IOSR Journal of Mathematics (IOSR-JM)
e-ISSN: 2278-5728, p-ISSN: 2319-765X. Volume 15, Issue 6 Ser. III (Nov – Dec 2019), PP 60-66
www.iosrjournals.org

Modified Crank-Nicolson Method for Solving One Dimensional Parabolic Equation

C.E. ABHULIMEN AND B.J OMOWO[1]

Department of Mathematics, Ambrose Ali University, Ekpoma, Nigeria
[1]Department of Mathematics, Nasarawa State University, Keffi, Nigeria
Corresponding Author: B.J Omowo

Abstract: *In this paper we developed a Modified Crank-Nicolson scheme for solving parabolic partial differential equations. The paper considers two solution methods for partial differential equations, one analytic and one numerical (finite difference method). The finite difference approximation, Modified Crank-Nicolson scheme, was implemented on the diffusion equation in order to solve it numerically. The aim was to compare exact solutions obtained by a classical method using separation of variables method, with the approximate solutions of Modified Crank-Nicolson method. Solutions of the numerical method were obtained manually since the method is easy and fast. The temperatures at specific time-steps were compared with their analytical result counterpart. The results were tabulated and presented also.*

Keyword: *Partial differential equation, Finite difference methods, Crank- Nicolson Method, Modified Crank-Nicolson Method, Parabolic Equations, Exact Solution.*
Mathematical Subject Classification: 35A20, 35A35, 35B35, 35K05

Date of Submission: 04-11-2019 Date of Acceptance: 20-11-2019

I. Introduction

Parabolic Partial Differential Equations are well known equations mostly, one dimensional heat equation (conduction equations). Problems involving time as one independent variable sometimes lead to this type of equations and it is said to be the simplest type of parabolic diffusion equation. This type of equation plays an important role in a broad range of practical application such as fluid mechanics. Solving parabolic equations is not easy analytically, only few can be solved in such way and the usefulness of these solutions is further restricted to problems involving shapes for which boundary conditions are satisfied. In such cases numerical methods are some of the very few means of solution.

Crank-Nicolson Method for solving parabolic partial differential equations was developed by John Crank and Phyllis Nicolson in 1956. A practical method for numerical solution to partial differential equations of heat conduction type was considered by [1]. [2] Modified the simple explicit scheme and prove that it is much more stable than the simple explicit case, enabling larger time steps to be used. [3] Considered the stability and accuracy of finite difference method for option pricing. The Crank-Nicolson scheme, which is forward time central space (FTCS), According to Kreyszig (1993), the time derivative was replaced by forward difference in time because we have no information for negative 't' at the start. The freedom to experiment with any value of r is one of the reasons the Crank-Nicolson scheme was chosen for this study, even though small values of 'r' yield more accurate results. Because of this unconditional stability and ease of implementation in a computer no matter how small 'r' becomes, in this work, we modified the classical Crank-Nicolson method and use it to solve a parabolic partial differential equations (Heat Equation in one dimension). We also compared the result obtained with the analytical solution. The results are presented in a table in this paper.
There are many exhaustive texts on this subject such as [2], [11] and [13] to mention few.

In this paper we considered the comparison of the modified Crank-Nicolson scheme to the exact solution.

II. Development Of Modified Crank-Nicolson Method

This section presents the method of separation of variable and the Modified Crank-Nicolson method for solving Parabolic Partial differential equations.

