IOSR Journal of Mathematics (IOSR-JM)
e-ISSN: 2278-5728, p-ISSN: 2319-765X. Volume 16, Issue 1 Ser. III (Jan – Feb 2020), PP 01-05
www.iosrjournals.org

Some Relations Connected To Order of Composite Functions and Relative Order of Entire and Meromorphic Functions

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Abstract: *In this paper, some relations of order, L-order, L*-order of composite entire and meromorphic functions with relative lower order, relative L-lower order, relative L*-lower order of a meromorphic function with respect to an entire function are established.*
***AMS Subject Classification (2010)** : 30D20,30D30,30D35.*
***Keywords and Phrases** : Entire function, Meromorphic function, Slowly changing function, order, L-order, L*-order, relative lower order, relative L-lower order, relative L*-lower order.*

Date of Submission: 08-01-2020 Date of Acceptance: 23-01-2020

I. Introduction

The maximum modulus $M_g(r)$ of the entire function g defined in the open complex plane ℂ is defined as $M_g(r) =$ max {|g(z)|:|z|=r}. For meromorphic function f defined in the open complex plane ℂ, $M_f(r)$ can not be defined as f is not analytic. In this case one may define another function $T_f(r)$, which is known as Nevanlinna's Characteristic function of f, playing the same role as maximum modulus.
All the standard notations and definitions in the theory of entire and meromorphic functions are available in the books of Hayman (1964) and Valiron (1949).
In this connection we just recall the following definitions which are relevant:

II. Definitions

Definition 1 The order ρ_f and lower order λ_f of an entire function f are defined as

$$\rho_f = \limsup_{r\to\infty} \frac{\log^{[2]} M_f(r)}{\log r} \quad \text{and} \quad \lambda_f = \liminf_{r\to\infty} \frac{\log^{[2]} M_f(r)}{\log r}$$

When f is meromorphic, then

$$\rho_f = \limsup_{r\to\infty} \frac{\log T_f(r)}{\log r} \quad \text{and} \quad \lambda_f = \liminf_{r\to\infty} \frac{\log T_f(r)}{\log r}$$

Bernal (1984, 1988) introduced the definition of relative order of an entire function f with respect to another entire function g, denoted by $\rho_g(f)$ to avoid comparing growth just with exp z as follows:

$$\rho_g(f) = \inf\{\mu > 0 : M_f(r) < M_g(r^{\mu}) \text{ for all } r > r_0(\mu) > 0.\}$$
$$= \limsup_{r\to\infty} \frac{\log M_g^{-1} M_f(r)}{\log r}$$

Similarly, one can define the relative lower order of an entire function f with respect to another entire function g denoted by $\lambda_g(f)$ as follows :

$$\lambda_g(f) = \liminf_{r\to\infty} \frac{\log M_g^{-1} M_f(r)}{\log r}$$

Extending this notion, Lahiri et.al.(1999) introduced the definition of relative order of a meromorphic function with respect to an entire function in the following way :
Definition 2 Let f be any meromorphic function and g be any entire function. The relative order of f with respect to g is defined as

$$\rho_g(f) = \inf\{\mu > 0 : T_f(r) < T_g(r^{\mu}) \text{ for all sufficiently large r }\}$$
$$= \limsup_{r\to\infty} \frac{\log T_g^{-1} T_f(r)}{\log r}$$

Likewise, one can define the relative lower order of a meromorphic function f with respect to an entire function g denoted by $\lambda_g(f)$ as follows :

$$\lambda_g(f) = \liminf_{r\to\infty} \frac{\log T_g^{-1} T_f(r)}{\log r}$$

It is known that if g(z)=exp z then **Definition 2** coincides with the classical definition of the order of a meromorphic function f .

