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Homotopy Analysis and AdomianDecomposition Methods for Approximating the Solution of Two Dimensional Partial Integro-Differential Equations of Fractional Order

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Abstract:*This work presents two computational semi-analytic methods for solving a class of two dimensional partial-intgro differential equations of fractional order.The fractional order derivative is described in Caputo sense. First, existence of a unique solution under certain condition is proved.Then,the ADM and HAM are employed to derive a general solution for the mentioned problem and the convergence analysis of the proposed methods is investigated.Finally, some examples are included for demonstrated the efficiency of the proposed methods.*

Keywords: *Caputo fractional order derivative, partial Integro-differential equations, Homotopy Analysis, Adomian decomposition Method.*

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I. Introduction

The subject of fractional calculus (integrals and derivatives of any arbitrary order) has gained considerable popularity and importance during the past three decades or so, due mainly to its demonstrated applications in numerous seemingly diverse and wide spread fields of science and engineering. It does provide several potentially useful tool for solving differential and integral equations, and various other problems involving special functions of mathematical physics, as well as, their extensions and generalizations from one variable to more variables [1].

Many problems can be treated by fractional Integro-differential equations from different sciences applications. In fact, most mathematical problems are hard to solve analytically, and therefore finding an approximate solution, by investigating several numerical methods, would be very convenient. In recent years, several numerical methods have been applied to solve fractional differential equations (FDEs) and fractional Integro-differential equations (FIDEs).

In this paper, the semi-analytic methods which are so called HAM and ADM will be applied together to solve fractional order partial integro differential equations(FPIDEs).

In [3], Kumer presents a comparative study among three numerical schemes such as linear, quadratic and quadratic-linear scheme of the fractional integro-differential equation. In [7],Singh study the numerical solution of nonlinear weakly singular partial integro-differential equation via operational matrices, he proposed and analyze an efficient matrix based on shifted Legender polynomials for the solution of non-linear volterra singular partial integro-differential equations (PIDEs).

In [2],Hecht was presenteda finite difference for one system of nonlinear integro-differential equations.In [8],Yanghave been introduced an exponential variance Gamma method to the valuation of convertible bond pricing with partial integro-differential equation. In [6], Ray was discussed for finding an approximate solution of two-dimensional wavelets operational method for solving volterra weakly singular partial integro-differential equations.

In this paper, HAM and ADM will be applied for solving partial integro-differential equations with fractional order derivative given by the following formula

$$^{C}D_t^{\alpha}\mathrm{u}(x,t)=\mathrm{g}(x,t)+\int_0^x\int_a^t \mathrm{k}(y,s)F[\mathrm{u}(y,s)]dsdy, \quad (1)$$

subject to

$$\mathrm{u}(x,0)=u_0(x), \quad x\in[a,\mathrm{b}]. \quad (2)$$

and

$$^{C}D_t^{\alpha}\mathrm{u}(x,t)=\mathrm{g}(x,t)+I_t^{\beta}I_x^{\alpha}\mathrm{k}(x,s)F[\mathrm{u}(x,s)], \quad (3)$$

subject to

$$\mathrm{u}(x,0)=u_0(x), \quad x\in[a,\mathrm{b}]. \quad (4)$$

II. Preliminaries

In this section, we present some necessary definitions and mathematical preliminaries of the fractional calculus theory that have been needed in the construction of this paper.

II.1 Fractional Order Derivatives and Integrals

In this part we shall give some basic definitions and properties of the fractional order derivatives and integrals [4].

Definition 1. The Riemann–Liouville (R-L) fractional integral of order $\alpha > 0$ is defined as follows:

$I_x^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (t-\tau)^{\alpha-1} f(\tau) d\tau$, $x > 0$,$\alpha \in R^+$.

Definition 2. The Caputo fractional derivative of order $\alpha > 0$ is defined as follows:

$$ {}^C D_x^{\alpha} f(x) = \begin{cases} \frac{1}{\Gamma(m-\alpha)} \int_0^x \frac{f^{(m)}(\tau)}{(x-\tau)^{\alpha+1-m}} d\tau \ , & m-1 < \alpha < m \ ; \\ \frac{d^m}{dx^m(x)} f(x) \ , & \alpha = m \ . \end{cases} $$

For $\alpha > 0$, *we have [5]:*

1- ${}^C D_x^{\alpha} (I_x^{\alpha} f(x)) = f(x)$.

2- $I_x^{\alpha} \left({}^C D_x^{\alpha} f(x) \right) = f(x) - \sum_{k=0}^{n-1} f^{(k)}(0^+) \frac{x^k}{k!}$.

3- ${}^C D_x^{\alpha}(c) = 0$, $c \in$ R.

III. Existence and Uniqueness Theorems of the solutions of two dimensional partial integro-differential equations of fractional order.

The existence and uniqueness of problems (1)-(2) and (3)-(4) will be obtained in this section under certain conditions using Banach fixed point theorem.