Partial differential equations occur frequently in Mathematics, natural science and engineering. These are problems involving rate of change of functions of several variables. For examples

- Advection equation: $\frac{\partial y}{\partial t} + v\frac{\partial x}{\partial x} = 0$
- Heat equation: $\frac{\partial f}{\partial t} = D\frac{\partial^2 f}{\partial x^2}$
- Poisson equation: $-\frac{\partial^2 f}{\partial x^2} - \frac{\partial^2 f}{\partial y^2} = u(x, y)$
- Wave equation: $\frac{\partial^2 f}{\partial t^2} - \frac{\partial^2 f}{\partial y^2} = 0$

In the above equations, *x, y* are space coordinates, *v, D, c,* are real positive constants and *t, x* are often said to be time and space coordinates respectively. The general second order linear partial differential equation with two independent variables and one dependent variable is given by

$$A\frac{\partial^2 f}{\partial x^2} + B\frac{\partial^2 f}{\partial x\partial y} + C\frac{\partial^2 f}{\partial y^2} + D = 0 \qquad (1)$$

Here, A, B, C are functions of independent variables, and x, y, D are functions of x, y, f, $\frac{\partial y}{\partial x}$ and $\frac{\partial f}{\partial y}$. It is important to note that for a parabolic partial differential equation to be parabolic, $b^2 - 4ac = 0$ is required.

The one dimensional heat conduction equation of the form

$$\frac{\partial f}{\partial t} = \frac{\partial^2 f}{\partial x^2} \qquad (2)$$

is a well known example of a parabolic partial differential equation. The solution of these equation is a temperature function u(x, t) which is defined for values of x from 0 to l and for values of t from 0 to ∞. The solution is not defined in a closed domain but advances in an open-ended region from initial values satisfying the prescribed boundary conditions.

2.1 DERIVATION OF MODIFIED CRANK-NICOLSON SCHEME:

In this work, we derive a modified Crank-Nicolson scheme from the classical Crank-Nicolson scheme and use it to solve some problems on parabolic differential equations, we then compare the results obtain from the scheme with the exact solution.

We first show the derivation of the classical Crank-Nicolson scheme using the implicit and explicit scheme

$$\frac{f_{i,j+1} - f_{i,j}}{k} = \frac{f_{i-1,j} - 2f_{i,j} + f_{i+1,j}}{h^2} \qquad (3)$$

and

$$\frac{f_{i,j+1} - f_{i,j}}{k} = \frac{f_{i-1,j+1} - 2f_{i,j+1} + f_{i+1,j+1}}{h^2} \qquad (4)$$

adding (3) and (4) we get

$$\frac{f_{i,j+1} - f_{i,j}}{k} = \frac{1}{2}\left[\frac{f_{i-1,j} - 2f_{i,j} + f_{i+1,j}}{h^2}\right] + \frac{1}{2}\left[\frac{f_{i-1,j+1} - 2f_{i,j+1} + f_{i+1,j+1}}{h^2}\right]$$

solving we have

$$2(f_{i,j+1} - f_{i,j}) = \frac{k}{h^2}(f_{i-1,j} - 2f_{i,j} + f_{i+1,j} + f_{i-1,j+1} - 2f_{i,j+1} + f_{i+1,j+1})$$

let $\frac{k}{h^2} = r$, then

$$2(f_{i,j+1} - f_{i,j}) = r(f_{i-1,j} - 2f_{i,j} + f_{i+1,j} + f_{i-1,j+1} - 2f_{i,j+1} + f_{i+1,j+1})$$

$$2f_{i,j+1} + rf_{i-1,j+1} + 2rf_{i,j+1} - rf_{i+1,j+1} = 2f_{i,j} + rf_{i-1,j} - 2rf_{i,j} + rf_{i+1,j}$$

$$rf_{i-1,j+1} + (2+2r)f_{i,j+1} - rf_{i+1,j+1} = rf_{i-1,j} + (2-2r)f_{i,j} - rf_{i+1,j}$$

or

$$2(1+r)f_{i,j+1} + r[f_{i-1,j+1} - f_{i+1,j+1}] = 2(1-r)f_{i,j} + r[f_{i-1,j} + f_{i+1,j}] \qquad (5)$$

Equation (5) is the Crank-Nicolson method.

