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Comparative Analysis of Richards, Gompertz and Weibull Models

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Abstract: *Traditional statistical methods for nonlinear models require a starting point (initial parameters or guess values) to begin the optimization process. The nonlinear model expression must be written, the parameter names declared, and initial parameter values specified, then the parameters are estimated through an iterative approach. A computer program for estimating three growth models (Richards, Gompertz and Weibull model) using the modified version of the Levenberg-Marquardt method for solving non-linear regression model was employed. The growth models were decomposed by additive and multiplicative error terms which help in identifying the most appropriate model for growth studies. The problem of the initial parameters was addressed by second-order regression techniques before an iterative approach was done. The result contain the final estimate of the parameters, standard errors, p-values and model adequacy criteria, used to determine the most suitable growth model. This study was able to identify the Weibull Growth Model with Additive Error Terms as the best growth model. These studies recommend/suggest the Weibull Growth Model for further growth studies.*

Keywords*: Growth Models; Modified Levenberg-Marquardt Algorithm; Initial Parameters; Additive and Multiplicative Error Terms and Model Adequacy Criteria.*

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I. Introduction

In many field of studies, biological growth models had played vital role. Many researchers have contributed in developing relevant models. There are several common models, such as Asymptotic Regression/Growth Model, Gompertz, Weibull, Hill, Richards, Logistic, S-shaped Curves etc. These models (or curves) referred to as Sigmoidal Growth Models (Sigmoidal curves) which arise in various applications including bioassay, signal detection theory, agriculture, engineering field, tree diameter, height distribution in forestry, fire size, high-cycle fatigue strength predication, seismological data analysis for earthquakes and economics [4,17,19,22,25].

The growth models have been widely used in many biological growth problems in several studies [2,15]. The growth models considered in this study are Richards, Gompertz and Weibull Models.

1.1 Richard Growth Model

The Richards function is defined as in the usual notations [20,21,22] *as*

$$y_i = \beta_0\left(1 - \beta_1 e^{-\beta_2 x_i}\right)^{\beta_3} \qquad (1)$$

where

e represents Euler number (e = 2.71828)

x_i represents time

$\beta_0, \beta_1, \beta_2, \beta_3$ are the parameters

β_0 represents upper asymptote when time approaches positive infinity (i.e. maximum growth response or scale parameter)

β_1 represents the shape parameter related to initial time

β_2 represents growth range (or intrinsic growth range)

β_3 represents growth rate

y_i is the i^th^ observation at specific time

and if $\beta_1 = 1$ in the four parameter model, then Equation (1) becomes three parameter Equation (2);

$$y_i = \beta_0\left(1 - e^{-\beta_1 x_i}\right)^{\beta_2} \qquad (2)$$

where, β_0 , $\beta_2 = \beta_1$ and $\beta_3 = \beta_2$

1.2 Gompertz Growth Model

$$y_i = \beta_0 e^{-\beta_1 e^{-\beta_2 x_i^{\beta 3}}} \quad (3)$$

where

x_i represent time

β_0 represent upper asymptote when time approaches $+\infty$

β_1 represent positive number of the shape parameter related to initial time (or displacement along the x axis)

β_2 represent positive number of the growth range

β_3 represents growth rate (or shape parameter)

y_i is the i^{th} observation at time t_i

similarly, if $\beta_3 = 1$ in the four parameters model, then Equation (3) model become three parameters model Equation (4);

$$y_i = \beta_0 e^{-\beta_1 e^{-\beta_2 x_i}} \quad (4)$$

where all parameters remain the same [Hint: β_0 , $\beta_2 = \beta_1$ and $\beta_3 = \beta_2$]

1.3 Weibull Growth Model

The Weibull model with four parameters is expressed as

$$y_i = \beta_0 x_i^{\beta_1 - 1} e^{-\beta_2 x_i^{-\beta_3}} \quad (5)$$

A simple rewrite of the four parameter Weibull model can be expressed as three parameters (if $\beta_1 = 1$), that is

$$y_i = \beta_0 e^{-\beta_1 x_i^{-\beta_2}} \quad (6)$$

[Hint: β_0 , $\beta_2 = \beta_1$ and $\beta_3 = \beta_2$]

Hence, this study will centre on comparing the relationship among Richards, Gompertz and Weibull models in growth analysis

II. Literature Review

2.1. Review of Works on Richards Growth Model

Amir applied Richards growth model to the description of growth of Green gram [3]. The objective of his study was to use a model to predict the growth process and derive growth parameters in green gram. However, three models of growth include Beta, Gompertz and Richards was used. Gompertz and Beta models have three parameters, while Richards function has additional parameter to describe growth kinetics. His result revealed that estimation from three models (Beta, Gompertz and Richard) were suitable for predict dry matter accumulation of green gram. Thus, the Richards growth model fitted is more flexible in describing asymmetrical growth patterns of the dry matter and age data of green gram in term coefficients of determination, mean square error, mean absolute percent error.

Pommerening and Muszta used absolute and relative growth rates in the analysis of plant growth relative to plant size, assessing the growth performance and growth efficiency of plants and plant populations. In their paper, they explained how these isolated methods can be combined to form a consistent methodology for modelling relative growth rates. The results indicate; 1) an improved analysis of growth performance and efficiency and 2) the prediction of future or past growth rates [20].

