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Stability Analysis of Prey-Predator in Inconsistent Habitations and Non-Selective Harvesting Factor.

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Abstract: *This study targeted to analyze the relationship and stability between prey and predator Population with respect to impact of variable environment and harvesting factor on prey population. Prey-Predator concatenation is an authoritative and imperative discipline in analyzing and predicting population growth for future purposes and Management. The classical logistic growth model and Brody growth function of prey were applied to derive the corresponding predator's growth functions. Additionally, a coupled prey-predator's logistic model, variable carrying capacity with Holling's type II functional response and non selective harvest factor on prey population applied to account for sigmoid change of environment support. The effects of other model parameters presented using dynamic behavior of equilibrium points on prey-predator dynamics. The stability of the models presented analytically and by numerical simulation.*

Keywords: *Mathematical model, Prey-predator, Holling's type II Functional response, stability, variable carrying capacity, Non-selective Harvesting factor*

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I. Introduction

Prey-Predator population dynamics is indispensable and a key tool in mathematical modeling of ecological dynamics [5]. Ecological and population dynamics modeling have been broadly studied and applied by using differential equations and their corresponding functions such as logistic, Brody, Exponential, Gompertz, Von-Bertalanffy growth function etc [1]&[7].In ecological system living organisms are classified into two groups namely Predator and prey in a particular physical environment. A predator is an organism that eats another organism while the prey is the organism which the predator eats [6]. To mention some of the prey-predators are lion -zebra, lion-gazelles, Bear-fish, fox- rabbit, birds-insects, shark-fish and grasshopper-leaf. These linkages are the prime movers of energy through food chain and food web [10].

Predator-prey relation is an important energetic force to develop the suitability and fitness of both predator and prey in terms of appraisal [4]. The predator-prey relationship continues to be beneficial in forcing both species to adapt and ensure that they feed without becoming a meal for another predator. Predator-prey relationships can be more complex than a straightforward relationship because the species that is the predator or the prey in one circumstance can be the opposite in a relationship with different species. Predator and prey populations counter dynamically to one another when the numbers of prey explode, the abundance at this level of the food chain supports higher numbers of predator population. The dynamics of Prey-predator population with harvesting refers to the most advantageous or optimal management and administration of population and non permanent resources [7].

The fitness of a prey Population and the number of individuals in the Population, chance of being able to reproduce and likelihood of survival is under control of the predator population. However most of the studies consider environmental support to be constant in modeling of population dynamics which is not realistic. Currently environmental change, climate change, resource depletion and Global warming are burning issues worldwide [12].

(a)Predator attack prey for food

(b) Hunting of Prey

The images (a) and (b) above represent how the prey's inhabitants suffer to complete the food chain in their habitat such as Game reserves and National Parks [14]. In this study we formulated the corresponding predator's function from the prey's function, estimate the parameters for sustainable and plausible grow population and the Holling's Type II functional response with the variable carrying capacity used to analyze the stability of population in the given habitat with changing the environmental support. An integrated nonlinear system based on the logistic equation and interaction of population with the surrounding environments and non-selective Harvesting factor applied.

Prey-predator model guided with the following assumptions; (a) In the limited environment support and absence of predators, the prey population will grow logistically. (b) The predator population will starve in the absence of prey population. (c) Both prey and predator populations can move randomly through a homogeneous environment. (d) There is a limited food supply for prey population.

II. Material And Methods

The Proposed Mathematical models

1. Classical Logistic Prey function

The population growth does not depend on density size but also how far the size from its maximum environment supports as described by A Mathematician Verhulst [13]. This study considered that the growth of prey population follows logistic growth function, the logistic growth function used to derive the corresponding predator's growth function. The logistic growth function is given by;

$$P(t) = \frac{K}{1 + Me^{-rt}} \quad (1)$$

Where $P(0) = P_0$ is initial population of prey, $M = \left(\frac{K}{P_0} - 1\right)$, r is a growth rate parameter of the prey population and K is the Maximum environment support of prey Population. The point of inflection of prey growth function occurs at time $t^* = \frac{1}{r}\ln\left(\frac{K}{P_0} - 1\right)$ and at t^* an asymptotic growth function reaches half value of K. i.e. $f(t) = \frac{K}{2}$. The corresponding Predator's population growth function is derived as follows;

$$\frac{dp}{dt} = -bq + apq$$

$$\frac{dp}{q} = (-b + ap)dt$$

$$\ln q = -bt + a\int P(t)dt$$

We have $P(t) = \frac{K}{1+Me^{-rt}}$, by direct substitution gives; $\ln q = -bt + a\int \frac{K}{1+Me^{-rt}} dt$

