k \leq minB$, then for all sequences of signs (ϵ_n),

$$\left| \left\| |B|^{-1} \sum_{n \in B} \epsilon_n\, v\ {}^{m}_{n} \right\| - \left\| |B|^{-1} \sum_{m,n \in B} \epsilon_n\, v\ {}^{m}_{n} \right\|_E \right| < \varepsilon/4.$$

We may assume that $n_k < n_{k+1}$ for all k. It follows that for the double sequence $(u_k'^m)$ with $u_k'^m = u_{n_k}^m$, all $B \subset \mathbb{N}$ with $|B| \leq 2^k$and $k \leq minB$, and all sequences of signs (ϵ_n),

$$K - \varepsilon/4 \leq \left\| |B|^{-1} \sum_{m,n \in B} \epsilon_n\, u_n'^m \right\| \leq K + \varepsilon/2.$$

Let $A \subset \mathbb{N}$ be finite and $A_0 = \{n \in A : n < \log_2 |A|\}$. Then

$$\left\| \sum_{m,n \in A_0} \epsilon_n\, u_k'^m \right\| \leq |A_0|\,(K + \varepsilon/2) \text{ and } \left\| \sum_{m,n \in A \setminus A_0} \epsilon_n\, u_k'^m \right\| \geq |A \setminus A_0|\,(K - \varepsilon/4)\,.$$

Of course, we assume that the sum over the empty set is 0.Consequently,

$$\left\| |A|^{-1} \sum_{m,n\in A} \epsilon_n \, u_k^{'m} \right\| \geq \left\| |A|^{-1} \sum_{m,n\in A\setminus A_0} \epsilon_n \, u_k^{'m} \right\| - \left\| |A|^{-1} \sum_{m,n\in A_0} \epsilon_n \, u_n^{'} \right\|$$
$$\geq K - \varepsilon/4 - |A_0|\,|A|^{-1}(2K + \varepsilon/4).$$

There is an $m_0 \in \mathbb{N}$ such that if $|A| \geq m_0$, then $|A_0|\,|A|^{-1}(2K + \varepsilon/4) \leq \varepsilon/4$. Then

$$K - \varepsilon/2 \leq \left\| |A|^{-1} \sum_{m,n\in A} \epsilon_n \, u_k^{'m} \right\| \leq K + \varepsilon/2.$$

Let $w_n = m_0^{-1} \sum_{i=1}^{m_0} \mathcal{Z}^{'}_{(n-1)m_0+i}$for every $n \in \mathbb{N}$. Then for all finite subsets$A \subset \mathbb{N}$ and all sequences of signs (ϵ_n),

$$K + \varepsilon/2 \geq \left\| |A|^{-1} \sum_{m,n\in A} \epsilon_n \, w_{\ n}^{\ m} \right\| \geq | \left\| |A|^{-1} m_0^{-1} \sum_{m,n\in A} \sum_{i=1}^{m_0} \epsilon_n \, u^{'m}_{(n-1)m_0+i} \right\|$$
$$\geq K \frac{\varepsilon}{2}.$$

Thus

$\left\| |A|^{-1} \sum_{m,n\in A} \epsilon_n \, w_{\ n}^{\ m} \right\| \leq \phi_{vsm}(w_{\ n}^{\ m}) + \varepsilon$for all finite subsets $A \subset \mathbb{N}$ and all sequences of signs (ϵ_n). Of course, $(w_{\ n}^{\ m})$ is boublea sequence of svsmfor$(v_{\ n}^{\ m})$.

Definition 5. Let V, Ybe Banach spaces and $T \in \mathcal{L}(V, Y)$. Define

$$\Phi_{ABS}(T) = \sup\{\phi_{vsm}(Tv_{\ n}^{\ m}) : (v_{\ n}^{\ m}) \subset B(V)\} .$$

Proposition 6. $\Phi_{ABS}$is a seminorm in$\mathcal{L}(V, Y)$. $\Phi_{ABS}(T) = 0$if and only if $T \in ABS(V, Y)$.