III.1 The Existence and uniqueness of the solution of problem (1)-(2).

Lemma 1:

The function $u \in C([a,b] \times [0,T]) = X$ is a solution of problem (1)-(2) if and only if $u(x,t)$ is satisfying

$$ u(x,t) = u_0(x) + I_t^{\alpha} g(x,t) + I_t^{\alpha} \int_a^x \int_0^t k(y,s) F[u(y,s)] ds dy. \quad (5) $$

Proof:

Apply I_t^{α} on both sides of equation (1), yields

$$ I_t^{\alpha} \, {}^C D_t^{\alpha} u(x,t) = I_t^{\alpha} g(x,t) + I_t^{\alpha} \int_a^x \int_0^t k(y,s) F[u(y,s)] ds dy, $$

according to equation (2), we have

$$ u(x,t) = u_0(x) + I_t^{\alpha} g(x,t) + I_t^{\alpha} \int_a^x \int_0^t k(y,s) F[u(y,s)] ds dy. $$

Hence the result is obtained.

Theorem 1

Let $A: X \rightarrow X$ be defined as

$$ Au = u_0(x) + I_t^{\alpha} g(x,t) + I_t^{\alpha} \int_a^x \int_0^t k(y,s) F[u(y,s)] ds dy, \quad (6) $$

such that k verified a Lipschitz condition w.r.t. $u(x,t)$with a Lipschitz constant $L \geq 0$,and$|k(x,t)| \leq M$,

Furthermore, if $\frac{ML(b-a)T^{\alpha+1}}{\Gamma(\alpha+2)} < 1$, then T has a unique solution.

Proof:

Define the supermum norm, which will be needed later in the proof as

$$ \|u(x,t)\| = \sup_{\substack{x \in [a,b] \\ t \in [0,T]}} |u(x,t)| $$

Now to prove that T is a contractive mapping

Let$u_1(x,t), u_2(x,t) \in X$, *then*

$|Au_1(x,t) - Au_2(x,t)|$

$$ \leq \frac{1}{\Gamma(\alpha)} \int_0^t \int_a^x \int_0^v (t-v)^{\alpha-1} |k(y,s)| |F[u_1(y,s)] - F[u_2(y,s)]| dy ds dv $$

Hence

$$\|Au_1(x,t) - Au_2(x,t)\| \le \frac{ML}{\Gamma(\alpha)}\|u_1(x,t) - u_2(x,t)\| \int_0^t \int_a^x \int_0^v (t-v)^{\alpha-1}\, dydsdv.$$
$$\le \frac{ML}{\Gamma(\alpha)}\|u_1(y,s) - u_2(y,s)\| \frac{(x-a)t^{\alpha+1}}{\alpha(\alpha+1)}$$
$$\le \frac{ML(b-a)T^{\alpha+1}}{\Gamma(\alpha+2)}\|u_1(y,s) - u_2(y,s)\|$$

Since

$$\frac{ML(b-a)T^{\alpha+1}}{\Gamma(\alpha+2)} < 1,$$

Then A is a contractive mapping therefore, the problem (1)-(2) has a unique solution.

III.2 Existence and Uniqueness Theorems of the solutions of Problem (3)-(4):

In this subsection the existence and uniqueness of problem (3)-(4) will be introduced.

Lemma 2:

The function $u \in X$ is a solution of problem (3)-(4) if and only if $u(x,t)$ is satisfying

$$u(x,t) = u_0(x) + I_t^{\alpha} g(x,t) + I_t^{\alpha} I_t^{\beta} I_x^{\alpha} k(x,t) F[u(x,t)] \quad (7)$$

Proof:

Apply I_t^{α} on both sides of equation (3), yields

$$I_t^{\alpha}\, {}^{C}D_t^{\alpha} u(x,t) = I_t^{\alpha} g(x,t) + I_t^{\alpha} I_t^{\beta} I_x^{\alpha} k(x,t) F[u(x,t)],$$

According to equation (4), we have

$$u(x,t) = u_0(x) + I_t^{\alpha} g(x,t) + I_t^{\alpha} I_t^{\beta} I_x^{\alpha} k(x,t) F[u(x,t)].$$

Hence the result is obtained.

Theorem 2:

Let $A: X \to X$ be defined as

$$Au = u_0(x) + I_t^{\alpha} g(x,t) + \frac{1}{\Gamma(w)\Gamma(\alpha)} \int_0^t \int_a^x (t-s)^{w-1} (x-y)^{\alpha-1} F[u(y,s)]\, dyds \quad (8)$$

such that k verified a Lipschitz condition w.r.t. $u(x,t)$with a Lipschitz constant $L \ge 0, w = \alpha + \beta$and$|k(x,t)| \le M$.

Furthermore, if $\frac{LM\, T^{w} (b-a)^{\alpha}}{\Gamma(w+1)\Gamma(\alpha+1)} < 1$, thenA has a unique solution.