2.2 Modified Crank-Nicolson Method:

We derive the modified Crank-Nicolson scheme as follows; we replace the left hand sides of both (3) and (4) by $\frac{f_{i,j} - f_{i,j-1}}{k}$ and add to get

$$\frac{f_{i,j} - f_{i,j-1}}{k} = \frac{1}{2}\left[\frac{f_{i+1,j-1} - 2f_{i,j-1} + f_{i-1,j-1}}{h^2}\right] + \frac{1}{2}\left[\frac{f_{i+1,j} - 2f_{i,j} + f_{i-1,j}}{h^2}\right] \tag{6}$$

$$2(f_{i,j} - f_{i,j-1}) = \frac{k}{h^2}(f_{i+1,j-1} - 2f_{i,j-1} + f_{i-1,j-1} + f_{i+1,j} - 2f_{i,j} + f_{i-1,j})$$

$$2f_{i,j} - 2f_{i,j-1} = \mathrm{r}f_{i+1,j-1} - 2rf_{i,j-1} + rf_{i-1,j-1} + rf_{i+1,j} - 2rf_{i,j} + rf_{i-1,j})$$

where r = $\frac{k}{h^2}$

$$-2rf_{i,j-1} + 2f_{i,j-1} + rf_{i+1,j-1} + rf_{i-1,j-1} = -rf_{i+1,j} + 2f_{i,j} + 2rf_{i,j} - rf_{i-1,j}$$

$$2(1+r)f_{i,j} - r(f_{i+1,j} + f_{i-1,j}) = 2(1-2r)f_{i,j-1} + r(f_{i+1,j-1} + f_{i-1,j-1}) \tag{7}$$

Equation (7) is the modified Crank-Nicolson scheme.

2.3 Stability of Modified Crank-Nicolson Scheme:

Consider the equation (6) given as

$$\frac{f_{i,j} - f_{i,j-1}}{k} = \frac{1}{2}\left[\frac{f_{i+1,j-1} - 2f_{i,j-1} + f_{i-1,j-1}}{h^2}\right] + \frac{1}{2}\left[\frac{f_{i+1,j} - 2f_{i,j} + f_{i-1,j}}{h^2}\right]$$

Worse case solution is given as

$$f_{i,j} = \lambda^{n+1}(-1)^i \tag{8}$$

substituting (8) into (6) to get

$-r\lambda^j(-1)^{i-1}[(-1)^{i-1} + (-1)^{i+1}] + 2(1+r)\lambda^j(-1)^i = r\lambda^{j-1}(-1)^{i-1}[(-1)^{i-1} + (-1)^{i-1}] + 2(1+$
$r\lambda j\!-\!1\ (-1)i$

which gives

$$\lambda[\text{-r (-1)} - 1 + (1+2r) - r(-1) + 1] = r(-1) - 1 + 2(1- r) + r(-1) + 1 \tag{9}$$

the equation (9) above can be written as

$$\lambda = \frac{1-2r}{1+2r}$$

then

$$|\ \lambda = \frac{1-2r}{1+2r}\ | \rightarrow |\lambda| < 1,\ \forall\ \lambda > 0 \tag{10}$$

III. Numerical Examples

In this section, we present some numerical examples of the modified Crank-Nicolson Method and compared the results with the exact solutions.

Example 1: Solve the partial differential equation using Modified Crank-Nicolson Method and compare the results with the exact solutions:

$$\begin{cases} \frac{1}{2}\frac{\partial^2 f}{\partial x^2} = \frac{\partial f}{\partial t}, 0 \le x \le 4 \\ with \quad f(x,0) = x(4-x) \\ amd\ f(0,t) = 0 = f(4,t) \end{cases} \tag{11}$$

Solution:

Let

$$f = XT \tag{12}$$

where X is a function of x and T is a function of t only. Differentiating (12) w.r.t t we have $\frac{\partial f}{\partial t}$ = XT' and $\frac{\partial^2 f}{\partial x^2}$ = X''T. Let XT' = c^2X''T = z^2 and separating variables, we get

$$\frac{X''}{X} = \frac{1}{C^2}\frac{T''}{T} = -z^2$$
$$T' = -c^2\ z^2\ T$$
$$m = -c^2 z^2$$
$$T = c_1 e^{-c^2 z^2 t}$$

