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An Extended Generalized Estimator in Double Sampling for the Estimation of Finite Population Variance

S. Athar Hasan Rizvi and S. Ali Manzar Rizvi[1]
Department of Statistics, Shia P. G. College,
Lucknow-226020, INDIA.

Abstract: *For the estimation of finite population variance in double sampling, an extended generalized class of estimators is proposed. After obtaining the bias and mean square error of the proposed estimator, it is compared with the usual conventional estimator of the finite population variance in the sense of having lesser mean square error. A subclass of estimators relying on optimum values for which the subclass achieves the minimum mean square error is investigated and further a subclass of estimators depending upon estimated optimum value is also explored and its properties are studied.*
KEY-WORDS: Bias and mean square error, Generalized class of estimators, Optimum values, Estimated optimum values and Efficient estimator.

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I. Introduction

Let a first phase simple random sample without replacement be drawn from a population of size N, and a second phase simple random sample of size n without replacement be drawn from the first phase sample of size n'. At first phase sample of size n', only the auxiliary character x is observed and at the second phase subsample of size n, both the study variable y and the auxiliary character x are observed.

Let $(\bar{Y}, \bar{X})$ be the population means of (y, x) respectively and $S_y^2 = \frac{1}{(N-1)}\sum_{i=1}^{N}(Y_i - \bar{Y})^2$, $S_x^2 = \frac{1}{(N-1)}\sum_{i=1}^{N}(X_i - \bar{X})^2$ where (Y_i, X_i) are the population values on (y, x) respectively, for the i^{th} unit $(i = 1,2,\dots,N)$ of the population. Also, let $(\bar{y}, \bar{x})$ based on second phase sample of size n be the sample means of (y, x), $\bar{x}'$ be the sample mean of the first phase n' sample values on the auxiliary character x.

Let $s_x'^2 = \frac{1}{(n'-1)}\sum_{i=1}^{n}(x_i' - \bar{x}')^2$ based on the first phase sample observations $(x_1', x_2', \dots, x_{n'}')$, $s_x^2 = \frac{1}{(n-1)}\sum_{i=1}^{n}(x_i - \bar{x})^2$ based on the second phase sample observations $(x_1, x_2, \dots, x_n)$ on x be the unbiased estimator of $S_x^2 = \frac{N}{N-1}\sigma_x^2$ and $s_y^2 = \frac{1}{(n-1)}\sum_{i=1}^{n}(y_i - \bar{y})^2$ based on thesecond phase sample observations $(y_1, y_2, \dots, y_n)$ on y be the conventional estimator of the finite population variance σ_y^2 of the study variable y.

For estimating σ_y^2, Rizvi, S.A.M and Rizvi S.A.H. (2017) proposed the following generalized double sampling estimator:

$$\hat{\sigma}_d^2 = g(s_y^2, \bar{x}, \bar{x}') \tag{1.1}$$

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[1]***Author for Correspondence***

where $g\left(s_y^2, \bar{x}, \bar{x}'\right)$ satisfying the validity conditions of Taylor's series expansion is a bounded function of $\left(s_y^2, \bar{x}, \bar{x}'\right)$ such that at the point $Q = \left(\sigma_y^2, \bar{X}, \bar{X}\right)$,

(i) $$g\left(\sigma_y^2, \bar{X}, \bar{X}\right) = \sigma_y^2 \quad ; \tag{1.2}$$

(ii) first order partial derivative of $g\left(s_y^2, \bar{x}, \bar{x}'\right)$ with respect to s_y^2 at the point Q is unity, that is,

$$g_0 = \left.\frac{\partial g\left(s_y^2, \bar{x}, \bar{x}'\right)}{\partial s_y^2}\right]_Q = 1 \quad , \tag{1.3}$$

(iii) $$g_1 = -g_2 \tag{1.4}$$

for first order partial derivatives

$$g_1 = \left.\frac{\partial g\left(s_y^2, \bar{x}, \bar{x}'\right)}{\partial \bar{x}}\right]_Q \quad , \qquad g_2 = \left.\frac{\partial g\left(s_y^2, \bar{x}, \bar{x}'\right)}{\partial \bar{x}'}\right]_Q \; ;$$

(iv) second order partial derivative

$$g_{00} = \left.\frac{\partial^2 g\left(s_y^2, \bar{x}, \bar{x}'\right)}{\partial \left(s_y^2\right)^2}\right]_Q = 0 \tag{1.5}$$

(v) $$g_{01} = -g_{02} \tag{1.6}$$

for $$g_{01} = \left.\frac{\partial^2 g\left(s_y^2, \bar{x}, \bar{x}'\right)}{\partial s_y^2 \partial \bar{x}}\right]_Q \quad , \qquad g_{02} = \left.\frac{\partial^2 g\left(s_y^2, \bar{x}, \bar{x}'\right)}{\partial s_y^2 \partial \bar{x}'}\right]_Q .$$