Proof:

$$\|Au_1(x,t) - Au_2(x,t)\|$$
$$\le \frac{LM}{\Gamma(w)\Gamma(\beta)}\|u_1(x,t) - u_2(x,t)\| \int_0^t \int_a^x (t-s)^{w-1}(x-y)^{\beta-1} F[u(y,s)]\, dyds$$
$$\|Au_1(x,t) - Au_2(x,t)\| \le \frac{LM}{\Gamma(w)\Gamma(\beta)}\|u_1(x,t) - u_2(x,t)\| \left\| \frac{t^{w}(x-\alpha)^{\alpha}}{w^{\alpha}} \right\|$$
$$\|Au_1(x,t) - Au_2(x,t)\| \le \frac{LMT^{w}(b-a)^{\alpha}}{\Gamma(w+1)\Gamma(\beta+1)}\|u_1(x,t) - u_2(x,t)\|$$

Since $\frac{LM\, T^{w} (b-a)^{\beta}}{\Gamma(w+1)\Gamma(\beta+1)} < 1$,

Then A is a contractive mapping therefore, the problem (3)-(4) has a unique solution.

IV. ADM for solving two dimensional partial Integro-differential Equations of Fractional order:

In this section the implementation of the ADM for solving two dimensional FPIDEs will be presented.

IV.1 ADM for solving problem (1)-(2):

To apply the ADM for solving problem (1)-(2) first operating I_t^{α} on both sides of equation (1) to get:

$$u(x,t) = u_0(x) + I_t^{\alpha} g(x,t) + I_t^{\alpha} \left\{ \int_0^x \int_a^t k(y,s) F[u(y,s)]\, dyds \right\} \quad (9)$$

according to ADM, we let the solution to be :

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t) \quad (10)$$

and the nonlinear term $F[u(x,t)]$, in equation (1), will be decomposed as:

$$F[u(x,t)] = \sum_{n=0}^{\infty} A_n \quad (11)$$

where

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[F\left(\sum_{n=0}^{\infty} u_n \lambda^n\right)\right]_{\lambda=0} \quad (12)$$

Substituting equations (10) and (11) into equation (9), we get:

$$\sum_{n=0}^{\infty} u_n(x,t) = u_0(x) + I_t^{\alpha} g(x,t) + I_t^{\alpha}\left\{\int_0^x \int_a^t k(y,s) \sum_{n=0}^{\infty} A_n \, dyds\right\}$$

consequently, we can write

$$u_0(x,t) = u_0(x) + I_t^{\alpha} g(x,t)$$

and

$$u_{n+1}(x,t) = I_t^{\alpha}\left[\int_0^x \int_a^t k(y,s) A_n dyds\right], \qquad n \geq 1.$$

Truncating the summation into equation (10), after m terms, so we have the m^{th} approximate solution of the problem (1)-(2)as:
$u_m(x,t) = \sum_{n=0}^{m} u_n(x,t).$

IV.2 ADM for solving problem (3)-(4):

In this subsection a similar manner that have been given in subsection (IV.1) will be implemented in order formulate a recurrence formula for finding the approximate solution of the problem (3)-(4) using ADMand as follows:

$u_0(x,t) = u_0(x) + I_t^{\alpha} g(x,t)$ (13)

$u_{n+1}(x,t) = I_t^{\alpha}\left[I_t^{\beta} I_x^{\alpha} \; k(y,s) A_n\right], n \geq 1.$ (14)

So, the m^{th} order approximate solution of the problem (3)-(4) is given by
$u_m(x,t) = \sum_{n=0}^{m} u_n(x,t)$ (15)

V. HAM for solving two dimensional partial Integro-differential Equations of Fractional order.

In this section the implementation of the HAM for solving two dimensional PIDEs will be presented.

V.1 HAM for solving problem (1)-(2):

Rewriting equation (1) in an operator equation, so we have
$N[u(x,t)] = 0$ (16)
where
$N[u(x,t)] = {}^{C}D_t^{\alpha} u(x,t) - g(x,t) - \int_0^x \int_a^t k(y,s) F[u(y,s)] dyds.$ (17)

According to the HAM, we construct the so called zero-order deformation equation
$(1-q)\mathcal{L}[\emptyset(x,t,q) - u_0(x)] = q\hbar H(x) N[\emptyset(x,t,q)]$ (18)
where$q \in [0,1]$ is an embedding parameter, $\hbar \neq 0$ is a nonzero auxiliary parameter, $H(x) \neq 0$ is an auxiliary function $u_0(x)$ is an initial guess of $u(x,t)$ and $\mathcal{L}$ is an auxiliary linear operator defined by:
$\mathcal{L} = {}^{C}D_t^{\alpha}$ (19)
Obviously, when $q = 0$and 1, it holds
$\emptyset(x,t,0) = u_0(x), \qquad \emptyset(x,t,1) = u(x,t)$respectively.
Expanding $\emptyset(x,t,q)$ in Taylor series with respect to q,we have