Similarly,

$$X'' = -z^2\, x$$
$$m = \pm iz$$
$$X = c_2\, cos zx + c_3\, sin zx$$

from (11), $c^2 = \frac{1}{2}$, substitute for the values of X and T in (12) to get,

$$f = (c_2 \cos zx + c_3 \sin zx)\, c_1 e^{\frac{-z^2 t}{2}} \tag{13}$$

now, put the value of $f(0,t) = 0$, where $x = 0$ in equation (13), to get

$$0 = c_2 c_1 e^{\frac{-z^2 t}{2}}$$

Let $c_2 = 0$, arbitrary constant, we have

$$T = c_1 e^{-c^2 z^2 t}$$

so, $c_1 \neq 0$,

$$f = (c_3 \sin zx)\, c_1 e^{\frac{-z^2 t}{2}} \tag{14}$$

we now put $f(4,t) = 0$, in equation (14), to get

$$0 = c_3 \sin 4z\, c_1 e^{\frac{-z^2 t}{2}} = sin4z = 0 = \sin n\pi$$

then

$$z = \frac{n\pi}{4}$$

Put the value of z in equation (14) so that

$$f = c_3 \sin(\frac{n\pi}{4})x\ \ c_1 e^{(\frac{n\pi}{4})^2 t}$$
$$f = c_1 c_3\, sin(\frac{n\pi}{4})x\ \ e^{\left(\frac{n^2\pi^2}{32}\right)t}$$

where $c_1 c_3 = \sum bn$ therefore,

$$f = \sum b_n\, sin(\frac{n\pi}{4})\, x\, e^{-\left(\frac{n^2\pi^2}{32}\right)t}$$

solving for *bn using Fourier series we have ,*

$$b_n = \frac{2}{l}\int_0^l f(x)\sin(\frac{n\pi x}{l})\,dx$$

$$= \frac{2}{4}\int_0^4 x(4-x)\sin(\frac{n\pi x}{4})\,dx$$

$$b_n = \frac{1}{2}\int_0^4 4x \sin(\frac{n\pi x}{4})\,dx - \frac{1}{2}\int_0^4 x^2\, sin(\frac{n\pi x}{4})\,dx \tag{15}$$

Integration the first part of the R.H.S of (15) we have

$$I_1 = 2\left[-x\frac{cos\left(\frac{n\pi x}{4}\right)}{\left(\frac{n\pi}{4}\right)} + \frac{sin\left(\frac{n\pi x}{4}\right)}{\left(\frac{n\pi}{4}\right)^2}\right]_0^4$$

$$I_1 = -32\frac{cos\ n\pi}{n\pi} \tag{16}$$

Integration second part we have

$$I_2 = \frac{1}{2}\left[\left\{-x^2\frac{cos\left(\frac{n\pi x}{4}\right)}{\left(\frac{n\pi}{4}\right)}\right\} - \frac{8}{\pi}\left\{-x\ \frac{sin\left(\frac{n\pi x}{4}\right)}{\left(\frac{n\pi}{4}\right)}\right\} + \frac{8}{\pi}\left\{\frac{\cos\left(\frac{n\pi x}{4}\right)}{\left(\frac{n\pi}{4}\right)^2}\right\}\right]$$

$$= \frac{1}{2}\left[-16\,\frac{\cos n\pi}{\left(\frac{n\pi}{4}\right)} + \frac{8}{n\pi}\left\{\frac{\cos n\pi}{\left(\frac{n\pi}{4}\right)^2}\right\}\right] - \frac{64}{n^3\pi^3}$$