Proof. Clearly, $\Phi_{ABS}(\lambda T) = |\lambda| \Phi_{ABS}(T)$ for all scalars λ.We show that for all $S, T \in \mathcal{L}(V, Y)$, $\Phi_{ABS}(S + T) \leq \Phi_{ABS}(S) + \Phi_{ABS}(T)$. Let $\varepsilon > 0$ and $(v_{\ n}^{\ m}) \subset B(V)$. By Proposition 4, there exists a sequence $(v_n^{'m})$ of svsm for $(v_{\ n}^{\ m})$ such that for thesequence $(Sv_n^{'m})$ of svsm for $(Sv_{\ n}^{\ m})$,

$$\left\| |A|^{-1} \sum_{m,n\in A} \epsilon_n \, Sv_n^{'m} \right\| \leq \phi_{vsm}(Sv_n^{'m}) + \varepsilon$$

for all finite subsets $A \subset \mathbb{N}$ and all sequences of signs (ϵ_n). Alsoby Proposition 4, we get a sequence $(v_n^{''m})$ of svsm for $(v_n^{'m})$, such that for all finite subsets $A \subset \mathbb{N}$ and all sequences of signs (ϵ_n),

$$\left\| |A|^{-1} \sum_{m,n\in A} \epsilon_n \, Tv_n^{''m} \right\| \leq \phi_{vsm}(Tv_{\ n}^{\ m}) + \varepsilon.$$

Since the relation svsm is transitive,

$$\phi_{vsm}\left((S + T)v_{\ n}^{\ m}\right) \leq \phi_{vsm}\left((S + T)v_n^{''m}\right) \leq \left\| |A|^{-1} \sum_{m,n\in A} \epsilon_n \, (S + T)v_n^{''m} \right\|$$
$$\leq \left\| |A|^{-1} \sum_{m,n\in A} \epsilon_n \, Sv_n^{''m} \right\| + \left\| |A|^{-1} \sum_{m,n\in A} \epsilon_n \, Tv_n^{''m} \right\|$$
$$\leq \phi_{vsm}(Sv_n^{'m}) + \phi_{vsm}(Tv_n^{''m}) + 2\varepsilon \leq \Phi_{ABS}(S) + \Phi_{ABS}(T) + 2\varepsilon.$$

By an arbitrary choice of $\varepsilon > 0$ and $(v_{\ n}^{\ m}) \subset B(V)$, we obtain the conclusion.

T has the ABS property if and only if for every bounded sequence $(v_{\ n}^{\ m})$ in X there exist a subsequence $(v_n^{'m})$ of v_n and a sequence of signs (ϵ_n) such that the Cesàro means of $(\epsilon_n \, Tv_n^{'m})$ converge to 0 in Y . From this T has the ABS property if and only if for every bounded sequence $(v_{\ n}^{\ m})$ in V, $\phi_{vsm}(Tv_{\ n}^{\ m}) = 0$. By positive homogeneity of Φ_{ABS} ,T has the ABS property if and only if $\Phi_{ABS}(T) = 0$.

III. Operators on invariant spaces and Banach-Saks property and $l_p(X)$ spaces

Let X be a Banach space, $1 < p < \infty$and let (e_i) be the unit vector basis of l_p. We denote by $l_p(V)$ the Banach space of all sequences

$v = (v(i))$such that $v(i) \in V$for every $i \in \mathbb{N}$ and

$$\|v\|_{l_p(V)} = \left\| \sum_{i=1}^{\infty} \|v(i)\|_V e_i \right\|_{l_p} < \infty .$$

In the sequel, we also deal with $l_p(V)$ of the families $(v(i))_{i\in\mathbb{Z}}$ indexed by integers. Partington [6] proved that $l_p(V), 1 < p < \infty$, has the BS property if and only if so has V(in fact, a more general setting of direct sums was used). We use similar arguments as in the proof of Theorem 3 of [6] to show the next lemma.