2.2. Review of Works on Gompertz Model

The statistical mechanics for the Gompertz model whose system consists of interacting species was considered more than two decades ago [14,24]. Gompertz model is recent developments as a stochastic model, a stochastic model that incorporating environmental fluctuations was investigated in [16]. His attempts have much to do with statistical description for the tumour growth phenomena and are the main motivation behind his work.

Measuring biological growth has been important in many fields. Many researchers have contributed in developing relevant models: Purnachandra and Ayele [20] for Gompertz function; [10,12]. The growth models have been widely used in many biological growth problems including: in animal sciences [10,12].

2.3. Review Research on Weibull Growth Model

The Weibull model is a flexible and simple function with great potential for application to biological data, particularly if the system to be modeled can exist in either of two states (e.g., germinated or ungerminated). It is often suitable where conditions of strict randomness of the exponential distribution are not satisfied [8]. The Weibull has been used by several authors for analyzing and describing seed germination [6,7,9]. The function's parameters are biologically interpretable, reflecting maximum germination (M), germination rate (K), lag in onset of germination (L), and the shape of the cumulative distribution (C).

The Weibull model was first introduced by Ernst Hjalmar Waloddi Weibull in 1951. Initially it was described as a statistical distribution. It has many applications in population growth, agricultural growth, height growth and is also used to describe survival in cases of injury or disease or in population dynamic studies [1,18,25].

III. Methods

3.1.1 The Three Growth Models with Additive Error Terms

Equation (2), (4) and (6) can be expressed as

$$y_i = \beta_0\left(1 - e^{-\beta_1 x_i}\right)^{\beta_2} + \varepsilon_i \quad (7)$$

$$y_i = \beta_0 e^{-\beta_1 e^{-\beta_2 x_i}} + \varepsilon_i \quad (8)$$

$$y_i = \beta_0 e^{-\beta_1 x_i^{-\beta_2}} + \varepsilon_i \quad (9)$$

where ε_i are the error terms (or ε_i are normal measurement errors with 0 means independent of the random effects).

3.1.2. The Three Growth Models with Multiplicative Error Terms

Likewise Equation (2), (4) and (6) can be expressed as

$$y_i = \beta_0\left(1 - e^{-\beta_1 x_i}\right)^{\beta_2} \varepsilon_i \quad (10)$$

$$y_i = \beta_0 e^{-\beta_1 e^{-\beta_2 x_i}} \varepsilon_i \quad (11)$$

$$y_i = \beta_0 e^{-\beta_1 x_i^{-\beta_2}} \varepsilon_i \quad (12)$$

where ε_i are the error terms

- **Modified Levenberg-Marquardt Algorithm**

The research used the modified version of the Levenberg-Marquardt method. That is

1) Obtain partial derivative of the model with respect to the three parameters $(\beta_0, \beta_1, \beta_2)$.

2) To developed a program in the Gretl software using Equation (7) to (12) and input the initial values by fitting second-order polynomial (quadratic model), using Minitab 17 software.

3) Then, substitute the second-order polynomial coefficients $(\beta_0^{(0)}, \beta_1^{(0)}, \beta_2^{(0)})$ as the initial guess values for iteration process.

4) Input the data and initial guess values on the developed program. Then, run the iteration to obtain the results.

Let $\beta = (\beta_0^{(0)}, \beta_1^{(0)}, \beta_2^{(0)})$ be the initial parameters. Thus, we take the Logarithm transformation of Equation (2)]. We have

$$\ln(y_i) = \ln(\beta_0) + \beta_2 \ln(1 - e^{-\beta_i x_i}) \quad (13)$$

The NLS estimation using a modified version of the Levenberg-Marquardt, we take the derivative with respect to the parameters $(\beta_0, \beta_1, \beta_2)$. Then, substitute $(\beta_0^{(0)}, \beta_1^{(0)}, \beta_2^{(0)})$ as the initial guess vector for iteration process.

In Equation (13), let $\ln(y_i) = Z_i$, $\ln(\beta_0) = \hat{\beta}_0$, $\beta_1 = \hat{\beta}_1$ *and* $\beta_2 = \hat{\beta}_2$

$$Z_i = \hat{\beta}_0 + \hat{\beta}_2 \ln(1 - e^{-\hat{\beta}_1 x_i}) \quad (14)$$

By take partial derivative of Equation (14) with respect to $(\hat{\beta}_0, \hat{\beta}_1, \hat{\beta}_2)$ we have

$$\frac{\partial z_i}{\partial \hat{\beta}_0} = 1 \quad (15)$$

$$\frac{\partial z_i}{\partial \hat{\beta}_2} = \ln(1 - e^{-\beta_i x_i}) \quad (16)$$

$$\frac{\partial z_i}{\partial \hat{\beta}_1} = \beta_2 \times X_i \times \frac{e^{-\beta_i X_i}}{(1 - e^{-\beta_i X_i})} \quad (17)$$

Likewise, let $\beta = (\beta_0^{(0)}, \beta_1^{(0)}, \beta_2^{(0)})$ be the initial parameters. Thus, we take the Logarithm transformation of Equation (4)]. We have

$$\ln(y_i) = \ln(\beta_0) - \beta_1 e^{-\beta_2 X_i} \quad (18)$$

The NLS estimation using a modified version of the Levenberg-Marquardt, we take the derivative with respect to the parameters$(\beta_0, \beta_1, \beta_2)$. Then, substitute $(\beta_0^{(0)}, \beta_1^{(0)}, \beta_2^{(0)})$ as the initial guess vector for iteration process.