$$I_2 = \left[-32\,\frac{\cos n\pi}{n\pi} + \frac{64}{n^3\pi^3}\,(\cos n\pi - 1)\right] \quad (17)$$

substituting for (17), (16) in (15) we get,

$$b_n = -32\frac{\cos n\pi}{n\pi} + 32\,\frac{\cos n\pi}{n\pi} - \frac{64}{n^3\pi^3}(\cos n\pi - 1)$$

$$b_n = -\frac{64}{n^3\pi^3}(\cos n\pi - 1)$$

so at $n = 1$, $b_1 = \frac{128}{\pi^3}$ and finally, we have

$$f = \frac{128}{\pi^3} sin(\frac{n\pi}{4})\ x\ e^{-\left(\frac{n^2\pi^2}{32}\right)t} \quad (18)$$

Using Modified Crank-Nicolson method;

we have

$2(1 + r)f_{i,j} - r(f_{i-1,j} + f_{i+1,j}) = 2(1 - r)f_{i,j-1} + r(f_{i-1,j-1} + f_{i+1,j-1})$

at $i = 1, j = 1$

$$2(1 + \tfrac{1}{2})f_{1,1} - \tfrac{1}{2}f_{2,1} - \tfrac{1}{2}f_{0,1} = 2(1 - \tfrac{1}{2})f_{1,0} + \tfrac{1}{2}f_{2,0} + \tfrac{1}{2}f_{0,0}$$

$$2\tfrac{3}{2}f_{1,1} - 0.5f_{0,1} - 0.5f_{2,1} = f_{1,0} + 0.5f_{2,0}$$

$$3f_{1,1} - 0,5f_{2,1} = 5 \quad (19)$$

at $i = 2, j = 1$

$$3f_{2,1} - 0.5f_{1,1} - 0.5f_{3,1} = f_{2,0} + 0.5f_{1,0} + 0.5f_{3,0}$$

$$3f_{2,1} - 0.5f_{1,1} - 0.5f_{3,1} = 7 \quad (20)$$

at $i = 3, j = 1$

$$3f_{3,1} - 0.5f_{2,1} = 5 \quad (21)$$

solving the equations above we have, $f_{1,1} = 2.1765, f_{2,1} = 3.0588, f_{3,1} = 2.1765$ next step is at $i = 1, j = 2$

$$3f_{1,2} - 0.5f_{0,2} - 0.5f_{2,2} = f_{1,1} + 0.5f_{0,1} + 0.5f_{2,1}$$

$$3f_{1,2} - 0.5f_{2,2} - = 3.7059 \quad (22)$$

at $i = 2, j = 2$

$$3f_{2,2} - 0.5f_{1,2} - 0.5f_{3,2} = 5.2353 \quad (23)$$

at $i = 3, j = 2$

$$3f_{3,2} - 0.5f_{2,2} = 3.7059 \quad (24)$$

Table 1: Results of Exact solution and Modified Crank-Nicolson scheme

Exact solution	2.1356	3.0326	2.1356
Modified Crank-Nicolson	2.1765	3.0588	2.1765
i=1, j=2 Modified Crank-Nicolson Method	1.6159	2.2837	1.6159
i=1, j=3 Modified Crank-Nicolson Method	1.2027	1.7008	1.2027

Example 2:
Solve the partial differential equation using modified Crank-Nicolson method

$$\begin{cases} \frac{\partial^2 f}{\partial x^2} = \frac{\partial f}{\partial t} \ , 0 \ \leq x \ \leq 1 \\ with \quad f(x,0) = \sin \pi x \\ amd \ \ f(0,t) = 0 = f(1,t) \end{cases} \qquad (25)$$