Transformation of the equation, let $u = 1 + Me^{-rt}$, $du = -rMe^{-rt}dt = -r(u-1)dt$

Thus, $\ln q = -bt - \frac{Ka}{r}\int \left[\frac{1}{u(u-1)}\right] du = -bt - \frac{Ka}{r}\int\left(\frac{1}{u-1} - \frac{1}{u}\right) du.$

$$\ln q(t) = -bt - \frac{Ka}{r}\left[\ln\left(\frac{u-1}{u}\right)\right] + \ln c$$

where c is an integral constant which can be determined as follows;

$$q(t) = ce^{-bt}\left(\frac{1+Me^{-rt}}{Me^{-rt}}\right)^{\frac{Ka}{r}}$$

At $t = 0, q(o) = q_0$ then$c = q_0\left(\frac{M}{1+M}\right)^{\frac{Ka}{r}}$, $q(t) = q_0(P_0)^{\frac{Ka}{r}} Exp\left((Ka - b)t\left[\frac{1+Me^{-rt}}{K}\right]^{\frac{Ka}{r}}\right)$.

Therefore the predator's population growth function at time t is given by;

$$q(t) = q_0(P_0)^{\frac{Ka}{r}} Exp\left((Ka - b)t\left[\frac{1+Me^{-rt}}{K}\right]^{\frac{Ka}{r}}\right) \quad (2)$$

Where a is a birth rate parameter and b is the death rate parameter of the predator's population.

2. Brody growth function.

[Brody,1945] defined as a special case of Richards' growth function when $m = 1$. The study considered that the prey population obeys the Brody function model, from this function we derive the corresponding predator growth function. Richard's growth function is given by; $P(t) = K(1 - Be^{-rt})^m$. For $m = 1$,a function is reduced to Brody growth function.

$$P(t) = K(1 - Be^{-rt}) \quad (3)$$

where r is the growth rate parameter, K is Asymptotic growth of the prey Population and $B = \left(\frac{P_0}{K}\right)$, $P(0) = P_0$,while the relative growth is given by; $r_t = r\left(\frac{K}{P(t)} - 1\right)$.

Recall predator equation,

$$\frac{dp}{dt} = -bq + apq$$
$$\frac{dp}{q} = (-b + ap)dt$$
$$\ln q = -bt + a\int P(t)dt$$

We have $P(t) = K(1 - Be^{-rt})$, direct substitution of a Brody function into the predator equation. ;

$\ln q = -bt + aK\int (1 - Be^{-rt})\, dt$. Integrating and introduce the initial condition at $t = 0, q_0 = e^{\frac{KaB}{r}}$ gives the predator's population growth function as;

$$q(t) = Exp\left(\frac{KaB}{r}\right) Exp\left(-bt + aKt + \frac{KaB}{r} e^{-rt}\right) \qquad (4)$$

3. A Coupled Logistic environment support model

A non-linear system based on the logistic equation that shows the interactions between prey's population and time varying environment in the absence of predator. In this study we consider P to Population density, r growth rate, k Carrying capacity, b_1 development rate, c_1 rate at which the population depletes a carrying capacity through interaction.

$$\frac{dP}{dt} = rP\left(1 - \frac{P}{K}\right)$$
$$\frac{dK}{dt} = b_1 K\left(1 - \frac{c_1}{b_1} P\right)$$

If the carrying capacity is influenced by the population or vice versa, this can be interpreted as an inter-specific relationship between the prey's population and its carrying capacity. Non-dimensional parameters introduced to give the system below.

$$\frac{dp}{d\tau} = p\left(1 - \frac{p}{k}\right)$$
$$\frac{dk}{d\tau} = \mu k(1 - p)$$

