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Compact Finite Difference Schemes for One, Two and Three Dimensional Helmholtz Equation Using Pade Approximation

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Abstract: *This paper is designed to develop high order compact finite differences schemes for solving Helmholtz equation using Pade approximation.The developed schemes are fourth order in one, two and three dimensional cases. Test problems were conducted to validate the efficiency and accuracy of the schemes and resultsobtained from the proposed schemes are compared with the exact solution , the traditional second order and any other fourth order schemes developed by Crank- Nicolson. The proposed schemes can be applied to any fractional problems. The results obtained have high degree of accuracy than usual second order difference scheme and canbe appliedto any other type of partial differential equations.*

Keywords*: Helmholtz equation, Pade approximation, compact finite difference schemes, fourth order schemes and Numerical experiment.*

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I. Introduction

Helmholtz equation was developed by Herman Von Helmholtz of the form

$$\nabla^2 A + k^2 A = 0 \qquad (1)$$

where ∇^2 is the Laplacian, k is the wave number and A is the amplitude. Helmholtz equation often arises in the study of physical problems involving partial differential equations in both space and time. Linear standing wave is governed by Helmholtz equation. Intensive research has been performed in recent years to developed efficient and accurate numerical schemes for solving Helmholtz equation because of its relevance. In light of the problems caused by non –compact finite difference schemes, it is desirable to develop a class of schemes that are both high order and compact. The emergence and growing popularity of compact schemes have brought a renewed interest towards the finite difference approach. As such a great deal of effort towards numerical approximation of Helmholtz equation using compact finite difference approach can now be seem in literature. A compact FD schemes is one which utilizes grid points located only directly adjacent to the node about which the differences are often taken. These schemes offer higher accuracy even when the grid size is small. They are able to determine the flow with information solely from the nearest neighbors. The major advantage of compact discretization is that it leads to a system of linear equation with coefficient matrix having smaller band – width as compared to non – compact schemes. For solving Helmholtz equation, the high order compact (HOC) discretizations have been utilized in a number of different ways and varieties of specialized techniques have been developed. The pioneering works on Helmholtz equation were done by Lele (1995), presented a paper titled spectral – like resolution of the classical Pade schemes and the flexibility of compact finite difference schemes. Spotz (1995) developed high order compact finite difference schemes for computational mechanics. Singer and Turkel (1998, 2006) developed high order finite difference method for Helmholtz equation and sixth –order accurate finite difference schemes for the Helmholtz equation using Taylor series of expansion. The equations were used to calculate the higher order correction terms. Nabavi et.al (2007) demonstrated A new 9 point sixth order accurate compact finite difference method for the Helmholtz equation. Godehand (2007) presented compact finite difference schemes of sixth order for the Helmholtz equation. Okoro and Owoloko (2010) developed one- way dissection of high order schemes for the solution of 2D Poisson equation. Mohammed and Othman (2010) presented eighth order compact finite differences for one dimensional Helmholtz equation. 0koro and oyakhire (2016) published paper titled compact finite difference schemes for Poisson equation using Pade approximation method. The outline of the paper are as follows, section 2 treats the derivation or formulation compact schemes, section 3 is concerned with numerical analysis, in section 4 deals with teat problems and results analysis and finally conclusion in section 5.

II. Derivation Of Modified Form Of Pade Approximation

Given the function $u(x)$ with Maclaurin's series expansion

$$u(x)=\sum_{i=0}^{\infty}u(x)^{n}, \qquad 0\le x\le 1 \tag{2}$$

$u(x)$ is of order $[M,N]$ which is defined by M/N in Baker (1975). Therefore, Pade approximation of a function $u(x)$ as $h^{n,} \quad ,\Re\partial_x$

$$\Re_{m,n}[u(x)] = \frac{\sum_{i=0}^{M}P_i x^i}{\sum_{i=0}^{N}q_i x^i} \tag{3}$$