Lemma7. Suppose V be a Banach space and $(v\ {}_n^m)$ a boundeddobule sequence in l_p(X),$1 < p < \infty$. Then for every $\varepsilon > 0$ there exist m $\in \mathbb{N}$ and double a sequence $(w\ {}_n^m)$ of sam for $(v\ {}_n^m)$ such that for all finite subsets A $\subset \mathbb{N}$ and all sequences of signs(ϵ_n),

$$\left\| \sum_{i=m+1}^{\infty} \left\| |A|^{-1} \sum_{m,n\in A} \epsilon_n\, w\ {}_n^m(i) \right\|_V e_i \right\|_{l_p} < \varepsilon \ .$$

Proof. For $v\ {}_n^m = (v\ {}_n^m(i)) \in l_p(V)$,put $t\ {}_n^m = \sum_{i=1}^{\infty} \|v\ {}_n^m(i)\|_V\, e_i \in l_p$. Since $l_p$has the BS property, by ErdÖs-Magidor's theorem in [2], there exists a subsequence $(t\ {}_n^{'m})$ of $(t\ {}_n^m)$ such that the Cesàro means of all subsequences of $(t\ {}_n^{'m})$ converge to the same limit $t$in l_p. Then $\phi_{am}(s\ {}_n^m - t) = 0$ for every sequence $(s\ {}_n^m)$ of sam for $\left(t\ {}_n^{'m}\right)$. By Proposition 4, there exists a sequence$(s\ {}_n^m)$ of sam for $\left(t\ {}_n^{'m}\right)$ such that for everyfinite subset $A \subset \mathbb{N}$,

$$\left\| \sum_{i=1}^{\infty} (s\ {}_n^m) - t \right\|_{l_p} < \varepsilon/2\,.$$

There exist $k_0 \in \mathbb{N}$ and a sequence(A_n) of finite subsets of $\mathbb{N}$ with max$A_n <$ min$A_n + 1$ and $|A_n| = k_0$ for all $n$such that

$s\ {}_n^m = k_0^{-1} \sum_{k\in A_n} t_k^{'}$. Let $(w\ {}_n^m)$ be thecorresponding sequence of sam for $(v\ {}_n^m)$. That is, first we take the subsequence $(v\ {}_n^{'m})$of $(v\ {}_n^m)$ such that

$t\ {}_n^{'m} = \sum_{i=1}^{\infty} \|v\ {}_n^{'m}(i)\|_V e_i$, and then we put$w_n = k_0^{-1} \sum_{k\in A_n} v_k^{'}$.

Let $t = \sum_{i=1}^{\infty} \alpha_i e_i$and let $m \in \mathbb{N}$ satisfy$\|\sum_{i=m+1}^{\infty} \alpha_i e_i\|_{l_p} < \varepsilon/2$.Then for every finite subset $A \subset \mathbb{N}$,

$$\left\| \sum_{i=m+1}^{\infty} \left(|A|^{-1} \sum_{n\in A} k_0^{-1} \sum_{k,m\in A_n} \left\|v\ {}_k^{'m}(i)\right\|_V - \alpha_i \right) e_i \right\|_{l_p} < \varepsilon/2\,.$$

It follows that

$$\left\| \sum_{i=m+1}^{\infty} \left(|A|^{-1} \sum_{n\in A} k_0^{-1} \sum_{k,m\in A_n} \left\|v\ {}_k^{'m}(i)\right\|_V \right) e_i \right\|_{l_p} < \varepsilon \ .$$

By hyperorthogonality of the basis (e_i), for all sequences of signs (ϵ_n),

$$\left\| \sum_{i=m+1}^{\infty} \left\| |A|^{-1} \sum_{m,n\in A} \epsilon_n\, w\ {}_n^m(i) \right\|_V e_i \right\|_{l_p} < \varepsilon.$$

Theorem 8. PutV, Y *be Banach spaces and* $1 < p < \infty$. *If*
$T \in \mathcal{L}(V, Y)$*and*if$\tilde{T} \in \mathcal{L}(l_p(V), l_p(Y))$*is given by* $\tilde{T}v = (Tv(i))$for every$v = (v(i))$, *then* $\Phi_{ABS}(T) = \Phi_{ABS}(\tilde{T})$.