Where: $p = \alpha P,\ \alpha = \frac{c_1}{b_1},\ k = \alpha K,\ \tau = rt\ and\ \mu = \frac{b_1}{r}$. The stability of the system can be observed using the given Jacobian matrix at $(p^*, k^*) = (1,1)$ as follows.

$$J = \begin{pmatrix} 1-\frac{2p}{k} & \frac{p^2}{k^2} \\ -\mu k & \mu(1-p) \end{pmatrix}$$

Therefore, the system is stable at (1,1) if $\mu > 0.25$.The decay of population,the resources become richer which allow the population to grow back, this means the system exhibits damped Oscillation until reaches the stable environment.

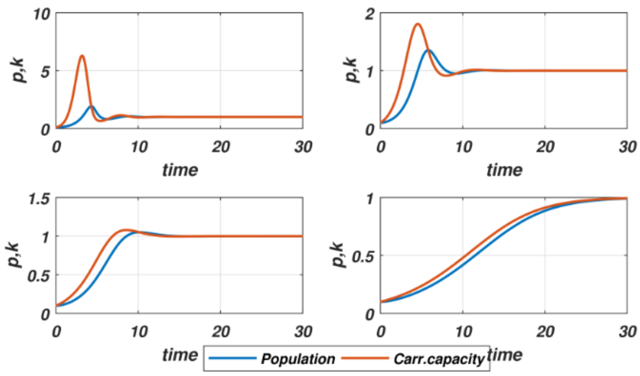


Figure: 1.

Figure 1: Describe the above explanation at initial value of $(p_0, k_0) = (0.1,0.1)$ with different value of $\mu = [2,1,0.5,0.2]$.

4. Coupled Prey-Predator's Logistic with variable carrying capacity, Holling's Type II functional response and non-selective Harvesting factor.

Lotka (1925) described the dynamics of prey-predator using mathematical population model [9] &[11]. According to C.S.Buzz Holling(1959) conducted an experiment on predator's rate of prey Capture (C)named as Holling Type II functional response for vertebrates; $C = \frac{aPq}{1+\gamma P}$,The prey-predator logistic model with Holling's Type II functional response of logistic carrying capacity and non-selective Harvesting factor is given by;

$$\begin{cases} \frac{dP(t)}{dt} = rP(t)\left(1-\frac{P(t)}{K(t)}\right) - \frac{a_1 P(t)q(t)}{1+\gamma P(t)} - \beta H P(t) \\ \frac{dK(t)}{dt} = \alpha(K(t)-K_1)\left(1-\frac{K(t)-K_1}{K_2}\right) \\ \frac{dq(t)}{dt} = \frac{a_2 P(t)q(t)}{1+\gamma P(t)} - cq(t) \end{cases} \quad (5)$$

$P(0) = P_0, K(0) = K_0 \text{ and } q(0) = q_0.$

Numerical Simulation and Discussion of the study

The simulation of the study is designed by changing the model parameters P_0, K *and* $r$for Prey and q_0, a *and* b for predator population.The two models classical Logistic and Brody growth function by including their corresponding derived equation (2) and (4) in Table I. The table II used to simulate proposed model in equations (5).

Case I: $Ka < b$		**Case II:** $Ka > b$		**Case III:** $Ka = b$	
a	b	a	b	a	b
0.0001	0.11	0.001	0.08	0.00025	0.25
0.0001	0.12	0.001	0.07	0.001	1.0
0.0001	0.13	0.001	0.06	0.0001	0.1
0.0001	0.14	0.001	0.05	0.00001	0.01
0.0001	0.15	0.001	0.04	1×10^{-10}	1×10^{-7}