Where p_i and q_i are chosen in a way that $\frac{\partial^i\Re}{\partial x^i}\Big|_x=0=\frac{\partial^i f}{\partial x^i}\Big|_x=0$. It is constructed in such a way that $\left[\frac{M}{N}\right]u(x)$ can be expressed in matrix form. One can argue that there are an infinite number of finite difference schemes which may be derived on the bases of Pade approximation. Deriving finite difference schemes of order $o(h)$ with approximation $\Re_{m,n}$ where $m+n+2>n$. And keeping only the terms in the resulting expression which are $o(h)$ will lead to increasingly worse results the larger the $m+n$ .If $(U_{\alpha\alpha})_{i,j,k}$ is an approximation to the second partial derivatives with respect to the co-ordinates directed α and where (i,j,k) denoted three dimensional lattices indices, (i, j) two dimension and i one dimensional lattices indices respectively. The simplest approximation can be obtained by

$$(U_{\alpha\alpha})_{i,j,k}=\frac{1}{h_\alpha^2}\partial_\alpha^2 u_{i,j,k} \tag{4}$$

where h_α is the grid spacing in the direction α and ∂_α^2 is the second – order difference.

$$\delta_\alpha^2=u_{i-1,j,k}-2u_{i,j,k}+u_{i+1,j,k.} \tag{5}$$

The explicit expression in terms of $u_{i,j,k}$ reads for the x - component

$$(u_{xx})_{ijk}=-\frac{1}{h_x^2}\frac{1}{12}(u_{i-2,j,k}-16u_{i-1,j,k}+30u_{i+1,j,k}-16u_{i+1,j,k}+u_{i+2,j,k})\,. \tag{6}$$

If $\Lambda_{i,j,k}$ is defined as the difference scheme and $h_x=h\forall\alpha,$ the resulting 13- points stencil can be written as

$$\Lambda_{i,0,k}=\frac{1}{12h^2}\begin{pmatrix} 0 & 0 & 1 & 0 & 0\\ 0 & 0 & -16 & 0 & 0\\ -1 & -16 & 90 & -16 & -1\\ 0 & 0 & -16 & 0 & 0\\ 0 & 0 & 1 & 0 & 0 \end{pmatrix},$$

$$\Lambda_{i,\pm 1} = \frac{1}{12h^2}\begin{matrix} & & 0 & 0 & 0 & 0 & 0 \\ & & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & & & 16 & 0 & 0 \\ & & 0 & 0 & 0 & 0 & 0 \\ & & 0 & 0 & 0 & 0 & 0 \end{matrix}$$

$$\Lambda_{i,\pm 1,k} = \frac{1}{30h^2}\begin{pmatrix} 1 & 3 & 1 \\ 3 & 14 & 3 \\ 1 & 3 & 1 \end{pmatrix}$$

The source term function is thereby modified to

$$f \to f + \frac{h^2}{12}\left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}\right) + \frac{h^4}{360}\left(\frac{\partial^4 f}{\partial x^4} + \frac{\partial^{4f}}{\partial y^4} + \frac{\partial^4 f}{\partial z^4}\right)$$
$$+ \frac{h^4}{180}\left(\frac{\partial^4 f}{\partial x^2 \partial y^2} + \frac{\partial^4 f}{\partial x^2 \partial z^2} + \frac{\partial^4 f}{\partial y^2 \partial z^2}\right). \quad (7)$$

Pade approximation to the bracketed expression in equation (9) will be used to derive difference form of compact stencil forHelmholtz equations. The terms compact will be used in the numerical schemes, which need less neighbor grid points than the straight – forward expansion approach neighbor of equation (9).The proposed method has the advantage of flexibility and highaccuracy comparedwith other classical finite difference formulations neighbors, which cause problems at the boundary.

III. Numerical Analysis:

Considering the function

$$\frac{\partial^2 u}{\partial x^2}\Big|_{x+ih} = \frac{4}{h_\alpha^2}\left[\sinh^{-1}\left(\frac{\partial}{2}\right)\right]^2 \quad (8)$$

The expansion of equation (8) yields

$$= \frac{1}{h_\alpha^2}\partial_\alpha^2\left\{1 - \frac{1}{12}\partial_\alpha^2 + \frac{1}{90}\partial_\alpha^4 - \frac{1}{560}\partial_\alpha^6 + \frac{1}{3150}\partial_\alpha^8 - \frac{1}{16632}\partial_\alpha^{10} \pm \ldots\right\}u_{i,j,k} \quad (9)$$

One dimensional case:

Considering one dimensional Helmholtz equation of the form

$$\Delta u(x) + k^2 u(x) = f(x), \qquad x \in \Omega \quad (10)$$

A Fourth –order accurate scheme may be derived from equation (9) when considering only the first two terms in the expansion and substituting in equation (3)