Proof. Since $l_p(V)$ contains isometric copies of $V, \Phi_{ABS}(T) \leq \Phi_{ABS}(\tilde{T})$. Fix $\varepsilon > 0$. There exists $(v_n) \subset B(l_p(V))$ such that

$\Phi_{ABS}(\tilde{T}) - \varepsilon \leq \emptyset_{vsm}(\tilde{T}v_n)$. By Lemma 7, there exist $m \in \mathbb{N}$ and a sequence $(v\ {}_n^{'m})$ of samfor $(v\ {}_n^m)$ such that for the sequence $(\tilde{T}v_n^{'})$of sam for $(\tilde{T}v_n)$, and for all finite subsets $A \subset \mathbb{N}$ and all sequences of signs (ϵ_n),

$$\left\| \sum_{i=m+1}^{\infty} \left\| |A|^{-1} \sum_{m,n\in A} \epsilon_n\, Tv\ {}_n^{'m}(i) \right\|_V e_i \right\|_{l_p} < \varepsilon.$$

There exists a subsequence $(v\ {}_n^{''m})$ of $(v\ {}_n^{'m})$ such that for each $1 \leq i \leq m$the limit $\beta_i = \lim_{m,n} \|v\ {}_n^{''m}(i)\|_V$exists and $\|v\ {}_n^{''m}(i)\|_V < \beta_i + \frac{\varepsilon}{m}$for every n. Putting$v_n(i) = \left(\beta_i + \frac{\varepsilon}{m}\right)^{-1} Tv\ {}_n^{''m}(i)$, we have $\left(v\ {}_n^m(i)\right) \subset T(B(V))$ for every $1 \leq i \leq m$.By Proposition 4, there exists a sequence (v_n^{m1}) of svsm for $(v_n^{''m})$ such thatfor the sequence $(v_n^{m1}(1))$ of svsm for $\left(v\ {}_n^m(1)\right)$, where $v_n^1(i) = \left(\beta_i + \frac{\varepsilon}{m}\right)^{-1} Tv_n^1(i), 1 \leq i \leq m$, we have

$$\left\| |A|^{-1} \sum_{m,n\in A} \epsilon_n \, v_n^{m1}(1) \right\|_Y \le \emptyset_{vsm}(v_n^{m1}(1)) + \varepsilon$$

for all finite subsets $A \subset \mathbb{N}$ and all sequences of signs (ϵ_n).

Proceeding in this way consecutively for $i = 2, \ldots, m$, in the kth step, we obtain a sequence (v_n^k) of svsm for (v_n^{k-1}) such that for the sequence $(v_n^k(k))$ of svsm for $(v_n^{k-1}(k))$, where $v_n^k(i) = (\beta_i + \varepsilon/m)^{-1} T v_n^k(i)$, $1 \le i \le m$, we have

$$\left\| |A|^{-1} \sum_{n\in A} \epsilon_n \, v_n^k(k) \right\|_Y \le \emptyset_{vsm}(v_n^k(k)) + \varepsilon$$

for all finite subsets $A \subset \mathbb{N}$ and all sequences of signs (ϵ_n). In this way, all sequences $(v_n^m(i)), 1 \le i \le m$, are built on the common sequence (v_n^m) of svsm for (v_n), and for all finite subsets $A \subset \mathbb{N}$ and all sequences of signs (ϵ_n),

$$\left\| |A|^{-1} \sum_{n\in A} \epsilon_n \, v_n^m(i) \right\|_Y \le \emptyset_{vsm}(v_n^m(i)) + \varepsilon, 1 \le i \le m.$$