If the exact solution of equation (25) is given by $e^{\pi^2 t} \ \sin \pi x$, then from (7) we have at i=1, j=1 and using $r = \frac{k}{h^2}$,

$$2.2f_{1,1} - 0.1f_{2,1} = 0.61498$$

at i=2, j=1

$$2.2f_{2,1} - 0.1f_{3,1} - 0.1f_{1,1} = 1.16984$$

for 3≤ $i \ \leq 10$, we have

$$2.2f_{3,1} - 0.1f_{4,1} - 0.1f_{2,1} = 1.61009$$
$$2.2f_{4,1} - 0.1f_{5,1} - 0.1f_{3,1} = 1.89288$$
$$2.2f_{5,1} - 0.1f_{6,1} - 0.1f_{4,1} = 1.99022$$
$$2.2f_{6,1} - 0.1f_{7,1} - 0.1f_{5,1} = 1.89288$$
$$2.2f_{7,1} - 0.1f_{8,1} - 0.1f_{6,1} = 1.61009$$
$$2.2f_{8,1} - 0.1f_{9,1} - 0.1f_{7,1} = 1.16984$$
$$2.2f_{9,1} - 0.1f_{10,1} - 0.1f_{8,1} = 0.61498$$

solving the above tridiagonal matrix, we have that $f_{1,1}$= 0.3060, $f_{2,1}$= 0.5821, $f_{3,1}$ = 0.8011, $f_{4,1}$ = 0.9418, $f_{5,1}$= 0. 9903, $f_{6,1}$ = 0.9418, $f_{7,1}$= 0.8011, $f_{8,1}$=0.5821, $f_{9,1}$= 0.3060
for next step we have,
at i=1, j=2

$$2.2f_{1,2} - 0.1f_{2,2} = 0.60901$$

at i=2, j=2

$$2.2f_{2,2} - 0.1f_{3,2} - 0.1f_{1,2} = 1.87438$$

for 3≤ $x \leq 10$, we have

$$2.2f_{3,2} - 0.1f_{4,2} - 0.1f_{2,2} = 1.59437$$
$$2.2f_{4,2} - 0.1f_{5,2} - 0.1f_{3,2} = 1.87438$$
$$2.2f_{5,2} - 0.1f_{6,2} - 0.1f_{4,2} = 1.9709$$
$$2.2f_{6,2} - 0.1f_{7,2} - 0.1f_{5,2} = 1.87438$$
$$2.2f_{7,2} - 0.1f_{8,2} - 0.1f_{6,2} = 1.59437$$
$$2.2f_{8,2} - 0.1f_{9,2} - 0.1f_{7,2} = 1.15849$$
$$2.2f_{9,2} - 0.1f_{10,2} - 0.1f_{8,2} = 0.60901$$

on solving, we have that $f_{1,2}$ = 0.3030, $f_{2,2}$= 0.5764, $f_{3,2}$=0.7933, $f_{4,2}$=0.9326, $f_{5,2}$= 0.9802, $f_{6,2}$=0. 9326, $f_{7,2}$=0.7933, $f_{8,2}$=0.5764, $f_{9,2}$ =0.3030
The table below show the results for I = 1,2,…0 and 3 $\leq j \ \leq 8$

Table 2: Results of Modified Crank-Nicolson Solution

i	$f_{i,j=1}$	$f_{i,j=2}$	$f_{i,j=3}$	$f_{i,j=4}$	$f_{i,j=5}$	$f_{i,j=6}$	$f_{i,j=7}$	$f_{i,j=8}$
1	0.3060	0.3030	0.3001	0.2972	0.2943	0.2914	0.2886	0.2858
2	0.5821	0.5764	0.5708	0.5652	0.5597	0.5542	0.5488	0.5435
3	0.8011	0.7933	0.7856	0.7779	0.7703	0.7628	0.7554	0.7480
4	0.9418	0.9326	0.9235	0.9145	0.9056	0.8968	0.8880	0.8793
5	0.9903	0.9806	0.9710	0.9615	0.9521	0.9428	0.9336	0.9245
6	0.9418	0.9326	0.9235	0.9145	0.9056	0.8968	0.8880	0.8793
7	0.8011	0.7933	0.7856	0.7779	0.7703	0.7628	0.7554	0.7480
8	0.5821	0.5764	0.5708	0.5652	0.5597	0.5542	0.5488	0.5435
9	0.3060	0.3030	0.3001	0.2972	0.2943	0.2914	0.2886	0.2858