A $\wp_{[0,2]}$ *compact fourth -order schemes*

$$\left(U_{\alpha\alpha}\right)_i = \frac{1}{h_\alpha^2}\partial_\alpha^2\left(1 + \frac{1}{12}\partial_\alpha^2\right)^{-1} \quad (11)$$

where $A_X = \frac{1}{h_\alpha^2}\partial_\alpha^2$

This yields symbolically

$$(D_{[0,2]_\alpha}^{-1} A_X + k^2)u_i = -f_i \qquad (12)$$

Applying the symbolic operators for the fourth –order compact approximation as

$$\left((D_{[0,2]_\alpha}^{-1} A_x + k^2)u_i\left(D_{[0,2]}\right)\right) = \left(D_{[0,2]}\right) - f_i \qquad (13)$$

Multiply both sides of equation (13) by D_x which is $\left(1+\frac{1}{12}\right)\partial x^2$

$$\left\{\left(1+\frac{1}{12}\partial_x^2\right)^{-1}\left(1+\frac{1}{12}\partial_x^2\right) + k^2\left(1+\frac{1}{12}\partial_x^2\right)\right\}u_i = \left(1+\frac{1}{12}\partial_x^2\right) - f_i$$

(14)Keeping terms on the Left hand side LHS up to fourth –order equation (14) may be written finally as

$$\left\{\partial x^2 + k^2\left(1+\frac{1}{12}\partial x^2\right)\right\}u_i = -\left(1+\frac{1}{12}\partial x^2\right)f_i \qquad (15)$$

. Since the operator commute this leads to

$$\frac{1}{h_x^2}\left\{D_{[0,2],X} + k^2(D_{[0,2]})\right\}u_i = -g_i \qquad (16)$$

where an effective term g_i was introduced

$$g_i = -\left(1+\frac{1}{12}\right)\partial x^2 \qquad (17)$$

$$\partial_\alpha^2 = \frac{\delta_\alpha^2}{1+\frac{1}{12}\delta_\alpha^2} \qquad (18)$$

Recall from equation (1) and applying the fourth order schemes yields

$$\delta_\alpha^2 = -k^2 u - \frac{k^2}{12}\delta_\alpha^2 u + f + \frac{1}{12}\delta_\alpha^2 f \qquad (19)$$

$$\delta_\alpha^2 + k^2(1+\frac{1}{12}\delta_\alpha^2)\,u_i = (1+\frac{1}{12}\delta_\alpha^2) - f_i \qquad (20)$$

Note that $\delta_\alpha^2 u = \frac{u_{i+1} - 2u_i + u_{i-1}}{h^2}$ and $\delta_\alpha^2 f = \frac{f_{i+1} - 2f_i + f_{i-1}}{h^2}$ (21)

Substituting expressions (21) into equation (20), yields

$$u_{i+1} - 2u_i + u_{i-1} + k^2h^2\left(1+\frac{h^2}{12}\left(\frac{u_{i+1} - 2u_i + u_{i-1}}{h^2}\right)\right) + h^2\left(1+\frac{h^2}{12}\left(\frac{f_{i+1} - 2f_i + f_{i-1}}{h^2}\right)\right) \qquad (22)$$

Simplifying equation (22) a system of N linear equations are obtained for $i = 0$

$$a_{41}u_1 + a_{42}u_{-1} - \left(2 + k^2h^2(1-\frac{h^2}{6}\right)a_{40}u_0 = h^2\left(1-\frac{h^2}{6}\right)b_0 f_0 + \frac{h^2}{12}b_{11}\left(f_1 + f_{-1}\right) \qquad (23)$$

where

$$a_{40} = -2 + k^2h^2\left(1-\frac{h^2}{6}\right), \quad a_{41} = 1+\frac{k^2h^2}{12}, \quad a_{42} = 1-\frac{k^2h^2}{12}, b_0 = h^2\left(1-\frac{h^2}{6}\right), \quad b_{11} = \frac{1}{12}, \ b_{12} = \frac{1}{12}$$

Two dimensional cases:

The two dimensional case Helmholtz equation is of the form

$$\Delta u(x,y) - k^2 u(x,y) = f(x,y)$$
$$u(x,y) = u_0(x,y), \qquad x,y \in \partial\Omega \tag{24}$$

With the discretized form as

$$(u_{\alpha\alpha})_{i,j} + (u_{\alpha\alpha})_{i,j} + k^2 u_{i,j} = f_{i,j} \tag{25}$$

Where i, j denote the two dimensional lattice indices and $(u_{\alpha\alpha})_{i,j}$ is an approximation to the second partial derivatives with respect to the coordinates direction x.