In Equation (18), let $\ln(y_i) = Z_i$, $\ln(\beta_0) = \hat{\beta}_0$, $\beta_1 = \hat{\beta}_1$ *and* $\beta_2 = \hat{\beta}_2$

$$Z_i = \hat{\beta}_0 - \hat{\beta}_1 e^{-\hat{\beta}_2 X_i} \quad (19)$$

By take partial derivative of Equation (19) with respect to $(\hat{\beta}_0, \hat{\beta}_1, \hat{\beta}_2)$, we have

$$\frac{\partial z_i}{\partial \hat{\beta}_0} = 1 \quad (20)$$

$$\frac{\partial z_i}{\partial \hat{\beta}_1} = -e^{-\hat{\beta}_2 X_i} \quad (21)$$

$$\frac{\partial z_i}{\partial \hat{\beta}_2} = \beta_2 \times X_i \times e^{-\beta_2 X_i} \quad (22)$$

Similarly, let $\beta = (\beta_0^{(0)}, \beta_1^{(0)}, \beta_2^{(0)})$ be the initial parameters. Thus, we take the Logarithm transformation of Equation (6). We have

$$\ln(y_i) = \ln(\beta_0) - \beta_1 X_i^{-\hat{\beta}_2} \quad (23)$$

The NLS estimation using a modified version of the Levenberg-Marquardt, we take the derivative with respect to the parameters$(\beta_0, \beta_1, \beta_2)$. Then, substitute $(\beta_0^{(0)}, \beta_1^{(0)}, \beta_2^{(0)})$ as the initial guess vector for iteration process.

In Equation (23), let $\ln(y_i) = Z_i$, $\ln(\beta_0) = \hat{\beta}_0$, $-\beta_1 = \hat{\beta}_1$ and $\beta_2 = \hat{\beta}_2$

$$Z_i = \hat{\beta}_0 + \hat{\beta}_1 X_i^{\hat{\beta}_2} \quad (24)$$

By take partial derivative of Equation (24) with respect to $(\hat{\beta}_0, \hat{\beta}_1, \hat{\beta}_2)$, we have

$$\frac{\partial z_i}{\partial \hat{\beta}_0} = 1 \quad (25)$$

$$\frac{\partial z_i}{\partial \hat{\beta}_1} = X_i^{\hat{\beta}_2} \quad (26)$$

$$\frac{\partial z_i}{\partial \hat{\beta}_2} = \beta_2 \times (X_i^{\hat{\beta}_2}) \times \ln(X_i^{\hat{\beta}_2}) \quad (27)$$

The initial guess values $(\beta_0^{(0)}, \beta_1^{(0)}, \beta_2^{(0)})$ were estimated by fitting a second-order polynomial (quadratic mode $\hat{\varepsilon}_i = \frac{\epsilon_i - \bar{\epsilon}}{\sigma_{e_i}{}^2}$, using Minitab 17 software.

$$Z_i = \hat{\beta}_0^{(0)} + \hat{\beta}_1^{(0)} X_i + \hat{\beta}_2^{(0)} X_i{}^2 \quad (28)$$

The Gretl statistical software was used for the analysis, by adding the data set with additive and multiplicative error terms $[\varepsilon_i , N(0,1)]$ and the initial values for the parameters basis of Equation (28). The error terms $[\varepsilon_i , N(0,1)]$ were standardized, using the Equation (29) below;

$$\hat{\varepsilon}_i = \frac{\epsilon_i - \bar{\epsilon}}{\sigma_{e_i}{}^2} \quad (29)$$

Thus, since the growth models considered are positive value and appreciating (or increasing) growth process. Therefore, we used the maximum error value to add to all the errors which in turn make the entire errors value positive.

In determining the suitable model (or most appropriate model) for growth studies, models selection criteria were employed.

3.3. Applications

Three data sets were obtained with small and large sample size. Three primary data sets were considered in this research: an experiment used to determine the amount of transmitted voltage against time collected from the Department of Electrical/Electronic Engineering, University of Port-Harcourt; data for dry matter of green gram which exhibit sigmoid shape ~~[4]~~in black gram reported; and Nigeria Population Growth from 1960 to 2017.

Five errors terms $(\varepsilon_i , N(0,1))$ were stimulated and standardized using Equation (29) and MINITAB 17 statistical software.

IV. Results

Table 1: Summary of the Coefficients and p-values of the three growth models with additive and multiplicative error terms ($\hat{\mathcal{E}}_1$) for Data 1

Model	**Model Statistics**	**Estimated Coefficients (P-values)**		Remark
		Additive Error Terms	Multiplicative Error Terms	
Weibull	Alpha Beta	$\beta_0 = -1.05 \pm 0.9923$ (0.3107) $\beta_1 = 1.0756 \pm 0.8033$ (0.2054)	$\beta_0 = -0.6937 \pm 0.6994$(0.3408) $\beta_1 = 1.1389 \pm 0.6337$(0.0975*)	MET