predator's population.
q_o
$1.5K$
$0.5K$
$1.25P_o$
$0.5P_o$

Prey's Population		
K	P_o	r
1000 or 100	200 0r 20	0.1 or 0.01

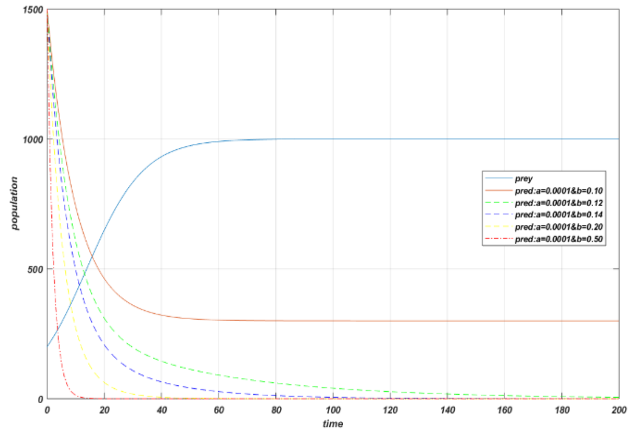
Table:I

Parameters	**Descriptions**	**Approx.Values**
r	Prey's per capital growth rate	0.25
a_1	Decrements factor of prey	0.0025
γ	Handling Parameter	0.002
β	Capability coefficient of Harvesting	1 [7]
H	Hunting effort/catching efficiency of prey	0.029
α	Growth rate of carrying capacity	[0.8,0.4,0.2,0.008]
K_1	First Carrying capacity	300
K_2	Second Carrying Capacity	500
a_2	Increments factor of predator	0.0025
c	Death rate of the predator	0.3
K_0	Initial carrying capacity	500
P_0	Initial Prey's Population	400
q_0	Initial Predator's Population	300

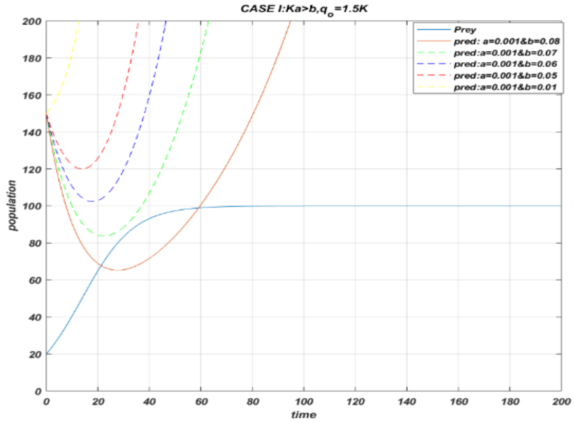
Table: II

III. Result

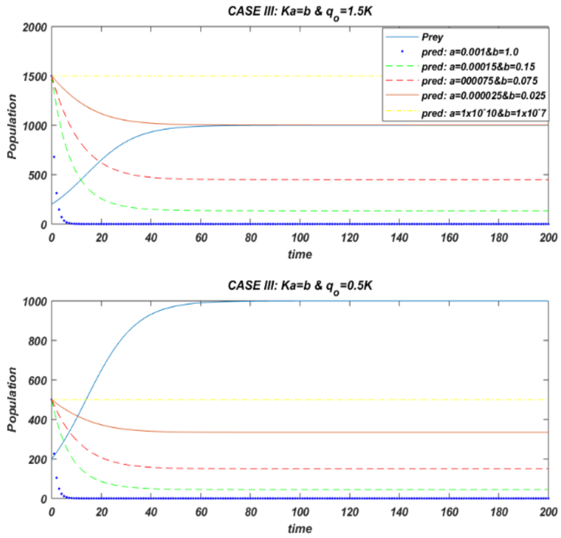
Part One

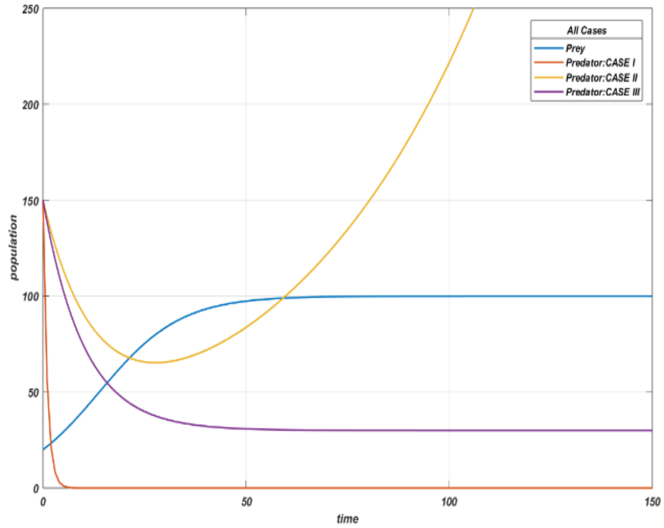


(a) Case I: $Ka < b$



(b) Case II: $Ka > b$





(c) Case III: $Ka = b$ (d) All three cases

Figure 2. Shows the variation and nature of the population at different parameters.