A $\wp_{[0,2]}$ *compact fourth -order schemes*

Considering equation. (9) approximated by an [$0,2$],Pade approximation through

$$(u_{\alpha\alpha})_{i,j} = \frac{1}{h_\alpha^2}\partial_\alpha^2\left(1+\frac{1}{12}\partial_\alpha^2\right)^{-1} \tag{26}$$

where in equation (9) the operator $L_{[0,2],\alpha}$ was defined. Inserting this approximation in equation (25), it can be written as

$$\frac{1}{h_\alpha^2}\partial_\alpha^2\left(1+\frac{1}{12}\partial_\alpha^2\right)^{-1} \tag{27}$$

Recall that $D_{[0,2]}^{-1} = \frac{1}{h_{\alpha=x,y}}\partial_\alpha^2\left(1+\frac{1}{12}\partial_\alpha^2\right)^{-1}$ and $D_{[0,2]} = \frac{1}{h_{\alpha=x,y}}\partial_\alpha^2\left(1+\frac{1}{12}\partial_\alpha^2\right)$ (28)

This yields symbolically

$$\left\{\left(D_{[0,2]_X}^{-1} A_x + D_{[0,2]_Y}^{-1} A_Y + k^2\right)u_{i,j}\right\} = -f_{i,j} \tag{29}$$

Applying the symbolic operators for the fourth –order compact approximation, we have

$$\left\{\left(1+\frac{1}{12}\partial y^2\right)^{-1}\partial x^2 + \left(1+\frac{1}{12}\partial x^2\right)^{-1}\partial y^2 + k^2\left(1+\frac{1}{12}\partial x^2\right)\left(1+\frac{1}{12}\partial y^2\right)\right\}u_{i,j}$$
$$= \left\{\left(1+\frac{1}{12}\partial x^2\right)\left(1+\frac{1}{12}\partial y^2\right)\right\} - f_{i,j} \tag{30}$$

Simplifying equation (30), we obtained

$$\left\{\partial x^2 + \partial y^2 + \frac{1}{6h^2}\partial x^2\partial y^2 + k^2\left(1+\frac{1}{12}(\partial x^2+\partial y^2)+\frac{1}{144}\partial x^2\partial y^2\right)\right\}u_{i,j}$$
$$= -\left\{1+\frac{1}{12}(\partial x^2+\partial y^2)+\frac{1}{144}\partial x^2\partial y^2\right\}f_{i,j} \tag{31}$$

Both sides of equation (31) are multiplied by D_x, D_y which are $\left(1+\frac{1}{12}\right)\partial x^2$ and $\left(1+\frac{1}{12}\right)\partial y^2$. Since the operators commute this leads to

$$\frac{1}{h_\alpha^2}\left\{\partial y^2 D_{[0,2]x} + \partial x^2 D_{[0,2]_y} + k^2 D_{[0,2]_{X,Y}}\right\}u_{i,j} = -g_{i,j} \tag{32}$$

where an effective source term $g_{i,j}$ was introduced.

$$g_{i,j} = -\left\{1+\frac{1}{12}\left(\partial x^2+\partial y^2\right)+\frac{1}{144}\partial x^2\partial y^2\right\} \tag{33}$$

Keeping terms on the left hand side LHS up to fourth – order equation (32) may be written finally as

$$\frac{1}{h^2}\left\{\partial x^2+\partial y^2+\frac{1}{6h^2}\partial x^2\partial y^2+k^2\left(1+\frac{1}{12}(\partial x^2+\partial y^2)+\frac{1}{144}\partial x^2\partial y^2\right)\right\}u_{i,j}$$
$$=-\left\{1+\frac{1}{12}(\partial x^2+\partial y^2)+\frac{c_0}{144}\partial x^2\partial y^2\right\}f_{i,j} \tag{34}$$

where

$$\partial x^2\partial y^2=\frac{1}{h^4}\left\{4u_{i,j}-2\left(u_{i-1,j}+u_{i+1,j}+u_{i,j+1}+u_{i,j+1}\right)+u_{i-1,j-1}+u_{i+1,j-1}+u_{i-1,j+1}+u_{i+1,j+1}\right\} \tag{35}$$