It follows that

$$\emptyset_{vsm}(\tilde{T}v_n) \le \emptyset_{vsm}(\tilde{T}v_n^m) \le \left\| \sum_{i=1}^{m} \left\| |A|^{-1} \sum_{n\in A} \epsilon_n \, T v_n^m(i) \right\|_Y e_i \right\|_{l_p} + \varepsilon$$

$$= \left\| \sum_{i=1}^{m} \left\| (\beta_i + \varepsilon/m)|A|^{-1} \sum_{n\in A} \epsilon_n \, v_n^m(i) \right\|_Y e_i \right\|_{l_p} + \varepsilon$$

$$\left\| \sum_{i=1}^{m} |\beta_i + \varepsilon/m| e_i \right\|_{l_p} \max_{1\le i\le m} \left\| |A|^{-1} \sum_{n\in A} \epsilon_n \, v_n^m(i) \right\|_Y + \varepsilon$$

$$\le \left(1 + \varepsilon m^{1/p-1}\right) \max_{1\le i\le m} \{\emptyset_{vsm}(v_n^m(i)) + \varepsilon\} + \varepsilon.$$

There exists $1 \le j \le m$such that $\emptyset_{vsm}(v_n^m(j)) = \max_{1\le i\le m} \emptyset_{vsm}(v_n^m(i))$.

Since$(v_n^m(j))$ is a sequence of svsm for $\left(v_n(j)\right)$,we have $(v_n^m(j)) \subset T(B(V))$ andconsequently,

$$\Phi_{ABS}(\tilde{T}) - 2\varepsilon \le \left(1 + \varepsilon m^{1/p-1}\right)(\Phi_{ABS}(T) + \varepsilon).$$

Letting $\varepsilon \to 0$, we get $\Phi_{ABS}(\tilde{T}) \le \Phi_{ABS}(T)$.

Corollary 9. The space$l_p(V), 1 < p < \infty$,has the ABS property if and only if V has the ABS property.

IV. Invariant spaces and Banach-Saks property and real interpolation

We recall briefly some basic definitions and facts concerning real interpolation. For a thorough treatment we refer to [4,5,10].

If two Banach spaces A_0 and A_1 are linearly and continuously embedded in a common Hausdorff topological vector space V, we call $\vec{A} = (A_0, A_1)$ an interpolationpair. Then $\Delta(\vec{A}) = A_0 \cap A_1, \Sigma(\vec{A}) = A_0 + A_1$ are Banach spaces with norms

$$\|a\|_{\Delta(\vec{A})} = \max\{\|a\|_{A_0}, \|a\|_{A_1}\}, \|a\|_{\Sigma(\vec{A})} = \inf\{\|a_0\|_{A_0} + \|a_1\|_{A_1} : a_0 + a_1 = a\}.$$

We consider a discrete method of construction of the real interpolation spaces of Lions and Peetre [3]. For $0 < \theta < 1$ and $1 < p < \infty$, let

$$A_{\theta,p} = \left\{a \in \Sigma(A) : \|a\|_{A_{\theta,p}} < \infty\right\},$$

where

$$\|a\|_{A_{\theta,p}} = \inf\max\left\|(2^{i\theta} a_0(i))\right\|_{l_p(A_0)}, \left\|(2^{i(\theta-1)} a_1(i))\right\|_{l_p(A_1)},$$

the infimum being taken over all families $(a_0(i)) \subset A_0$and $(a_1(i)) \subset A_1$ with $a_0(i) + a_1(i) = a$for all $i \in \mathbb{Z}$. Then $\Delta(A) \subset A_{\theta,p} \subset \Sigma(A)$ with continuous embeddings. The Banach space $A_{\theta,p}$with norm $\|.\|_{A_{\theta,p}}$ is called a real interpolation space with respect to $A = (A_0, A_1)$. If $a \in A_{\theta,p}$,then