$\emptyset(x,t,q) = u_0(x) + \sum_{m=1}^{+\infty} u_m(x,t) q^m,$ (20)
where
$u_m(x,t) = \frac{1}{m!}\frac{\partial^m \emptyset(x,t,q)}{\partial q^m}\Big|_{q=0}$ (21)
If the initial guess, the auxiliary parameter $\hbar$ and the auxiliary function $H(x)$ are so properly chosen then the series (20) converges at $q = 1$.
Thus we have
$u(x,t) = u_0(x) + \sum_{m=1}^{+\infty} u_m(x,t),$ (22)
Define the vector

$$\overrightarrow{u_n} = \{u_0(x,t), u_1(x,t), \dots \dots \dots, u_n(x,t)\}.$$

Differentiating equation (18) m times with respect to the embedding parameter q and then setting $q = 0$ and finally dividing by m!, we have the so called m^{th} order deformation equation

$\mathcal{L}[u_m(x,t) - \chi_m u_{m-1}(x,t)] = \hbar H(x) R_m(\vec{u}_{m-1})$ (23)
where
$R_m(\vec{u}_{m-1}) = \left[\frac{1}{(m-1)!}\frac{\partial^{m-1} N[\emptyset(x,t,q)]}{\partial q^{m-1}}\Big|_{q=0}\right]$ (24)

Now, letting $\hbar = -1$and $H(x) = 1$, then the solution of the m^{th} order deformation equations (23),yields:

$$u_m(x,t) = \chi_m u_{m-1}(x,t) - I_t^\alpha \left[{}^C D_t^\alpha u_{m-1}(x,t) - (1-\chi_m)g(x,t) - \int_0^x \int_a^t k(y,s)F[u(y,s)]dyds \right] \quad (25)$$

and by means of the above iteration formula (25), we can obtain directly the other components in order one after one.

V.1.2Convergence analysis:

In this subsection, the convergence of the formula (25) to the exact solution $u(x,t)$, of problem (1)-(2), will be proved, it is remarkable that $F[u(x,t)]$ in equation (1) will be expressed as $F[u(x,t)] = [u(x,t)]^p, p \geq 1$.

Theorem 3 (convergence theorem of the solution of problem (1)-(2):

If the series $\sum_{m=0}^\infty u_m(x,t)$ is convergent, where $u_m(x,t)$ is produced by:

$$ {}^C D_t^\alpha [u_m(x,t) - \chi_m u_{m-1}(x,t)] = \hbar H R_m(\vec{u}_{m-1}) \quad (26)$$

Where

$$R_m(\vec{u}_{m-1}) = {}^C D_t^\alpha u_{m-1} - (1-\chi_m)g(x,t) - \int_0^x \int_0^t k(y,s)[u(y,s)]^p dsdy, p \geq 1 \quad (27)$$

and besides $\sum_{m=0}^\infty {}^C D_t^\alpha [u_m(x,t)]$ also converges, then it is the exact solution of problem (1)-(2).

Proof:

Suppose that $\sum_{m=0}^\infty u_m(x,t)$ converges uniformly to $u(x,t)$,

Then it is clear that

$$\lim_{m\to\infty} u_m(x,t) = 0, \text{ for all x and t} \in R^+ \quad (28)$$

since

${}^C D_t^\alpha$is a linear operator, we have:

$$\sum_{m=1}^n {}^C D_t^\alpha [u_m(x,t) - \chi_m u_{m-1}(x,t)] = \sum_{m=1}^n [{}^C D_t^\alpha u_m(x,t) - \chi_m {}^C D_t^\alpha u_{m-1}(x,t)].$$

$$= {}^C D_t^\alpha u_1(x,t) + \left({}^C D_t^\alpha u_2(x,t) - {}^C D_t^\alpha u_1(x,t) \right) + \cdots + \left({}^C D_t^\alpha u_n(x,t) - {}^C D_t^\alpha u_{n-1}(x,t) \right)$$

$$= {}^C D_t^\alpha u_n(x,t). \quad (29)$$

Then from equations (26), (28) and (29), we have

$$\sum_{m=1}^\infty {}^C D_t^\alpha [u_m(x,t) - \chi_m u_{m-1}(x,t)] = \lim_{n\to\infty} {}^C D_t^\alpha u_n(x,t)$$

$$= {}^C D_t^\alpha \left[\lim_{n\to\infty} u_n(x,t) \right] = 0$$

Hence

$$\hbar H \sum_{m=1}^\infty R_m(\vec{u}_{m-1}) = 0$$

since$\hbar$ and $H \neq 0$, then yields

$$\sum_{m=1}^\infty R_m(\vec{u}_{m-1}) = 0 \quad (30)$$

and since

$$R_m(\vec{u}_{m-1}) = {}^C D_t^\alpha u_{m-1} - (1-\chi_m)g(x,t) - \int_0^x \int_0^t k(y,s)[u_{m-1}(y,s)]^p dsdy.$$