This can be written explicitly as

$$\frac{1}{h^2}\left\{-\frac{10}{3}u_{i,j}+\frac{2}{3}\left(u_{i-1,j}+u_{i+1,j}+u_{i,j-1}+u_{i,j+1}\right)+\frac{1}{6}\left(u_{i-1,j-1}+u_{i-1,j+1}+u_{i+1,j-1}+u_{i+1,j+1}\right)\right\}=-g_{i,j} \tag{36}$$

and in stencil notation, we obtained

$$\Lambda_{i,j}=\frac{1}{6h^2}\begin{pmatrix}1 & 4 & 1\\ 4 & -20 & 4\\ 1 & 4 & 1\end{pmatrix},\quad \Gamma_{i,j}=\frac{1}{144}\begin{pmatrix}c_0 & -12+2c_0 & c_0\\ -12+2c_0 & 48-4c_0 & -12+2c_0\\ c_0 & -12+2c_0 & c_0\end{pmatrix} \tag{37}$$

Three dimensional cases:

$$\Delta u(x,y,z)-k^2u(x,y,z)=f(x,y,z)$$
$$u(x,y,z)=u_0(x,y,z),\qquad x,y,z\in\delta\Omega \tag{38}$$

With the discretized form as

$$(u_{\alpha\alpha})_{i,j,k}+(u_{\alpha\alpha})_{i,j,k}+k^2u_{i,j,k}=f_{i,j,k} \tag{39}$$

A $\wp_{[0,2]}$ *compact fourth -order schemes*

Considering equation. (9) approximated by an [0,2],Pade approximation through

$$\left(u_{\alpha\alpha}\right)_{i,j,k}=\frac{1}{h_\alpha^2}\partial_\varepsilon^2\left(1+\frac{1}{12}\partial_\alpha^2\right)^{-1} \tag{40}$$

where in equation (9) the operator $D_{[0,2],\alpha}$ was defined as

$$D_{[0,2]}=\frac{1}{h_{\alpha=x,y,z}}\partial_\alpha^2\left(1+\frac{1}{12}\partial_x^2\right),D_{[0,2]}=\frac{1}{h_{\alpha=x,y,z}}\partial_\alpha^2\left(1+\frac{1}{12}\partial_y^2\right)\ D_{[0,2]}=\frac{1}{h_{\alpha=x,y,z}}\partial_\alpha^2\left(1+\frac{1}{12}\partial_z^2\right) \tag{41}$$

Inserting this approximation in equation (38) symbolically yields

$$\left\{\left(D_{[0,2]_x}^{-1}A_x+D_{[0,2]_y}^{-1}A_y+D_{[0,2],z}^{-1}A_z+k^2\right)u_{i,j,k}\right\}=-f_{i,j,k} \tag{42}$$

Or in compact form as

$$\left\{\sum_{\alpha=x,y,z}\frac{1}{h_\alpha^2}\delta_\alpha^2D_{[0,2],\alpha}^{-1}+k^2(D_{[0,2]}^{-1})\right\}u_{i,j,k}=-f_{i,j,k} \tag{43}$$

Applying the symbolic operators for the fourth –order compact approximation and multiplied both sides of equation (43) D_x,D_y,D_z which are $\left(1+\frac{1}{12}\right)\partial x^2$, $\left(1+\frac{1}{12}\right)\partial y^2$ and $\left(1+\frac{1}{12}\right)\partial z^2$

$$\left\{\begin{array}{l}\left(1+\frac{1}{12}\partial y^2\partial z^2\right)^{-1}\partial x^2+\left(1+\frac{1}{12}\partial x^2\partial z^2\right)^{-1}\partial y^2\\ +\left(1+\frac{1}{12}\partial x^2\partial y^2\right)^{-1}\partial z^2+k^2\left(1+\frac{1}{12}\partial x^2\right)\left(1+\frac{1}{12}\partial y^2\right)\left(1+\frac{1}{12}\partial z^2\right)\end{array}\right\}u_{i,j,k} \quad (44)$$
$$=-\left(1+\frac{1}{12}\partial x^2\right)\left(1+\frac{1}{12}\partial y^2\right)\left(1+\frac{1}{12}\partial z^2\right)f_{i,j,k}$$