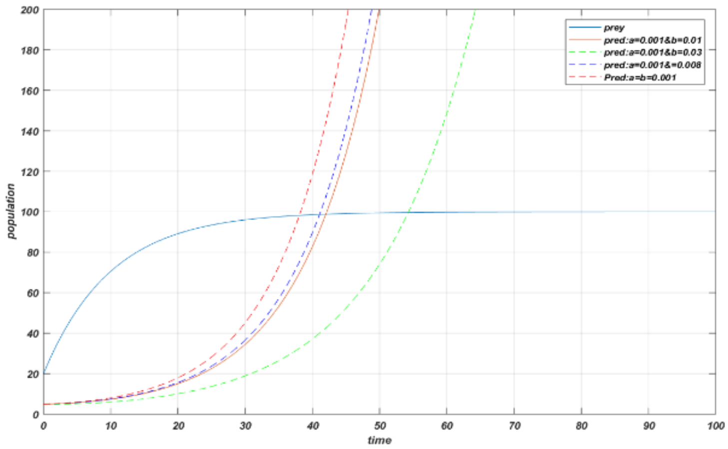


Figure: 3.

Figure 3. Shows the predator's population grow higher, eventually diverge to $+\infty$

Part Two

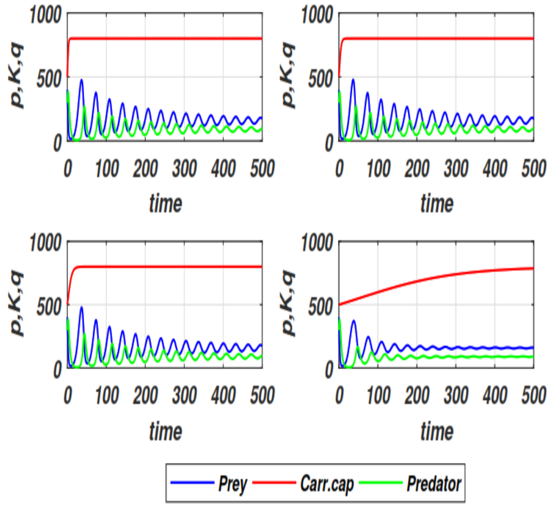


Figure 4(a): Prey-predator and Carrying Capacity

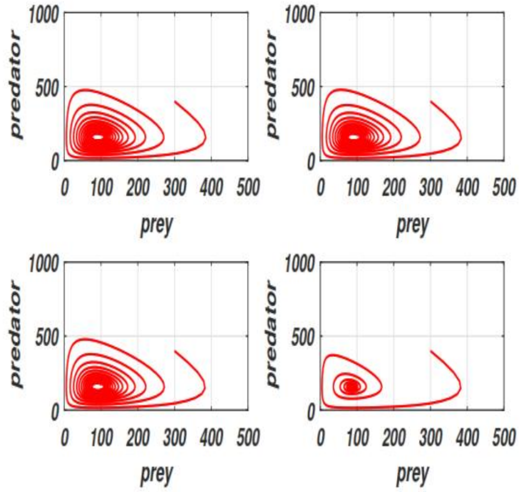


Figure 4(b): Prey-predator Phase plots

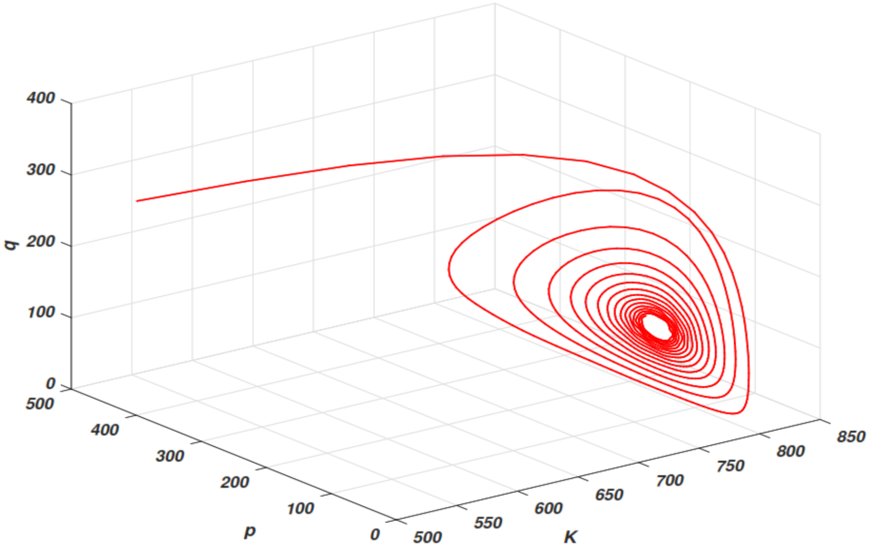


Figure: 5.
Figure: 5. Shows Co-existence against a carrying capacity.

IV. Discussion

(a) Equilibrium points, Nature and Stability in Part one.

Recall,
$$\frac{dP}{dt} = rP\left(1 - \frac{P}{K}\right)$$
$$\frac{dq}{dt} = -bq + aPq$$

From the system above, the equilibrium points found to be $(P_1^*, q_1^*) = (0,0)$ and $(P_2^*, q_2^*) = (K, 0)$, iff $\frac{dP}{dt} = 0$ and $\frac{dq}{dt} = 0$, The Jacobian (J) matrix of the system which used to determine the Eigenvalues given by;

$$J(P,q) = \begin{pmatrix} r\left(1-\frac{2P}{K}\right) & 0 \\ aq & -b+aP \end{pmatrix}$$
$$J(0,0) = \begin{pmatrix} r & 0 \\ 0 & -b \end{pmatrix}$$
$$\begin{vmatrix} r-\lambda & 0 \\ 0 & -b-\lambda \end{vmatrix} = 0$$
$$\lambda_1 = r, \lambda_2 = -b$$

And
$$J(K,0) = \begin{pmatrix} -r & 0 \\ 0 & -b+aK \end{pmatrix}$$
$$\Rightarrow \quad \lambda_3 = -r, \lambda_4 = -b + aK.$$

At equilibrium point $J(P_1^*, q_1^*) = (0,0)$ the eigenvalues are found to be $\lambda_1 = r, \lambda_2 = -b$