so, we have

$$0 = \sum_{m=1}^\infty \left[{}^C D_t^\alpha u_{m-1} - (1-\chi_m)g(x,t) - \int_0^x \int_0^t k(y,s)[u_{m-1}(y,s)]^p dsdy. \right]$$

$$= \sum_{m=1}^\infty {}^C D_t^\alpha u_{m-1}(x,t) - g(x,t) - \sum_{m=1}^\infty [\int_0^x \int_0^t k(y,s)[\sum_{r_{1=0}}^{m-1} u_{m-1-r_1}(y,s)$$

$$\sum_{r_{2=0}}^{r_1} u_{r_1-r_2}(y,s) \sum_{r_{3=0}}^{r_2} u_{r_2-r_3}(y,s) \ldots \sum_{r_{p-2=0}}^{r_{p-3}} u_{r_{p-3}-r_{p-2}}(y,s) \sum_{r_{p-1=0}}^{r_{p-2}} u_{r_{p-2}-r_{p-1}}(y,s)]dsdy].$$

Hence

$$0 = {}^C D_t^\alpha \sum_{m=1}^\infty u_{m-1}(x,t) - g(x,t) - \int_0^x \int_0^t k(y,s)(\sum_{r_{p-1=0}}^\infty u_{m-1-r_1} \sum_{r_{p-2=}r_{p-1}}^\infty u_{r_1-r_2}$$

$$\sum_{r_{p-2}=r_{p-1}}^{\infty} u_{r_2-r_3} \dots\dots \sum_{r_2=r_3}^{\infty} u_{r_{p-3}-r_{p-2}} \sum_{r_1=r_2}^{\infty} u_{r_{p-2}-r_{p-1}} \sum_{m=r_1}^{\infty} u_{m-r_1})(y,s)]dsdy.$$

$$0 = {}^{C}D_t^{\alpha} \sum_{m=0}^{\infty} u_m(x,t) - g(x,t) - \int_0^x \int_0^t k(y,s)(\sum_{i_1=0}^{\infty} u_{i_1}(y,s) \sum_{i_2=0}^{\infty} u_{i_2}(y,s)$$
$$\sum_{i_3=0}^{\infty} u_{i_3}(y,s) \dots\dots \sum_{i_{p-1}=0}^{\infty} u_{i_{p-1}}(y,s) \sum_{i_p=0}^{\infty} u_{i_p}(y,s))dsdy.$$

so from equation (30), we obtain

$$0 = {}^{C}D_t^{\alpha} u(x,t) - g(x,t) - \int_0^x \int_0^t k(y,s)[u(y,s)]^p dsdy.$$

Since

$\sum_{m=0}^{\infty} u_m(x,t)$also satisfies the initial condition

$$\sum_{m=0}^{\infty} u_m(x,0) = u(x,0) = u_0(x)$$

Therefore, we conclude that it's an exact solution of problem (1)-(2).

V.2 HAM for solving problem (3)-(4):

The HAM can be performed for solving problem (3)-(4)in a similar manner that have been given in subsection (V.1) and therefore, we have:

$$u_m(x,t) = \chi_m u_{m-1}(x,t) - I_t^{\alpha}\left[{}^{C}D_t^{\alpha} u_{m-1}(x,t) - (1-\chi_m)g(x,t) - I_t^{\beta} I_x^{\alpha} k(y,s) F[u(y,s)]\right] \qquad (31)$$

and by means of equation (31), we can obtain the other components in order one after one.

V.2.1. Convergence analysis:

In this subsection, the convergence of the formula (31) to the exact solution $u(x,t)$, of problem (3)-(4), will be proved, it is remarkable that $F[u(x,t)]$ in equation (3) will be expressed as $F[u(x,t)] = [u(x,t)]^p, p \geq 1$.

Theorem 4 (convergence theorem of problem (3)-(4)):

If the series $\sum_{m=0}^{\infty} u_m(x,t)$ is convergent , where $u_m(x,t)$ is produced by

$${}^{C}D_t^{\alpha}[u_m(x,t) - \chi_m u_{m-1}(x,t)] = \hbar H R_m(\vec{u}_{m-1}) \qquad (32)$$

where

$$R_m(\vec{u}_{m-1}) = {}^{C}D_t^{\alpha} u_{m-1} - (1-\chi_m)g(x,t) - I_t^{\beta} I_x^{\alpha} k(x,t)[u(x,t)]^p$$

and besides

$\sum_{m=0}^{\infty} {}^{C}D_t^{\alpha}[u_m(x,t)]$also converges, then it is the exact solution of problem (3)-(4).

Proof:

The proof of theorem (4) will be in similar manner to the proof of theorem (3).

VI. Applications:-

In this section, we shall introduce some illustrative numerical examples in order to confirm the applicability and accuracy of the HAM and the ADM, , for solving non- linear two dimensional partial integro-differential equation of fractional order.