Simplifying equation (44) gives

$$\left\{\begin{array}{l}\partial x^2+\partial y^2+\partial z^2+\frac{1}{4h^2}\partial x^2\partial y^2\partial z^2\\ +k^2\left(\begin{array}{l}1+\frac{1}{12}\left(\partial x^2+\partial y^2+\partial z^2\right)\\ +\frac{1}{144}\left(\partial y^2\partial z^2+\partial x^2\partial z^2+\partial x^2\partial y^2\right)+\frac{1}{1728}\partial x^2\partial y^2\partial z^2\end{array}\right)\end{array}\right\}u_{i.j.k}$$
$$=-\left\{\begin{array}{l}1+\frac{1}{12}\left(\partial x^2+\partial y^2+\partial z^2\right)\\ +\frac{1}{144}\left(\partial y^2\partial z^2+\partial x^2\partial z^2+\partial x^2\partial y^2\right)\\ +\frac{1}{1728}\left(\partial x^2\partial y^2\partial z^2\right)\end{array}\right\}f_{i,j,k} \quad (45)$$

Since the operators commute this leads to

$$\frac{1}{h_\alpha^2}\left\{\partial y^2 D_{[0,2]x}+\partial x^2 D_{[0,2]_y}+\partial z^2 D_{[0,2]Z}+k^2 D_{[0,2]}\right\}u_{i,j,K}=-g_{i,j,K} \quad (46)$$

where an effective source term $g_{i,j,k}$ was introduced.

$$g_{i,j,k}=-\left\{1+\frac{1}{12}\left(\partial x^2+\partial y^2+\partial z^2\right)+\frac{1}{144}\left(\partial y^2\partial z^2+\partial x^2\partial z^2+\partial x^2\partial y^2\right)+\frac{1}{1728}\left(\partial x^2\partial y^2\partial z^2\right)\right\} \quad (47)$$

Keeping terms on the left hand side LHS up to fourth – order equation (45) may be written finally as

$$\frac{1}{h^2}\left\{\begin{array}{l}\partial x^2+\partial y^2+\partial z^2+\frac{1}{4}\partial x^2\partial y^2\partial z^2\\ +k^2\left(\begin{array}{l}1+\frac{1}{12}\left(\partial x^2+\partial y^2+\partial z^2\right)\\ +\frac{1}{144}\left(\partial y^2\partial z^2+\partial x^2\partial z^2+\partial x^2\partial y^2\right)+\frac{1}{1728}\partial x^2\partial y^2\partial z^2\end{array}\right)\end{array}\right\}u_{i.j.k}$$
$$=-\left\{\begin{array}{l}1+\frac{1}{12}\left(\partial x^2+\partial y^2+\partial z^2\right)\\ +\frac{1}{144}\left(\partial y^2\partial z^2+\partial x^2\partial z^2+\partial x^2\partial y^2\right)\\ +\frac{1}{1728}\left(\partial x^2\partial y^2\partial z^2\right)\end{array}\right\}f_{i,j,k} \quad (48)$$

where $\partial x^2 \partial y^2 \partial z^2 = \frac{1}{h^8} + \begin{bmatrix} -8u_{i,j,k} + 4(u_{i+1,j,k} + u_{i,j+1,k} + u_{i,j-1,k} + u_{i-1,j,k} + u_{i,j,k+1} + u_{i,j,k-1}) \\ -2\begin{pmatrix} u_{i+1,j+1,k} + u_{i+1,j-1,k} + u_{i-1,j+1,k} + u_{i-1.j-1,k} + u_{i+1,j,k+1} + u_{i+1,j,k-1} \\ + u_{i,j+1,k+1} + u_{i,j+1,k-1} u_{i,j-1,k+1} + u_{i,j-1,k-1} + u_{i-1,j,k+1} + u_{i-1,j,k-1} \end{pmatrix} \\ + u_{i+1,j+1<k+1} + u_{i+1,j+1,k-1} + u_{i+1,j-1,k+1} + u_{i+1,j-1,k-1} \\ + u_{i-1,j+1,k+1} + u_{i-1,j+1,k-1} + u_{i-1,j-1,k+1} + u_{i-1,j-1,k-1} \end{bmatrix}$ (49)

IV. Set -Up Of The Numerical Experiments

Three standard measures of convergence, defined as follow: Let u^0, u, u^n denote the initial estimate, the urrent iterate and the analysis solution respectively.
The measure are

* rel- res (the relative residual $\|Au-b\|_2 / \|Au_0 - b\|_2$
* $L_2 - err$ (the relative L_2 error is $\|u^* - u\|_2 / \|u^*\|_2$
* Max-err denote the maximal component wise error $Max_i |u_i^* - u_i|$

Two and three dimensional problems are considered with nonzero on the right hand side and known analytic solutions. The example use the same variable k , with three parameters a, b,c
$k(x) = a - b\sin(x)$ with $a > b \geq 0$

Thus, a and b control he range of value of k and $|c|$ control the number of oscillation of k in the domain.