Table 3: Results of Modified Crank-Nicolson Solution and Exact solution at x=0.5,

t	Modified Crank-nicolson Method	Exact Solution	Error
0.005	0.9521	0.9518	0.0003
0.006	0.9428	0.9425	0.0003
0.007	0.9336	0.9332	0.0004
0.008	0.9245	0.9241	0.0004

IV. Discussion of Results

From table 1 and 3, we will observe that the Modified Crank-Nicolson method is effective for solving parabolic partial differential equations; both tables demonstrate the Modified Crank-Nicolson method performs well, consistent and agree with the exact solution. Better like the classical Crank-Nicolson method, it provides a fast convergence and better accuracy and also requires the solutions of tridiagonal system at every level.

V. Conclusion

On the basis of the above discussion we get the result obtained by analytical methods is always providing accurate solution and the Modified Crank-Nicolson method provides approximate results and fast convergence compared to the classical Crank-Nicolson method. Since it is not possible to solve every partial differential equation analytically so numerical methods providing a good agreement in those cases where solutions do not exist or where Partial differential equations cannot be solve analytically. The results of our method also agree with existing findings in literature that smaller time step produces more accurate results. This can always be achieved when the value of $r = \frac{k}{h^2}$ is kept reasonably small for a close approximation to the solution of the partial differential equation as seen in table 3.

References

[1]. Crank J and Philis N. A practical method for Numerical Evaluation of solution of partial differential equation of heat conduction type. Proc. camb. Phil. soc. 1 (1996), 50-57

[2]. Cooper J. Introduction to Partial differential Equation with Matlab, Boston, 1958

[3]. Duffy D.J Finite difference methods in financial engineering. A Partial differential approach. Wiley, 2006

[4]. DuFort E.C and Franel S.P Conditions in Numerical treatment of Partial differential equations. Math. comput. 7(43) (1953): 135-152

[5]. Emenogu George Ndubueze and Oko Nlia Solutions of parabolic partial differential equations by finite difference methods. Jornal Applied Mathematics, 8(2) (2015): 88-102

[6]. Fadugba S.E, Edogbanya O.H and Zelibe S.C Crank-nicolson method for solving parabolic partial differential equations. International journal of Applied mathematics and mod- elling IJA2M, vol 1, (2013) nos 3. 8-23

[7]. Fallahzadeh A. and Shakibi K. A method to solve Convection-Diffusion equation based on homotopy analysis method. Journal of interpolation and Approximation in scien- tific computing. 1 (2015) pp 1-8

[8]. Febi Sanjaya and Sudi Mungkasi A simple but accurate explicit finite difference method for Advection-diffusion equation,Journal of Phy. Conference series 909, (2017)

[9]. Gerald W. Recktennwald Finite-Difference Approximations to the Heat Equation, Math- ematical Method, 8(34) 2004 pp 747-760

[10]. Isede H.A Several examples of the Crank-Nicolson method for parabolic partial dif- ferential equations. Academia Journal of Scientific Research 1(4) (2013), 063-068

[11]. Kreyszig E. Advanced Engineering Mathematics. USA , John Wiley and sons PP 861-865

[12]. Mitchell A.R and Gridffiths D.F A Finite difference method in partial differential equations, John Wiley and Sons, (1980)

[13]. Williams F. Ames, Numerical methods for Partial differential Equations, Academic press, Inc, Third Edition, 1992

OMOWO B.J " Modified Crank-Nicolson Method for Solving One Dimensional Parabolic Equation." IOSR Journal of Mathematics (IOSR-JM) 15.6 (2019): 60-66.