Example 1

Consider the following linear two dimensional FPIDEs:-

$${}^{C}D_t^{3/4} u(x,t) = g(x,t) + \int_0^x \int_0^t (y-s)u(y,s)dsdy, \qquad (32)$$

subject to

$$u(x,0) = 0, \qquad (33)$$

where$g(x,t) = \frac{xt^{1/4}}{\Gamma(5/4)} - \frac{x^3t^2}{6} + \frac{x^2t^3}{6}$, and the exact solution of problem (32)-(33) is $u(x,t) = xt$.

Following figures (1)-(2) represent a comparison between the approximate solution of problem (32)-(33) using HAM and ADM up to 4-terms and the exact solution.

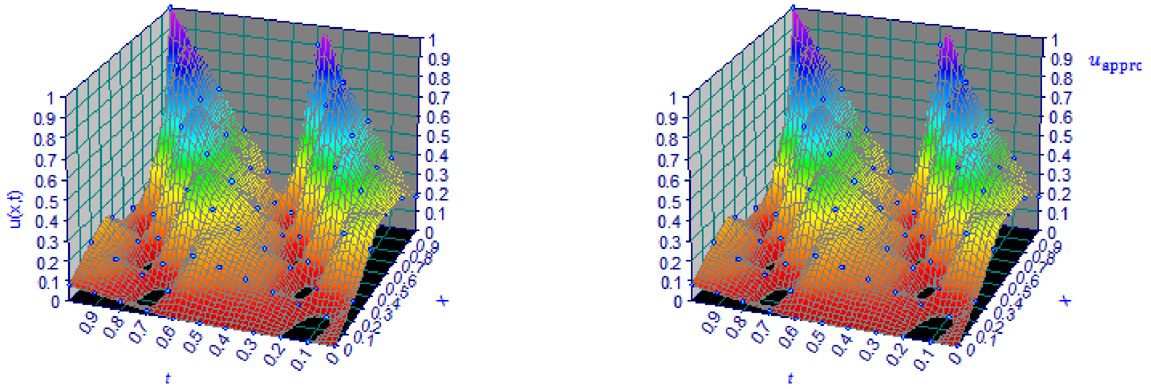

Figure1: comparison between the approximate solution of problem(32)-(33) using HAM up to 4-terms and the exact solution.

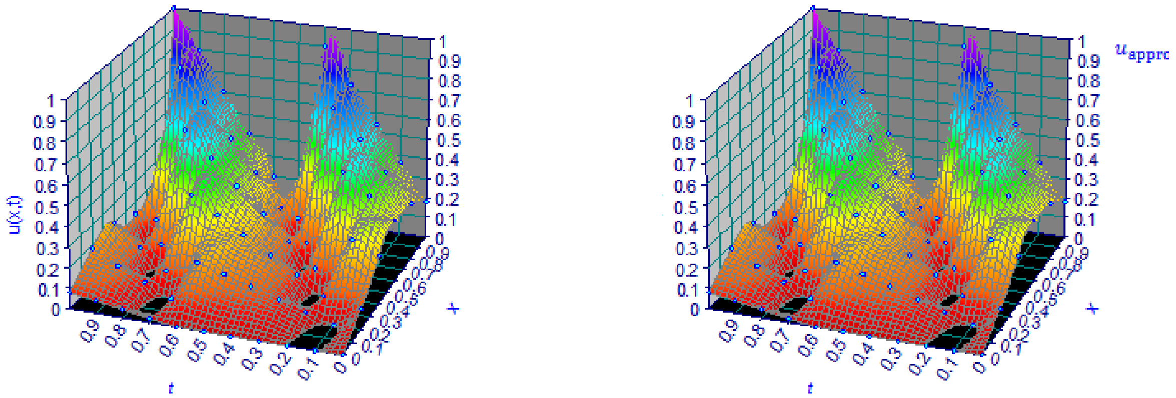

Figure2: comparison between the approximate solutions of problem (32)-(33) using ADM up to 4-terms and the exact solution.

Example 2

Consider the following linear two dimensional FPIDEs:-

$${}^{C}D_t^{1/2}u(x,t) = g(x,t) + I_t^{1/2}I_x^{3/4}(x-t)u(x,t), \qquad (34)$$

subject to

$$u(x,0) = 0, \qquad (35)$$

where $g(x,t) = \frac{xt^{1/2}}{\Gamma(3/2)} - \frac{\Gamma(3)x^{11/4}t^{3/2}}{\Gamma(15/4)\Gamma(5/2)} + \frac{\Gamma(3)x^{7/4}t^{5/2}}{\Gamma(11/4)\Gamma(7/2)}$, and the exact solution of problem (34)-(35) is $u(x,t) = xt$.

Following figures (3)-(4) represent a comparison between the approximate solution of problem (34)-(35) using HAM and ADM up to 4-terms and the exact solution.