Let L≡L(r) be a positive continuous function increasing slowly i.e., L(ar)~L(r) as $r \to \infty$ for every positive constant a. Singh et. al. (1977) defined it in the following way:

Definition 3 A positive continuous function L(r) is called a slowly changing function if for ε(>0),

$$\frac{1}{k^{\varepsilon}} \le \frac{L(kr)}{L(r)} \le k^{\varepsilon} \text{ for } r \ge r(\varepsilon) \text{ and uniformly for k}(\ge 1).$$

If further, L(r) is differentiable, the above condition is equivalent to

$$\lim_{r\to\infty} \frac{rL'(r)}{L(r)} = 0.$$

Somasundaram et.al. (1988) introduced the notions of L-order and L-lower order for entire functions.

Definition 4 The L-order $\rho_f^{\ L}$ and the L-lower order $\lambda_f^{\ L}$ of a meromorphic function f are defined as follows:

$$\rho_f^{\ L} = \limsup_{r\to\infty} \frac{\log T(r,f)}{\log[rL(r)]} \text{ and } \lambda_f^{\ L} = \liminf_{r\to\infty} \frac{\log T(r,f)}{\log[rL(r)]}$$

The more generalised concept of L-order for a functions is L^* -order.

Definition 5 [Somasundaram et.al. (1988)]The L^* -order $\rho_f^{\ L^*}$ and the L^* -lower order $\lambda_f^{\ L^*}$ of a meromorphic function f are defined as

$$\rho_f^{\ L^*} = \limsup_{r\to\infty} \frac{\log T(r,f)}{\log\left[re^{L(r)}\right]} \text{ and } \lambda_f^{\ L^*} = \liminf_{r\to\infty} \frac{\log T(r,f)}{\log\left[re^{L(r)}\right]}$$

In the line of Somasundaram et.al. (1988) and Bernal (1984, 1988), one may define the relative L-order and the relative L^*-order of a meromorphic function in the following manner :

Definition 5 The relative L-order $\rho_g^{\ L}(f)$ and the relative L-lower order $\lambda_g^{\ L}(f)$ of a meromorphic function f with respect to an entire function g are defined as follows:

$$\rho_g^{\ L}(f) = \limsup_{r\to\infty} \frac{\log T_g^{-1} T_f(r)}{\log[rL(r)]} \text{ and } \lambda_g^{\ L}(f) = \liminf_{r\to\infty} \frac{\log T_g^{-1} T_f(r)}{\log[rL(r)]}$$

Definition 6 The relative L^*-order $\rho_g^{\ L^*}(f)$ and relative L^*-lower $\lambda_g^{\ L^*}(f)$ of a meromorphic function f with respect to an entire function g are defined as

$$\rho_g^{\ L^*}(f) = \limsup_{r\to\infty} \frac{\log T_g^{-1} T_f(r)}{\log\left[re^{L(r)}\right]} \text{ and } \lambda_g^{\ L^*}(f) = \liminf_{r\to\infty} \frac{\log T_g^{-1} T_f(r)}{\log\left[re^{L(r)}\right]}$$

III. Preliminaries

In this section we present some lemmas which will be needed in the sequel.

Lemma 1 [Bergweiler (1990)] Let f be meromorphic and g be entire and suppose that $0 < \mu < \rho_g \le \infty$. Then for a sequence of values of r tending to infinity,

$$T_{f\circ g}(r) \geq T_f(\exp(r^{\mu}))$$

Lemma 2 [Lahiri et.al. (1995)] Let f be meromorphic and g be entire such that $0 < \rho_g < \infty$ and $0 < \lambda_f$. Then for a sequence of values of r tending to infinity,

$$T_{f\circ g}(r) > T_g(\exp(r^{\mu})) \text{ where } 0 < \mu < \rho_g.$$

IV. Main Results

In this section we present the main results of the paper.