$$\|a\|_{A_{\theta,p}} \le 2^{\theta(1-\theta)} \left\|(2^{i\theta} a_0(i))\right\|_{l_p(A_0)}^{1-\theta} \left\|(2^{i(\theta-1)} a_1(i))\right\|_{l_p(A_1)}^{\theta}$$

for all families $(a_0(i)) \subset A_0$ and $\left(a_1(i)\right) \subset A_1$with $a_0(i) + a_1(i) = a$for all $i \in \mathbb{Z}$ (see [1, 5 ,7]) .

Let $A_{\theta,p}$and $B_{\theta,p}$be two interpolation spaces with respect to the interpolation pairs $\vec{A} = (A_0, A_1)$ and $\vec{B} = (B_0, B_1)$, and let

$T : \Sigma(\vec{A}) \to \Sigma(\vec{B})$be a linear operator. We write $T : \vec{A} \to \vec{B}$, if for
$j = 0, 1$,the restriction $T|A_j$ is a bounded operator into B_j.
For every $T : \vec{A} \to \vec{B}$,

$$\|T : A_{\theta,p} \to B_{\theta,p}\| \le 2^{\theta(1-\theta)}\|T : A_0 \to B_0\|^{1-\theta}\|T : A_1 \to B_1\|^{\theta} .$$

we show that this classical inequality concerning boundedness has its counterpart for the ABS property.

Lemma 10 Let W be an invariant subspace of V under T.Then mTW divides mT.

If $A = \begin{pmatrix} B & C \\ O & D \end{pmatrix}$, then $A^k = \begin{pmatrix} B^k & C_k \\ O & D^k \end{pmatrix}$.

Example 11 *Let $W = W_1, \ldots, W_K$ be the space generated by all eigenvectors of T. Then W is invariant under T. Let $B' = \{\alpha_1, \ldots, \alpha_r\}$ be the basis for W and extend it to a basis $\mathcal{B}$ for V. Then*

$$[T]_{\mathcal{B}} = \begin{pmatrix} B & C \\ O & D \end{pmatrix}$$

and

$$B = [T_W]_{\mathcal{B}'} = diag(c_1, \ldots, c_1, c_2, \ldots, c_2, \ldots, c_k, \ldots, c_k).$$

Corollary 12.Φ_{ABS} isaseminorm in$\mathcal{L}(X, Y)$. $\Phi_{\mathrm{ABS}}(T) = 0$ if and only if $T \in \mathrm{ABS}(X, Y)$.

Proof. Clearly, $\Phi_{\mathrm{ABS}}(\lambda T) = |\lambda|\Phi_{\mathrm{ABS}}(T)$ for all scalars λ.We show that for all $S, T \in \mathcal{L}(X, Y)$, $\Phi_{\mathrm{ABS}}(S + T) \le \Phi_{\mathrm{ABS}}(S) + \Phi_{\mathrm{ABS}}(T)$. Let $\varepsilon > 0$ and $(v_n + w_n) \subset B(V)$. By Proposition 4, there exists a sequence $(v_n' + w_n')$ of svsm for $(v_n + w_n)$ such that for thesequence $(S(v_n' + w_n'))$ of svsm for $(S(v_n + w_n))$,

$$\left\| |A|^{-1} \sum_{n\in A} \epsilon_n \, S(v_n' + w_n') \right\| \le \phi_{vsm}(S(v_n' + w_n')) + \varepsilon$$

for all finite subsets$A \subset \mathbb{N}$ and all sequences of signs (ϵ_n). Again applying Proposition 4, we get a sequence $(v_n'' + w''_n)$ of svsm for $(v'_n + w'_n)$, such that for all finite subsets $A \subset \mathbb{N}$ and all sequences of signs (ϵ_n),

$$\left\| |A|^{-1} \sum_{n\in A} \epsilon_n \, T(v_n'' + w_n'') \right\| \le \phi_{vsm}(T(v_n + w_n)) + \varepsilon.$$