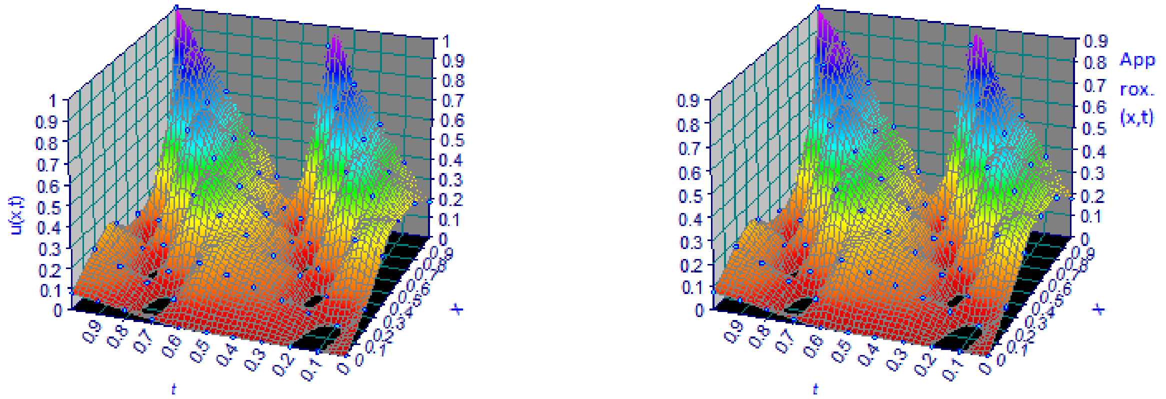

Figure3: comparison between the approximate solution of problem(44)-(45) using HAM up to 4-terms and the exact solution.

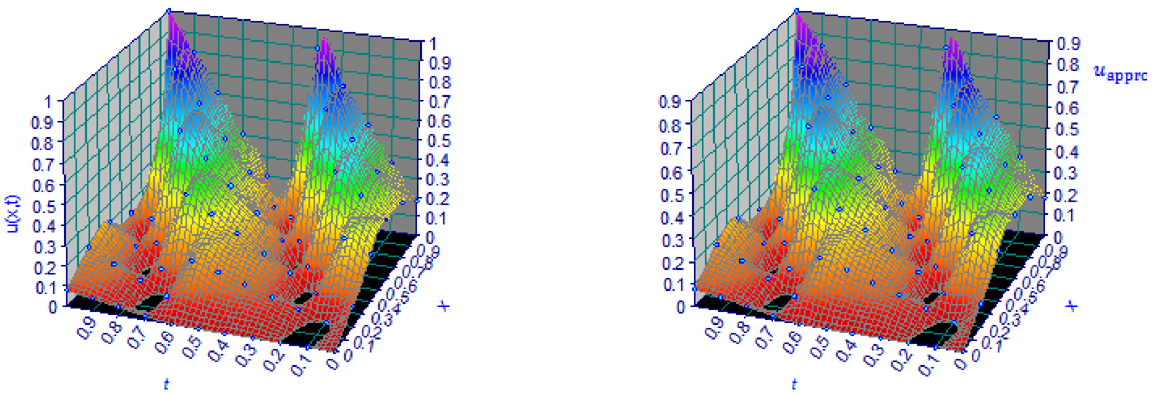

Figure4: comparison between the approximate solutions of problem (34)-(35) using ADM up to 4-terms and the exact solution.

Example 3

Consider the following nonlinear two dimensional FPIDEs:-

$${}^{C}D_t^{1/2}u(x,t) = g(x,t) + I_t^{1/2} I_x^{3/4}(x-t)u^2(x,t), \qquad (36)$$

subject to

$$u(x,0) = 0, \qquad (37)$$

where $g(x,t) = \frac{xt^{1/2}}{\Gamma(3/2)} - \frac{2\Gamma(4)x^{11/4}t^{5/2}}{\Gamma(19/4)\Gamma(7/2)} + \frac{2\Gamma(4)x^{11/4}t^{7/2}}{\Gamma(15/4)\Gamma(9/2)}$, and the exact solution of problem (36)-(37) is $u(x,t) = xt$.

Following figures (5)-(6) represent a comparison between the approximate solution of problem (36)-(37) using HAM and ADM up to 4-terms and the exact solution.