	Gamma	$\beta_2 = 0.4887 \pm 0.1894\ (0.0241^{**})$	$\beta_2 = 0.3174 \pm 0.1027(0.0093^{***})$	
	Iteration	21	21	
Model Selection Criteria (MSC)	BIC AIC R^2 R-Adj SSE	11.69766 9.5735 0.9297 0.9179 1.1145	10.8763 8.7521 0.9335 0.9223 1.0551	
Gompertz	Alpha Beta Gamma	$\beta_0 = -8.6251 \pm 79.6685(0.9156)$ $\beta_1 = -9.2137 \pm 79.2680(0.9094)$ $\beta_2 = -0.01229 \pm 0.0940(0.8982)$	$\beta_0 = 3.6963 \pm 0.46175(0.000^{***})$ $\beta_1 = 3.1717 \pm 0.41808(0.000^{***})$ $\beta_2 = 0.03709 \pm 0.012175(0.0101^{**})$	MET
	Iteration	40	**47**	
Model Selection Criteria (MSC)	BIC AIC R^2 R-adj SSE	32.9334 30.80929 0.71044 0.66218 4.59113	**9.5021** **7.3779** **0.9392** **0.92916** **0.9627**	
Richards	Alpha Beta Gamma	$\beta_0 = 2.4907 \pm 0.0418(0.000^{***})$ $\beta_1 = 0.3908 \pm 0.0805(0.000^{***})$ $\beta_2 = 1.8746 \pm 0.2301(0.000^{***})$	$\beta_0 = 2.61939 \pm 0.0417(0.000^{***})$ $\beta_1 = 0.11788 \pm 0.0235(0.000^{***})$ $\beta_2 = 0.770 \pm 0.04097(0.000^{***})$	MET
	Iteration	43	42	
Model Selection Criteria (MSC)	BIC AIC R^2 R-adj SSE	40.2866 38.16240 0.5272 0.4485 7.4957	35.86667 33.74251 0.6479 0.5892 5.5892	

Footnote: Sig. at * 0.10, **0.05, ***0.01; AET-Additive Error term; MET- Multiplicative Error term
The identified suitable model is the Gompertz Growth Model with Multiplicative Error Terms

Table 2: Summary of the Coefficients and p-values of the three growth models with additive and multiplicative error terms ($\hat{\varepsilon}_2$) for Data 1

Model	**Model Statistics**	**Estimated Coefficients (P-values)**		Remark
		Additive Error Terms	Multiplicative Error Terms	
Weibull	Alpha Beta Gamma	$\beta_0 = -1.0873 \pm 1.0164(0.3058)$ $\beta_1 = 1.2607 \pm 0.8767(0.1760)$ $\beta_2 = 0.4529 \pm 0.1764(0.0247^{**})$	$\beta_0 = -0.4800 \pm 0.6521(0.4758)$ $\beta_1 = 1.2053 \pm 0.5955(0.0658^{*})$ $\beta_2 = 0.3438 \pm 0.1052(0.0067^{***})$	MET
	Iteration	21	**18**	
Model Selection Criteria (MSC)	BIC AIC R^2 R-Adj SSE	11.4097 9.2855 0.9311 0.9196 1.0933	**9.1701** **7.0460** **0.9406** **0.9307** **0.9417**	
Gompertz	Alpha Beta Gamma	$\beta_0 = 2.9911 \pm 0.0269(0.000^{***})$ $\beta_1 = 4.4175 \pm 0.0912(0.000^{***})$ $\beta_2 = 0.4340 \pm 0.0178(0.000^{***})$	$\beta_0 = -1.5088 \pm 2.0438(0.4746)$ $\beta_1 = -2.6121 \pm 1.9604(0.2075)$ $\beta_2 = -0.0249 \pm 0.0133(0.0854^{*})$	MET
	Iteration	54	38	
Model Selection Criteria (MSC)	BIC AIC R^2 R-adj SSE	37.0323 34.9088 0.6194 0.5560 6.0341	33.7865 31.6624 0.6935 0.6424 4.8598	
Richards	Alpha Beta Gamma	$\beta_0 = 2.6267 \pm 0.0598(0.000^{***})$ $\beta_1 = 0.3179 \pm 0.0340(0.000^{***})$ $\beta_2 = 1.6493 \pm 0.0900(0.000^{***})$	$\beta_0 = 2.61939 \pm 0.0417(0.000^{***})$ $\beta_1 = 0.11788 \pm 0.0235(0.000^{***})$ $\beta_2 = 0.770 \pm 0.04097(0.000^{***})$	MET
	Iteration	45	49	
Model Selection Criteria (MSC)	BIC AIC R^2 R-adj SSE	36.7351 34.6109 0.6269 0.5647 5.9154	26.9004 24.7762 0.8063 0.7741 3.0708	

Footnote: Sig. at * 0.10, **0.05, ***0.01; AET-Additive Error term; MET- Multiplicative Error term
The identified suitable Model is the Weibull Growth Model with Multiplicative Error Terms

Table 3: Summary of the Coefficients and p-values of the three growth models with additive and multiplicative error terms ($\hat{\varepsilon}_3$) for Data 1