This implies that the nature of point is **Unstable.** $J(P_2^*, q_2^*) = (K, 0)$ if $aK < b$, equilibrium point is **Stable.** If $aK > b$, equilibrium point is **Unstable,** and if $aK = b$, it's **Unstable.**

(i): $\boldsymbol{aK} < b$:The predator population decay and eventually to zero while prey population grow logistically as shown in figure.2 (a).

$(\boldsymbol{ii})$: $\boldsymbol{aK} > b$: The prey population remain following logistic curve to maximum environment support K but predator population decay for a while then grow higher finally diverge to infinity as shown in figure.2(b)

$(\boldsymbol{iii})$: $\boldsymbol{aK = b}$:In this case, the growth function for predator is reduced to

$q(t) = q_0(P_0)^{\frac{aK}{r}}\left(\frac{K}{1+Me^{-rt}}\right)^{-\frac{aK}{r}}$, This can be manipulated as $q(t)P(t)^{\frac{aK}{r}} = q_0(P_0)^{\frac{aK}{r}}$.

As $t \to \infty, q(t) = q_0\left(\frac{P_0}{K}\right)^{\frac{aK}{r}}$,this signify that prey population grow logistically while predator population declines to lower the maximum carrying capacity.

(b) Boundedness of the system, equilibrium point, nature and stability in part two.

Since the carrying capacity is increasing sigmoidally, an initial value $K_0 > K_1$ and maximum value of $K_1 + K_2$, the boundedness of $P(t)$ and $q(t)$ can described by using the following lemma.