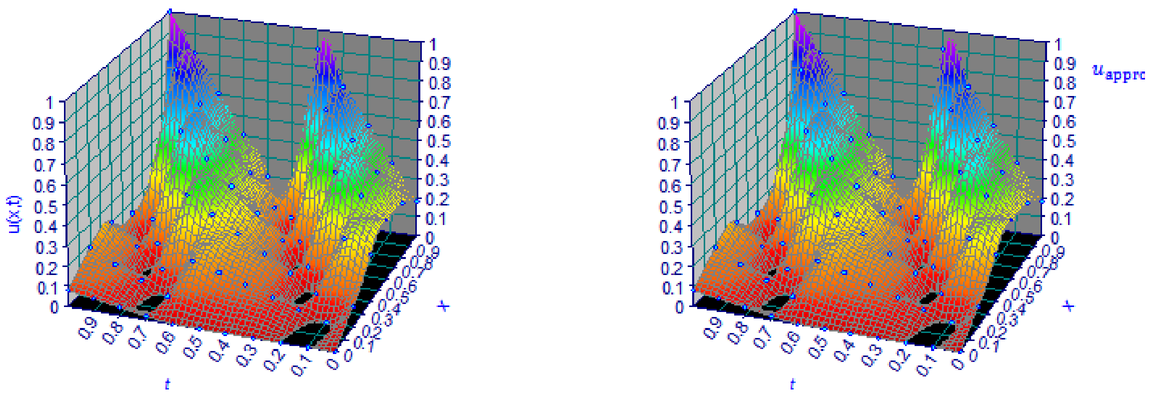

Figure5: comparison between the approximate solution of problem (36)-(37) using HAM up to 4-terms and the exact solution.

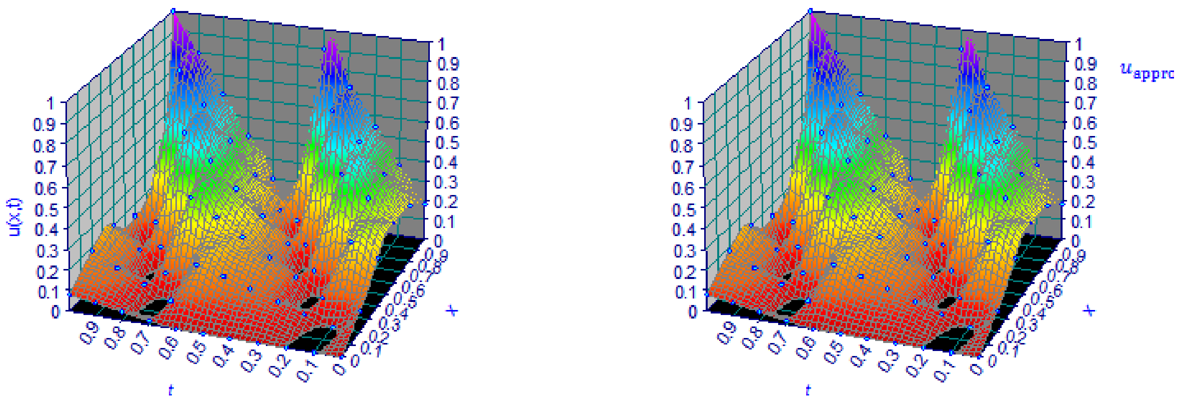

Figure6: comparison between the approximate solution of problem (36)-(37) up to 4-terms using ADM and the exact solution.

VII. Conclusions

The existence and uniqueness for the solution of a class of two dimensional partial integro-differential equationsoffractional orderis discussed.

Then two semi analytic methods which are so called ADM and HAM are introduced for approximating the solution of such kinds of problems. Moreover the convergence of the solution for the proposed methods is investigated.

The numerical results illustrate the efficiency and accuracy of the present schemes for solving two dimensional partial integro-differential equations of fractional order.

References

[1]. A. A. Kilbas, H. M. Srivastava and J. J. Trujillo, "Theory and Applications of Fractional Differential Equations", Elsevier, (2006).

[2]. F. Hecht, T. Jangveladze, Z. Kiguradze and O. Pironneau, "Finite Difference Scheme for One System of Nonlinear Partial Integro-Differential Equations", Applied Mathematics and Computation, 328, 287-300, (2018).

[3]. K. Kumar, R. K. Pandey and S. Sharma, "Comparative study of Three Numerical Schemes for Fractional Integro-Differential Equations", Journal of Computational and Applied Mathematics, Accepted (2016).

[4]. Mohammed. OH., "A Direct Method for Solving Fractional Order Variational Problems by Hat Basis Functions", Ain Shams Eng. J. http://dx.doi.org/10.1016/j.asej.2016.11.006. (2016).

[5]. Podlunny I., "Fractional Differential Equations", Academic Press, (1999).

[6]. S. S. Ray and S. Behera, "Two-Dimensional Wavelets Operational Method for Solving Volterra Weakly Singular Partial Integro-Differential Equations", Journal of Computational and Applied Mathematics, Accepted, (2019).

[7]. S. Singh, V. K. Patel, V. K. Singh and E. Tohidi, "Numerical Solution of Nonlinear Weakly Singular Partial Integro-Differential Equation Via Operational Matrices", Applied Mathematics and Computation, 298, 310-321, (2017).

[8]. X. Yang, J. Yu, M. Xu and W. Fan, " Convertible bond pricing with partial integro-differential equation model", Mathematics and Computers in Simulation, Accepted, (2018).

Ahmed K. Mohsin. " Homotopy Analysis and AdomianDecomposition Methods for Approximating the Solution of Two Dimensional Partial Integro-Differential Equations of Fractional Order." IOSR Journal of Mathematics (IOSR-JM) 15.6 (2019): 65-73.