Model	**Model Statistics**	**Estimated Coefficients (P-values)**		Remark
		Additive Error Terms	Multiplicative Error Terms	
Weibull	Alpha Beta Gamma	$\beta_0 = -1.1099 \pm 1.2715(0.3999)$ $\beta_1 = 1.2759 \pm 1.0542(0.2459)$ $\beta_2 = 0.4523 \pm 0.2053(0.0479^{**})$	$\beta_0 = -0.6937 \pm 0.6994(0.3408)$ $\beta_1 = 1.1389 \pm 0.6337(0.0975^{*})$ $\beta_2 = 0.3174 \pm 0.1027(0.0093^{***})$	MET
	Iteration	22	**18**	
Model Selection Criteria (MSC)	BIC AIC R^2 R-Adj SSE	12.6954 10.5712 0.9249 0.9124 1.1912	**12.2692** **10.1450** **0.9270** **0.9148** **1.1578**	
Gompertz	Alpha Beta Gamma	$\beta_0 = 3.6723 \pm 0.2125(0.000^{***})$ $\beta_1 = 4.4190 \pm 0.3374(0.000^{***})$ $\beta_2 = 0.1846 \pm 0.0343(0.002^{***})$	$\beta_0 = -1.4772 \pm 2.7834(0.6065)$ $\beta_1 = -2.5974 \pm 2.6944(0.3541)$ $\beta_2 = -0.0253 \pm 0.0191(0.2105)$	AET
	Iteration	39	39	
Model Selection Criteria (MSC)	BIC AIC R^2 R-adj SSE	32.9334 30.80929 0.71044 0.66218 4.59113	33.4540 31.3299 0.7002 0.6502 4.7533	
Richards	Alpha Beta Gamma	$\beta_0 = 2.7214 \pm 0.1319(0.000^{***})$ $\beta_1 = 0.2335 \pm 0.0703(0.0061^{***})$ $\beta_2 = 1.5323 \pm 0.1873(0.000^{***})$	$\beta_0 = 2.5281 \pm 0.0235(0.000^{***})$ $\beta_1 = 0.3256 \pm 0.0568(0.000^{***})$ $\beta_2 = 0.7182 \pm 0.0324(0.000^{***})$	MET
	Iteration	43	43	
Model Selection Criteria (MSC)	BIC AIC R^2 R-adj SSE	33.9179 31.7937 0.6908 0.6392 4.9025	34.2595 30.1354 0.7281 0.6495 4.4823	

Footnote: Sig. at * 0.10, **0.05, ***0.01; AET-Additive Error term; MET- Multiplicative Error term
The identified suitable Model is theWeibull Growth Model with Multiplicative Error Terms

Table 4.: Summary of the Coefficients and p-values of the three growth models with additive and multiplicative error terms ($\hat{\varepsilon}_1$) for Data 2

Model	**Model Statistics**	**Estimated Coefficients (P-values)**		Remark
		Additive Error Terms	Multiplicative Error Terms	
Weibull	Alpha Beta Gamma	$\beta_0 = 12.2073 \pm 3.7978(0.0147^{**})$ $\beta_1 = -62.0465 \pm 26.5410(0.0520^{*})$ $\beta_2 = 0.5362 \pm 0.2378(0.0588^{*})$	$\beta_0 = -2.8641 \pm 2.7513(0.3325)$ $\beta_1 = 1.7696 \pm 1.8519(0.3711)$ $\beta_2 = 0.2915 \pm 0.1353(0.0681^{*})$	AET
	Iteration	41	22	
Model Selection Criteria (MSC)	BIC AIC R^2 R-Adj SSE	7.6453 6.7375 0.9837 0.9790 0.6303	17.1974 16.2897 0.9576 0.9455 1.6383	
Gompertz	Alpha Beta Gamma	$\beta_0 = 7.0332 \pm 0.3291(0.000^{***})$ $\beta_1 = 15.0455 \pm 1.1657(0.000^{***})$ $\beta_2 = 0.0371 \pm 0.0044(0.000^{***})$	$\beta_0 = 5.3792 \pm 0.5019(0.0000^{***})$ $\beta_1 = 6.9465 \pm 2.0086(0.0016^{**})$ $\beta_2 = 0.0360 \pm 0.0220(0.1455)$	AET
	Iteration	**38**	44	
Model Selection Criteria (MSC)	BIC AIC R^2 R-adj SSE	**0.9412** **0.0334** **0.9917** **0.9893** **0.3224**	36.5998 35.6923 0.7051 0.6208 11.4036	
Richards	Alpha Beta Gamma	$\beta_0 = 2.4907 \pm 0.0418(0.000^{***})$ $\beta_1 = 0.3908 \pm 0.0805(0.000^{***})$ $\beta_2 = 1.8746 \pm 0.2301(0.000^{***})$	$\beta_0 = 2.61939 \pm 0.0417(0.000^{***})$ $\beta_1 = 0.11788 \pm 0.0235(0.000^{***})$ $\beta_2 = 0.770 \pm 0.04097(0.000^{***})$	AET
	Iteration	15	38	
Model Selection Criteria (MSC)	BIC AIC R^2 R-adj	4.9724 4.0646 0.9875 0.9839	37.2782 36.3705 0.6844 0.5942	

	SSE	0.4825	12.2039	

Footnote: Sig. at * 0.10, **0.05, ***0.01; AET-Additive Error term; MET- Multiplicative Error term

The identified suitable Model is the Gompertz Growth Model with Additive Error Terms

Table 5: Summary of the Coefficients and p-values of the three growth models with additive and multiplicative error terms ($\hat{\varepsilon}_2$) for Data 2