An example with a known exact solution is chosen in order to show the performance of the high order compact schemes developed in section 3 using compact programs that implement these schemes. Testing is conducted on the unit interval $[0,1]$ with a uniform mesh size h , and boundary condition are prescribed on ends of the until interval. The computation were performed in a MATLAB environment using version 7.and was executed on Pentium at 1.86 $GH_{Z,}$ RAM 1 GB. The computed solutions and the exact solution are compared with the use of ℓ^2 - norm of the error vector which is defined for $\ell = (\ell_1, \ell_2, \ell_3, \ldots, \ell_M)$ as

$$Error = (|u_{exact} - u_{approx}|_{i,j})$$

$$E = \sqrt{\frac{\sum_{i=1}^{N}\sum_{j=1}^{N}(U_{i,j} - u_{i.,j})^2}{(N)^2}}$$

Example 1: Considering 2D Helmholtz equation with the following analytic solution
$u(x,y) - e^{-\frac{k(x)}{c}} \sin(\beta y)$, where $\beta = \sqrt{a^2 + b^2}$
defined over the domain $D = \pi \ x \ \pi$,

Table 1:**Computational results using the developed compact finite difference schemes for second order and fourth order**.

N	2nd Order		4th Order	
	k	$L_2 - err$	k	$L_2 - err$
40	6.2996	$1.33x10^{-1}$	5-.7435	$8.47x10-4$
80	10	$1.21x10^{-1}$	10	$8.87x10-4$
120	13.1037	$1.23x10^{-1}$	13.8316	$9.07x10-4$
160	15.8740	$1.16x10^{-1}$	17.4116	$8.90x10-4$
200	18.4202	$1.19x10-1$	20.8138	$9.09x10-4$
240	20.3008	$1.14x10-1$	24.0830	$8.97x10-4$
280	23.0522	$1.20x10-1$	27.2430	$8.94x10-4$

320	25.1984	1.21x10−1	30.3143	8.95x10−4
	N=2.5298x $k^{\frac{3}{2}}$		N=4,4987x $k^{\frac{5}{4}}$	

Table 1: shows $L_2 -$ error obtained with the standard second order schemes and the fourth order schemes for different values of N and K,

Table 2: Comparison of the proposed fourth order schemes with any other fourth order schemes using finite difference methods.

N	K	$L_2 - error \quad for$ $Crank \quad Niclson$	$L_2 - error \quad for$ $Newc \quad schemes$
40	5.5204	4.49x10^{-6}	1.95x10^{-6}
80	10	5.13x10^{-6}	1.81x10^{-6}
120	14.1558	5.47x10^{-6}	1.42x10^{-6}
160	18.1145	4.49x10^{-6}	1.44x10^{-6}
200	21.9327	5.42x10^{-6}	1.59x10^{-6}
240	25.6425	5.43x10^{-6}	1.59x10^{-6}
280	29.2647	5.56x10^{-6}	1.36x10^{-6}
320	32.8134	5.45x10^{-6}	1.57x10^{-6}
$N = 5.4503x \;\; k^{\frac{7}{6}}$			

Table 2: shows the comparison of any fourth order schemes with the new compact finite difference schemes developed in this article

V. Conclusion

Derivation of fourth - order compact finite difference schemes method for 1D, 2D and 3DHelmholtz equation using Pade approximation was developed.Numerical experiments were conducted to test the validity, accuracy, efficiency and the robustness of the schemes. Matlab software was utilizes in this work. Computational experiment verified that the new compact finite difference schemes is much more efficient than the second-order scheme and the Crank- Nicolson method. Since the scheme is compact, no extra numerical boundary conditions are needed The proposed schemes can be applied to mathematical physics and computational finance with Neumann boundary condition because the schemes derived are flexible in terms of application to complex geometries and boundary condition when compared to other several methods like the traditional finite difference schemes that are non- compact.

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