Theorem 1 If f be a meromorphic function and g be an entire function such that $0 < \mu < \rho_g \leq \infty$, $\lambda_g(f) < \infty$. Then for a sequence of values of r tending to infinity,

$$\limsup_{r\to\infty} \frac{\log T_f(\exp(r^{\mu}))}{\log T_g^{-1} T_f(r)} \leq \frac{\rho_{f\circ g}}{\lambda_g(f)}$$

Proof In view of Lemma 1, for $0 < \mu < \rho_g \leq \infty$ and for a sequence of values of r tending to infinity,

$$\log T_{f\circ g}(r) \geq \log T_f(\exp(r^{\mu}))$$

$$\text{i.e., } \frac{\log T_f(\exp(r^{\mu}))}{\log T_g^{-1} T_f(r)} \leq \frac{\log T_{f\circ g}(r)}{\log T_g^{-1} T_f(r)}$$

$$= \frac{\log T_{f\circ g}(r)}{\log r} . \frac{\log r}{\log T_g^{-1} T_f(r)}$$

$$\text{i.e., } \limsup_{r\to\infty} \frac{\log T_f(\exp(r^{\mu}))}{\log T_g^{-1} T_f(r)} \leq \limsup_{r\to\infty} \left(\frac{\log T_{f\circ g}(r)}{\log r} . \frac{\log r}{\log T_g^{-1} T_f(r)} \right)$$

$$\text{i.e., } \limsup_{r\to\infty} \frac{\log T_f(\exp(r^{\mu}))}{\log T_g^{-1} T_f(r)} \leq \frac{\limsup_{r\to\infty} \frac{\log T_{f\circ g}(r)}{\log r}}{\liminf_{r\to\infty} \frac{\log T_g^{-1} T_f(r)}{\log r}} = \frac{\rho_{f\circ g}}{\lambda_g(f)}$$

In the line of Theorem 1 and Lemma 2, the following theorem can be stated without its proof:

Theorem 2 Let f be a meromorphic function and g be an entire function such that $0 < \mu < \rho_g < \infty$, $\lambda_g(f) < \infty$ and $0 < \lambda_f$. Then for a sequence of values of r tending to infinity,

$$\limsup_{r\to\infty} \frac{\log T_g(\exp(r^{\mu}))}{\log T_g^{-1} T_f(r)} \leq \frac{\rho_{f\circ g}}{\lambda_g(f)}$$

Theorem 3 If f be a meromorphic function and g be an entire function such that $0 < \mu < \rho_g \leq \infty$, $\lambda_g^{L}(f) < \infty$. Then for a sequence of values of r tending to infinity,

$$\limsup_{r\to\infty} \frac{\log T_f(\exp(r^{\mu}))}{\log T_g^{-1} T_f(r)} \leq \frac{\rho^{L}{}_{f\circ g}}{\lambda^{L}{}_{g}(f)}$$

Proof In view of Lemma 1, for $0 < \mu < \rho_g \leq \infty$, $\rho_g^{L}(f) < \infty$ and for a sequence of values of r tending to infinity,

$$\log T_{f\circ g}(r) \geq \log T_f(\exp(r^{\mu}))$$

$$\text{i.e., } \frac{\log T_f(\exp(r^{\mu}))}{\log T_g^{-1}T_f(r)} \le \frac{\log T_{f\circ g}(r)}{\log T_g^{-1}T_f(r)}$$

$$= \frac{\log T_{f\circ g}(r)}{\log\left[rL(r)\right]}.\frac{\log\left[rL(r)\right]}{\log T_g^{-1}T_f(r)}$$

$$\text{i.e., } \limsup_{r\to\infty} \frac{\log T_f(\exp(r^{\mu}))}{\log T_g^{-1}T_f(r)} \le \limsup_{r\to\infty}\left(\frac{\log T_{f\circ g}(r)}{\log\left[rL(r)\right]}.\frac{\log\left[rL(r)\right]}{\log T_g^{-1}T_f(r)}\right)$$

$$\text{i.e., } \limsup_{r\to\infty} \frac{\log T_f(\exp(r^{\mu}))}{\log T_g^{-1}T_f(r)} \le \frac{\limsup\limits_{r\to\infty}\frac{\log T_{f\circ g}(r)}{\log\left[rL(r)\right]}}{\liminf\limits_{r\to\infty}\frac{\log T_g^{-1}T_f(r)}{\log\left[rL(r)\right]}} = \frac{\rho^{L}{}_{f\circ g}}{\lambda^{L}{}_{g}(f)}$$