Model	**Model Statistics**	**Estimated Coefficients (P-values)**		Remark
		Additive Error Terms	Multiplicative Error Terms	
Weibull	Alpha Beta Gamma	$\beta_0 = 11.0089 \pm 2.5977(0.0038^{***})$ $\beta_1 = -74.9402 \pm 35.0908(0.0701^{*})$ $\beta_2 = -0.6343 \pm 0.2268(0.0267^{**})$	$\beta_0 = -3.0867 \pm 2.3480(0.2301)$ $\beta_1 = 0.2517 \pm 2.1374(0.2370)$ $\beta_2 = 0.2517 \pm 0.1108(0.0573^{*})$	AET
	Iteration	38	19	
Model Selection Criteria (MSC)	BIC AIC R^2 R-Adj SSE	6.5464 5.6387 0.9853 0.9812 0.5647	15.6104 14.7026 0.9638 0.9535 1.3979	
Gompertz	Alpha Beta Gamma	$\beta_0 = 9.0743 \pm 1.1124(0.0000^{***})$ $\beta_1 = 10.8988 \pm 0.6697(0.0000^{***})$ $\beta_2 = 0.0137 \pm 0.0030(0.0000^{***})$	$\beta_0 = 5.9910 \pm 0.1515(0.0000^{***})$ $\beta_1 = 6.6892 \pm 0.2088(0.0000^{***})$ $\beta_2 = 0.0750 \pm 0.0107(0.0002^{***})$	AET
	Iteration	47	42	
Model Selection Criteria (MSC)	BIC AIC R^2 R-adj SSE	34.9580 34.0502 0.7497 0.6782 9.6768	37.6848 36.7771 0.6713 0.5774 12.7103	
Richards	Alpha Beta Gamma	$\beta_0 = 7.33838 \pm 0.5486(0.0000^{***})$ $\beta_1 = 0.0276 \pm 0.0071(0.0061^{***})$ $\beta_2 = 8.7718 \pm 1.6175(0.00010^{***})$	$\beta_0 = 5.2680 \pm 0.2273(0.0000^{***})$ $\beta_1 = 0.0252 \pm 0.0153(0.1436)$ $\beta_2 = 1.4865 \pm 0.2182(0.0003^{***})$	AET
	Iteration	**12**	39	
Model Selection Criteria (MSC)	BIC AIC R^2 R-adj SSE	**3.2677** **2.3599** **0.9895** **0.9865** **0.4069**	34.3766 34.4689 0.7639 0.6964 9.1303	

Footnote: Sig. at * 0.10, **0.05, ***0.01; AET-Additive Error term; MET- Multiplicative Error term

The identified suitable Model is the Richards Growth Model with Additive Error Terms

Table 6: Summary of the Coefficients and p-values of the three growth models with additive and multiplicative error terms ($\hat{\varepsilon}_3$) for Data 2

Model	**Model Statistics**	**Estimated Coefficients (P-values)**		Remark
		Additive Error Terms	Multiplicative Error Terms	
Weibull	Alpha Beta Gamma	$\beta_0 = 11.0908 \pm 2.7203(0.0047^{***})$ $\beta_1 = -74.2928 \pm 35.4165(0.0741^{*})$ $\beta_2 = -0.6088 \pm 0.2326(0.0305^{**})$	$\beta_0 = -2.5354 \pm 2.3033(0.3074)$ $\beta_1 = 2.17453 \pm 2.0259(0.3187)$ $\beta_2 = 0.2948 \pm 0.1460(0.0832^{*})$	AET
	Iteration	39	17	
Model Selection Criteria (MSC)	BIC AIC R^2 R-Adj SSE	7.0405 6.1327 0.9847 0.9803 0.5933	19.8187 18.9109 0.9449 0.9292 2.1293	
Gompertz	Alpha Beta Gamma	$\beta_0 = 9.0396 \pm 1.1375(0.0000^{***})$ $\beta_1 = 10.9773 \pm 0.6640(0.0000^{***})$ $\beta_2 = 0.0141 \pm 0.0032(0.0032^{***})$	$\beta_0 = 6.6015 \pm 0.1640(0.0000^{***})$ $\beta_1 = 6.9416 \pm 0.2800(0.0000^{***})$ $\beta_2 = 0.0339 \pm 0.0047(0.0002^{***})$	MET
	Iteration	47	40	
Model Selection Criteria (MSC)	BIC AIC R^2 R-adj SSE	34.5163 33.6085 0.7606 0.6921 9.2587	25.5124 24.6046 0.9027 0.8748 3.7628	
Richards	Alpha Beta	$\beta_0 = 7.3565 \pm 0.5708(0.0000^{***})$ $\beta_1 = 0.0275 \pm 0.0073(0.0072^{***})$	$\beta_0 = 5.5057 \pm 0.5053(0.0000^{***})$ $\beta_1 = 0.0205 \pm 0.0178(0.2860)$	AET

	Gamma	$\beta_2 = 8.7695 \pm 1.6631(0.0012^{***})$	$\beta_2 = 1.6146 \pm 0.2851(0.0008^{***})$	
	Iteration	**12**	40	
Model Selection Criteria (MSC)	BIC AIC R^2 R-adj SSE	**3.8671** **2.9594** **0.9888** **0.9856** **0.4320**	33.4109 32.5032 0.7856 0.7244 8.2998	

Footnote: Sig. at * 0.10, **0.05, ***0.01; AET-Additive Error term; MET- Multiplicative Error term
The identified suitable Model is the Richards Growth Model with Additive Error Terms

Table 7: Summary of the Coefficients and p-values of the three growth models with additive and multiplicative error terms ($\hat{\varepsilon}_1$) for Data 3