Since the relation svsm is transitive,

$$\phi_{vsm}\big((S + T)(v_n + w_n)\big) \le \phi_{vsm}\big((S + T)(v_n'' + w_n'')\big) \le \left\| |A|^{-1} \sum_{n\in A} \epsilon_n \, (S + T)(v_n'' + w'')_n \right\|$$

$$\le \left\| |A|^{-1} \sum_{n\in A} \epsilon_n \, S(v_n'' + w_n'') \right\| + \left\| |A|^{-1} \sum_{n\in A} \epsilon_n \, T(v_n'' + w_n'') \right\|$$

$$\le \phi_{vsm}(S(v'_n + w'_n)) + \phi_{vsm}T(v_n'' + w_n'') + 2\varepsilon \le \Phi_{ABS}(S) + \Phi_{ABS}(T) + 2\varepsilon.$$

By an arbitrary choice of $\varepsilon > 0$ and $(v_n + w_n) \subset B(V)$, we obtain the conclusion.

Corollary 13. *Let $A_{\theta,p}$and $B_{\theta,p}$ with $0 < \theta < 1$ and $\varepsilon > 0$ be real interpolation spaces with respect to interpolation pairs*
$\vec{A} = (A_0, A_1)$and$\vec{B} = (B_0, B_1)$ *.Then for every $T : \vec{A} \to \vec{B}$,*

$$\Phi_{ABS}(T : A_{\theta,p} \to B_{\theta,p}) \le 2^{\theta(1-\theta)}\Phi_{ABS}^{1-\theta}(T : A_0 \to B_0)\Phi_{ABS}^{\theta}(T : A_1 \to B_1)$$

ProofFix $\varepsilon > 0$. Let (a_n) be a sequence in $B(A_{\theta,p})$.For each $a_n$there exist $v_{jn} = (2^{i(\theta-j)}a_{jn}(i))_{i\in\mathbb{Z}} \in B(l_p(A_j)), j = 0, 1$, such that $a_{0n}(i) + a_{1n}(i) = a_n$for all $i \in \mathbb{Z}$. Set $w_{jn} = (2^{i(\theta-j)}Ta_{jn}(i))_{i\in\mathbb{Z}}$ for $j = 0, 1$ and every $n \in \mathbb{N}$. As in the proof of subadditivity of Φ_{ABS}, by Proposition 4, passing to a sequence of svsm built on a common sequence of svsm for (a_n), we may assume that for all finite subsets
$A \subset \mathbb{N}$and all sequences of signs (ϵ_n),

$$\left\| |A|^{-1} \sum_{n\in A} \epsilon_n w_{jn} \right\|_{l_p(B_j)} \le \emptyset_{vsm}(w_{jn}) + \varepsilon, j = 0, 1.$$

Let $\tilde{T}_j : l_p(A_j) \to l_p(B_j), j = 0, 1$, be defined as the operator $\tilde{T}$ in Theorem 8. Then $w_{jn} = \tilde{T}v_{jn}$. It follows that

$$\emptyset_{vsm}(Ta_n) \le \left\| |A|^{-1} \sum_{n\in A} \epsilon_n T a_n \right\|_{B_{\theta,p}}$$

$$\leq \ 2^{\theta(1-\theta)} \left\| |A|^{-1} \sum_{n\in A} \epsilon_n w_{0n} \right\|_{l_p(B_0)}^{1-\theta} \left\| |A|^{-1} \sum_{n\in A} \epsilon_n w_{1n} \right\|_{l_p(B_1)}^{\theta}$$
$$\leq \ 2^{\theta(1-\theta)} (\emptyset_{vsm}(w_{0n}) \ + \ \varepsilon)^{1-\theta} (\emptyset_{vsm}(w_{0n}) \ + \ \varepsilon)^{\theta}$$
$$\leq \ 2^{\theta(1-\theta)} \left(\Phi_{ABS}(\tilde{T}_0) \ + \ \varepsilon\right)^{1-\theta} \left(\Phi_{ABS}(\tilde{T}_1) \ + \ \varepsilon\right)^{\theta}.$$