In the line of Theorem 3 and Lemma 2, the following theorem can be stated without its proof:

Theorem 4 Let f be a meromorphic function and g be an entire function such that $0 < \mu < \rho_g < \infty$, $0 < \lambda_f$ and $\lambda_g{}^{L}(f) < \infty.$ Then for a sequence of values of r tending to infinity,

$$\limsup_{r\to\infty} \frac{\log T_g(\exp(r^{\mu}))}{\log T_g^{-1}T_f(r)} \le \frac{\rho^{L}{}_{f\circ g}}{\lambda^{L}{}_{g}(f)}$$

Theorem 5 If f be a meromorphic function and g be an entire function such that $0 < \mu < \rho_g \le \infty$ and $\lambda_g{}^{L^*}(f) < \infty.$ Then for a sequence of values of r tending to infinity,

$$\limsup_{r\to\infty} \frac{\log T_f(\exp(r^{\mu}))}{\log T_g^{-1}T_f(r)} \le \frac{\rho^{L^*}{}_{f\circ g}}{\lambda^{L^*}{}_{g}(f)}$$

Proof In view of Lemma 1, for $0 < \mu < \rho_g \le \infty$, $\rho_g{}^{L^*}(f) < \infty$ and for a sequence of values of r tending to infinity,

$$\log T_{f\circ g}(r) \ge \log T_f(\exp(r^{\mu}))$$

$$\text{i.e., } \frac{\log T_f(\exp(r^{\mu}))}{\log T_g^{-1}T_f(r)} \le \frac{\log T_{f\circ g}(r)}{\log T_g^{-1}T_f(r)} = \frac{\log T_{f\circ g}(r)}{\log\left[re^{L(r)}\right]}.\frac{\log\left[re^{L(r)}\right]}{\log T_g^{-1}T_f(r)}$$

$$\text{i.e., } \limsup_{r\to\infty} \frac{\log T_f(\exp(r^{\mu}))}{\log T_g^{-1}T_f(r)} \le \limsup_{r\to\infty}\left(\frac{\log T_{f\circ g}(r)}{\log\left[re^{L(r)}\right]}.\frac{\log\left[re^{L(r)}\right]}{\log T_g^{-1}T_f(r)}\right)$$

$$\text{i.e., } \limsup_{r\to\infty} \frac{\log T_f(\exp(r^{\mu}))}{\log T_g^{-1}T_f(r)} \le \frac{\limsup\limits_{r\to\infty}\frac{\log T_{f\circ g}(r)}{\log\left[re^{L(r)}\right]}}{\liminf\limits_{r\to\infty}\frac{\log T_g^{-1}T_f(r)}{\log\left[re^{L(r)}\right]}} = \frac{\rho^{L^*}{}_{f\circ g}}{\lambda^{L^*}{}_{g}(f)}$$

In the line of Theorem 5 and Lemma 2, the following theorem can be stated without its proof:

Theorem 6 Let f be a meromorphic function and g be an entire function such that $0 < \mu < \rho_g < \infty$, $0 < \lambda_f$ and $\lambda_g{}^{L^*}(f) < \infty.$ Then for a sequence of values of r tending to infinity,

$$\limsup_{r\to\infty} \frac{\log T_g(\exp(r^{\mu}))}{\log T_g^{-1} T_f(r)} \leq \frac{\rho^{L^*}{}_{f\circ g}}{\lambda^{L^*}{}_g(f)}$$

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DR. Chinmay Biswas. "Some Relations Connected To Order of Composite Functions and Relative Order of Entire and Meromorphic Functions." *IOSR Journal of Mathematics (IOSR-JM)*, 16(1), (2020): pp. 01-05.