Model	**Model Statistics**	**Estimated Coefficients (P-values)**		Remark
		Additive Error Terms	Multiplicative Error Terms	
Weibull	Alpha Beta Gamma	$\beta_0 = 10.5768 \pm 0.0094(0.0000^{***})$ $\beta_1 = 0.00348 \pm 0.0018(0.0000^{***})$ $\beta_2 = 0.9226 \pm 0.0111(0.0000^{***})$	$\beta_0 = 10.5794 \pm 0.0260(0.0000^{***})$ $\beta_1 = 0.0755 \pm 0.0105(0.0000^{***})$ $\beta_2 = 0.5572 \pm 0.0246(0.0000^{***})$	AET
	Iteration	**16**	15	
Model Selection Criteria (MSC)	BIC AIC R^2 R-Adj SSE	**-375.3524** **-381.5337** **0.9996** **0.9996** **0.0043**	-251.3037 -245.1224 0.9964 0.9963 0.0403	
Gompertz	Alpha Beta Gamma	$\beta_0 = 14.4188 \pm 0.1129(0.0000^{***})$ $\beta_1 = 3.8706 \pm 0.1097(0.0000^{***})$ $\beta_2 = 0.0083 \pm 0.0003(0.0000^{***})$	$\beta_0 = 13.2164 \pm 0.1010(0.0000^{***})$ $\beta_1 = 2.4358 \pm 0.0923(0.0000^{***})$ $\beta_2 = 0.0037 \pm 0.0002(0.0000^{***})$	AET
	Iteration	43	47	
Model Selection Criteria (MSC)	BIC AIC R^2 R-adj SSE	-280.8538 -287.0351 0.9981 0.9980 0.0217	-166.6205 -172.8018 0.9862 0.9857 0.1556	
Richards	Alpha Beta Gamma	$\beta_0 = 13.0373 \pm 0.3486(0.0000^{***})$ $\beta_1 = 0.0013 \pm 0.0011(0.2188)$ $\beta_2 = 0.4686 \pm 0.0062(0.0000^{***})$	$\beta_0 = 12.3923 \pm 1.7806(0.0000^{***})$ $\beta_1 = 0.0004 \pm 0.0028(0.8982)$ $\beta_2 = 0.2379 \pm 0.0143(0.0000^{***})$	AET
	Iteration	45	44	
Model Selection Criteria (MSC)	BIC AIC R^2 R-adj SSE	-19.8209 -26.0023 0.8268 0.8205 1.9558	2.6092 -3.5720 0.7450 0.7357 2.8793	

Footnote: Sig. at * 0.10, **0.05, ***0.01; AET-Additive Error term; MET- Multiplicative Error term

The identified suitable Model is the Weibull Growth Model with Additive Error Terms

Table 8: Summary of the Coefficients and p-values of the three growth models with additive and multiplicative error terms ($\hat{\varepsilon}_2$) for Data 3

Model	**Model Statistics**	**Estimated Coefficients (P-values)**		Remark
		Additive Error Terms	Multiplicative Error Terms	
Weibull	Alpha Beta Gamma	$\beta_0 = 10.5640 \pm 0.0120(0.0000^{***})$ $\beta_1 = 0.0369 \pm 0.0023(0.0000^{***})$ $\beta_2 = 0.9104 \pm 0.0140(0.0000^{***})$	$\beta_0 = 10.5262 \pm 0.0372(0.0000^{***})$ $\beta_1 = 0.0920 \pm 0.0139(0.0000^{***})$ $\beta_2 = 0.5224 \pm 0.0245(0.0000^{***})$	AET
	Iteration	**13**	13	
Model Selection Criteria (MSC)	BIC AIC R^2 R-Adj SSE	**-359.8061** **-365.9875** **0.9995** **0.9994** **0.0056**	-231.4580 -237.6393 0.9954 0.9953 0.0509	
Gompertz	Alpha Beta Gamma	$\beta_0 = 17.5473 \pm 0.5269(0.0000^{***})$ $\beta_1 = 6.9805 \pm 0.5230(0.0000^{***})$ $\beta_2 = 0.0041 \pm 0.0003(0.0000^{***})$	$\beta_0 = 13.2164 \pm 0.1010(0.0000^{***})$ $\beta_1 = 2.4358 \pm 0.0924(0.0000^{***})$ $\beta_2 = 0.0037 \pm 0.0002(0.0002^{***})$	AET
	Iteration	43	47	
Model Selection	BIC AIC	-339.0767 -345.2580	-166.6205 -172.8018	

Criteria (MSC)	R^2 R-adj SSE	0.9993 0.9993 0.0080	0.9862 0.9857 0.1556	
Richards	Alpha Beta Gamma	$\beta_0 = 12.9135 \pm 0.3294(0.0000^{***})$ $\beta_1 = 0.0018 \pm 0.0014(0.0061^{***})$ $\beta_2 = 0.4810 \pm 0.0087(0.0000^{***})$	$\beta_0 = 11.8867 \pm 0.0969(0.0000^{***})$ $\beta_1 = 0.0042 \pm 0.0023(0.0701^{*})$ $\beta_2 = 0.2773 \pm 0.0201(0.0000^{***})$	AET
	Iteration	43	38	
Model Selection Criteria (MSC)	BIC AIC R^2 R-adj SSE	-21.2024 -27.3837 0.8309 0.8247 1.9098	6.8498 0.6685 0.7256 0.7157 3.0976	

Footnote: Sig. at * 0.10, **0.05, ***0.01; AET-Additive Error term; MET- Multiplicative Error term