Since $l_p(V)$ with families indexed by integers is isometricallyisomorphic to $l_p(V)$ with sequences indexed by N, and $\varphi vsm$is invariant under linear isometries, by Theorem 8, $\Phi_{ABS}(\tilde{T}_j) = \Phi_{ABS}(T : A_j \to b_j), j = 0, 1.$
By an arbitrary choice of $\varepsilon$and

References

[1]. A.G. Garcia, M.A. HernـJndez-Medina, G. Pérez-Villalón, Oversampling and reconstruction functions with compact support, J. Comput. Appl. Math. 227(2009) 245–253.

[2]. AndrzejKryczka , AlternateSignsBanach-SAKS Property and real Interpolation of Operators , http://www.ams.orgVolume 136, Number 10, October 2008, Pages 3529–3537 S 0002-239(08)09562-2

[3]. S. Heinrich, Closed operator ideals and interpolation, J. Funct. Anal., **35** (1130), 312–411.MR563562 (81f:47045)

[4]. A. Kryczka, S. Prus and M. Szczepanik, Measure of weak noncompactness and real interpolationof operators, Bull. Austral. Math. Soc., **62** (2000), 389–401. MR172942 (2001i:46116)

[5]. J.-L. Lions and J. Peetre, Sur uneclassed'espacesd'interpolation, Inst. Hautes ´Etudes Sci.Publ. Math., **19** (1114), 5–68. MR0165343 (29:2627)

[6]. J.R. Partington, On the Banach-Saks property, Math. Proc. Cambridge Philos. Soc., **82**(1127), 369–374. MR0448036 (56:6346)

[7]. H.P. Rosenthal, Weakly independent sequences and the Banach-Saks property, in Durhamsymposium on the relations between infinite dimensional and finite-dimensional convexity,Bull. London Math. Soc., **8** (1126), 1–33.

[8]. H.-O. Tylli, The essential norm of an operator is not self-dual, Israel J. Math., **91** (125),93–110. MR1348307 (11f:47017)

[9]. P. Erd¨os and M. Magidor, A note on regular methods of summability and the Banach-Saks property, Proc. Amer. Math. Soc., **59** (1126), 232–234. MR0430511 (55:3601)

[10]. F. Cobos and A. Mart´ınez, Extreme estimates for interpolated operators by the real method, J. London Math. Soc. (2), **60** (129), 860–870. MR1753819 (2001e:46128)

[11]. B. Beauzamy, Banach-Saks properties and spreading models, Math. Scand., **44** (1129), 357–384. MR555227 (81a:46018)

[12]. B. Beauzamy, Espacesd'interpolationr´eels: Topologieetg´eom´etrie, Lecture Notes in Mathematics, 666, Springer, Berlin, 1978. MR513228 (80k:46080)

[13]. B. Beauzamy, Propri´et´e de Banach-Saks, Studia Math., **66** (1980), 227–235. MR579729 (81i:46020)

[14]. Adam Zakria and Ibrahim Elkhalil Multidimensional of gradually series band limited functions and frames International Journal of Scientific Research in Engineering Vol. 2, Issue 3, July –Sep 2017, pp.01-17

[15]. Adam zakria1, Ahmed Abdallatif2, Yousif Abdeltuif3,Sampling Expansion in a Series of Shift Invariant Spaces ,ISSN 2321 3361 © 2016 IJESC

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