Lemma: *The positive quadrant integers (xy-plane) is invariant for the system*

Proof: To show that for all $t \in [0, A)$, $P(t) > 0$ and $q(t) > 0$, $A \in (0, \infty)$. Suppose that it is not true, then $there\ exist\ T, 0 < T < A$ such that for all $t \in [0, T), P(t) > 0$ and $q(t) > 0$ and either $P(T) = 0\ or\ q(T) = 0$ for all $t \in [0, T)$.

$$P(t) = P(0)Exp\ \int_0^t \left(r\left(1 - \frac{P(s)}{K}\right) - \frac{a_1 q(s)}{1+\gamma P(s)} - \beta H\right) ds$$

$$q(t) = q(0)Exp \int_0^t \left(\frac{a_2 P(s)}{1+\gamma P(s)} - c\right) ds$$

(P, q) is well defined and continuous on $[0, T)$ there exist $\delta \geq 0$ such that for all $t \in [0, T)$.

$$P(t) = P(0)Exp\ \int_0^t \left(r\left(1 - \frac{P(s)}{K}\right) - \frac{a_1 q(s)}{1+\gamma P(s)} - \beta H\right) ds \geq P(0)Exp(-\delta T)$$

$$q(t) = q(0)Exp \int_0^t \left(\frac{a_2 P(s)}{1+\gamma P(s)} - c\right) ds\ \geq q(0)Exp(-\delta T)$$

By taking limit at $t \to T$ and initials of $P(0)$ and $q(0)$, we have

$P(T) \geq P(0)Exp(-\delta T) > 0$ and $q(T) \geq q(0)Exp(-\delta T) > 0$.which contradicts the fact that either $P(T) = 0\ or\ q(T) = 0$ for all $t \in [0, T)$.

□

The Jacobian matrix of the system of equation (5) used to evaluate the equilibrium points given by;

$$J(P, K, q) = \begin{bmatrix} r - \frac{2rP}{K} - \frac{a_1 q}{1+\gamma P}\left(1 - \frac{\gamma P}{1+\gamma P}\right) - \beta H & \frac{rP^2}{K^2} & -\frac{a_1 P}{1+\gamma P} \\ 0 & \alpha\left(\frac{K_2 + 2K_1 - 2K}{K_2}\right) & 0 \\ \frac{a_2 q}{1+\gamma P}\left(1 - \frac{\gamma P}{1+\gamma P}\right) & 0 & \frac{a_2 P}{1+\gamma P} - c \end{bmatrix}$$

We have the following equilibrium points $E_1(0, K_1, 0)$: is Unstable, $E_2\left(\frac{K_1}{r}(r - \beta H), K_1, 0\right)$::is Unstable, $E_3(0, K_1 + K_2, 0)$: is Unstable, $E_4\left(\frac{K_1+K_2}{r}(r - \beta H), K_1 + K_2, 0\right)$: Conditionally local asymptotic stable, $E_5\left(\frac{c}{a_2 - \gamma c}, K_1, \frac{a_2(K_1(a_2-\gamma c)-c) - K_1(a_2-\gamma c)\beta H}{a_1 K_1 (a_2-\gamma c)^2}\right)$ is Unstable and

$E_6\left(\frac{c}{a_2-\gamma c}, K_1 + K_2, \frac{a_2((K_1+K_2)(a_2-\gamma c)-c) - (K_1+K_2)(a_2-\gamma c)\beta H}{a_1(K_1+K_2)(a_2-\gamma c)^2}\right)$: Conditionally local asymptotic stable.

V. Conclusion

The major motivating dynamics in biological and conservation, world have to do with connections of living organisms. The study found that the dynamical behavior is very sensitive to parameter estimations. The Prey-predator fluctuations can be defined and analyzed mathematically using systems of linear and non-linear ordinary differential equations. From figure 4 (a) Prey and predator dynamics changes from periodic behavior to damped oscillation and (b) represents co-existence of both population which implies asymptotic stable spiral equilibrium. The future research study will incorporate migration and movement of prey and predator under geospatial data simulation and effects of bifurcation parameters.

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