The identified suitable Model is the Weibull Growth Model with Additive Error Terms

Table 9: Summary of the Coefficients and p-values of the three growth models with additive and multiplicative error terms ($\hat{\varepsilon}_3$) for Data 3

Model	Model Statistics	Estimated Coefficients (P-values)		Remark
		Additive Error Terms	Multiplicative Error Terms	
Weibull	Alpha Beta Gamma	$\beta_0 = 10.5559 \pm 0.0137(0.0000^{***})$ $\beta_1 = 0.0401 \pm 0.0029(0.0000^{***})$ $\beta_2 = 0.8910 \pm 0.0157(0.0000^{***})$	$\beta_0 = 10.4588 \pm 0.0714(0.0000^{***})$ $\beta_1 = 0.1406 \pm 0.0349(0.0002^{***})$ $\beta_2 = 0.4507 \pm 0.0396(0.0000^{***})$	AET
	Iteration	**13**	22	
Model Selection Criteria (MSC)	BIC AIC R^2 R-Adj SSE	**-352.4893** **-358.6706** **0.9994** **0.9994** **0.0063**	-187.8935 -194.0748 0.9905 0.9901 0.1078	
Gompertz	Alpha Beta Gamma	$\beta_0 = 14.4188 \pm 0.1129(0.0000^{***})$ $\beta_1 = 3.8706 \pm 0.1097(0.0000^{***})$ $\beta_2 = 0.0083 \pm 0.0003(0.0000^{***})$	$\beta_0 = 13.2041 \pm 0.1654(0.0000^{***})$ $\beta_1 = 2.3852 \pm 0.1586(0.0000^{***})$ $\beta_2 = 0.0035 \pm 0.0004(0.0000^{***})$	AET
	Iteration	43	43	
Model Selection Criteria (MSC)	BIC AIC R^2 R-adj SSE	-242.2512 -248.4326 0.9963 0.9961 0.0422	-108.1707 -114.3521 0.9622 0.9609 0.4264	
Richards	Alpha Beta Gamma	$\beta_0 = 13.5148 \pm 0.0658(0.0000^{***})$ $\beta_1 = 3.0009 \pm 0.0621(0.0000^{***})$ $\beta_2 = 0.0119 \pm 0.0004(0.0000^{***})$	$\beta_0 = 11.6658 \pm 0.008(0.0000^{***})$ $\beta_1 = 0.0162 \pm 0.0011(0.0000^{***})$ $\beta_2 = 0.3064 \pm 0.00069(0.0000^{***})$	AET
	Iteration	42	36	
Model Selection Criteria (MSC)	BIC AIC R^2 R-adj SSE	-23.8720 -30.0534 0.8385 0.8326 1.8238	26.9774 20.7960 0.6119 0.5977 2.8793	

Footnote: Sig. at * 0.10, **0.05, ***0.01; AET-Additive Error term; MET- Multiplicative Error term
The identified suitable Model is the Weibull Growth Model with Additive Error Terms

V. Discussion Of The Result

For Data 1: The result identified Gompertz Growth model with Multiplicative error term for $\hat{\varepsilon}_1$ as the suitable model, since the parameters of alpha, beta and gamma are significant at 1% (Table 1). Weibull Growth model with Multiplicative error term was identified as the suitable model for $\hat{\varepsilon}_2$ because the parameters beta and gamma are significant at 10% and 1% respectively (Table 2). Weibull Growth model with Multiplicative error term was identified as the suitable model for $\hat{\varepsilon}_3$ with the parameters beta and gamma are significant at 10% and 1% respectively (Table 3).

For Data 2: Similarly, the result identified Gompertz Growth model with additive error term for $\hat{\varepsilon}_1$ as the suitable model, since the parameters of alpha, beta and gamma are significant at 1% (Table 4). Richard Growth model with additive error term was also identified as the suitable model for $\hat{\varepsilon}_2$ because all the parameters are significant at 1% (Table 5). Richard Growth model with additive error term was identified as the suitable model for $\hat{\varepsilon}_3$, since all the parameters are significant at 1% respectively (Table 6).

For Data 3: Furthermore, the result identified Weibull Growth model with additive error term for $\hat{\varepsilon}_1$ as the suitable model, since the parameters of alpha, beta and gamma are significant at 1% (Table 7). Weibull Growth model with additive error term was also identified as the suitable model for $\hat{\varepsilon}_2$ because all the parameters are significant at 1% (Table 8). Weibull Growth model with additive error term was identified as the suitable model for $\hat{\varepsilon}_3$, since all the parameters are significant at 1% respectively (Table 9).

Therefore, the three data sets result (Agricultural, Engineering and population data sets), Weibull Growth model with additive error term is identified as the best suitable model for growth analysis.

VI. Conclusion

This study was able to show the use of the three growth model using real data sets from Population, Engineering and Agricultural product. The problem of the initial parameters is addressed by second-order regression techniques before an iterative approach is done in this study. The three growth models were decomposed by additive and multiplicative error terms.

The modified version of the Levenberg-Marquardt method for solving non-linear regression model was used. A suitable Weibull growth model with Additive error term was determined using Model selection Criteria (like Mean squared error, R^2, BIC and AIC) among all the models fitted. It showed that the Weibull model is adequate and can be used for forecasting. Hence, this study recommended the Weibull Growth with Additive Error Terms as the best model for growth curves data set.

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