The optimum value of g_1 minimizing $MSE(\hat{\sigma}_d^2)$ is

$$g_1 = -\frac{\mu_{21}}{\mu_{02}} = -\frac{\sigma_y^2}{\bar{X}}\left(\frac{\mu_{21}/\sigma_y^2 \bar{X}}{C_x^2}\right) = D \tag{1.7}$$

and the minimum mean square error is

$$MSE\left(\hat{\sigma}_d^2\right)_{min} = MSE\left(s_y{}^2\right) - \left(\frac{1}{n} - \frac{1}{n'}\right)\frac{\mu_{21}^2}{\mu_{02}} \; . \tag{1.8}$$

The optimum value D in (1.7) may not be known always in practice, hence the alternative is to replace the parameters involved in the optimum value by their unbiased or consistent estimators to get the estimated optimum value depending upon sample observations as

$$\hat{D} = -\frac{s_y^2}{\bar{x}}\hat{\lambda} \tag{1.9}$$

where $\hat{\lambda} = \dfrac{m_{21}/s_y^2 \bar{x}}{\hat{C}_x^2}$, $\hat{C}_x^2 = \dfrac{m_{02}}{\bar{x}^2}$.

Using the estimated optimum value in (1.9), the estimator depending upon estimated optimum value is given as a function

$$g^*\left(s_y^2, \bar{x}, \bar{x}', \hat{D}\right) \tag{1.10}$$

attains the minimum mean square error given by (1.8).

Proposed Extended Generalized Estimator

For estimating σ_y^2, the proposed extended generalized double sampling estimator is:

$$\hat{\sigma}_{gd}^2 = g\left(s_y^2, \bar{x}, \bar{x}', s_x^2, s_x'^2\right) \quad (1.11)$$

where $g\left(s_y^2, \bar{x}, \bar{x}', s_x^2, s_x'^2\right)$ satisfying the validity conditions of Taylor's series expansion is a bounded function of $\left(s_y^2, \bar{x}, \bar{x}', s_x^2, s_x'^2\right)$ such that at the point $Q = \left(\sigma_y^2, \bar{X}, \bar{X}, S_x^2, S_x^2\right)$,

(i) $g\left(\sigma_y^2, \bar{X}, \bar{X}, S_x^2, S_x^2\right) = \sigma_y^2$; (1.12)

(ii) first order partial derivative of $g\left(s_y^2, \bar{x}, \bar{x}', s_x^2, s_x'^2\right)$ with respect to s_y^2 at the point Q is unity, that is,

$$g_0 = \left.\frac{\partial g\left(s_y^2, \bar{x}, \bar{x}', s_x^2, s_x'^2\right)}{\partial s_y^2}\right]_Q = 1 \quad , \quad (1.13)$$

(iii) $g_1 = -g_2$ for first order partial derivatives (1.14)

$$g_1 = \left.\frac{\partial g\left(s_y^2, \bar{x}, \bar{x}', s_x^2, s_x'^2\right)}{\partial \bar{x}}\right]_Q , g_2 = \left.\frac{\partial g\left(s_y^2, \bar{x}, \bar{x}', s_x^2, s_x'^2\right)}{\partial \bar{x}'}\right]_Q ;$$

(iv) and $g_3 = -g_4$ for (1.15)

$$g_3 = \left.\frac{\partial^2 g\left(s_y^2, \bar{x}, \bar{x}', s_x^2, s_x'^2\right)}{\partial s_x^2}\right]_Q , \quad g_4 = \left.\frac{\partial^2 g\left(s_y^2, \bar{x}, \bar{x}', s_x^2, s_x'^2\right)}{\partial s_x'^2}\right]_Q ,$$

(v) second order partial derivative

$$g_{00} = \left.\frac{\partial^2 g\left(s_y^2, \bar{x}, \bar{x}', s_x^2, s_x'^2\right)}{\partial \left(s_y^2\right)^2}\right]_Q = 0 \quad , \quad (1.16)$$

(vi) $g_{01} = -g_{02}$ for (1.17)

$$g_{01} = \left.\frac{\partial^2 g\left(s_y^2, \bar{x}, \bar{x}', s_x^2, s_x'^2\right)}{\partial s_y^2 \partial \bar{x}}\right]_Q , \quad g_{02} = \left.\frac{\partial^2 g\left(s_y^2, \bar{x}, \bar{x}', s_x^2, s_x'^2\right)}{\partial s_y^2 \partial \bar{x}'}\right]_Q ,$$

and (vii) $g_{03} = -g_{04}$ for (1.18)

$$g_{03} = \left.\frac{\partial^2 g\left(s_y^2, \bar{x}, \bar{x}', s_x^2, s_x'^2\right)}{\partial s_y^2 \partial s_x^2}\right]_Q , \quad g_{04} = \left.\frac{\partial^2 g\left(s_y^2, \bar{x}, \bar{x}', s_x^2, s_x'^2\right)}{\partial s_y^2 \partial s_x'^2}\right]_Q .$$

Some particular members belonging to the class of estimators represented by the generalized estimator $\hat{\sigma}_{gd}^2$ are listed below:

(i) $\hat{\sigma}_1^2 = s_y^2 \left(\frac{\bar{x}}{\bar{x}'}\right)\left(\frac{s_x^2}{s_x'^2}\right)$

(ii) $\hat{\sigma}_2^2 = s_y^2 \left(\frac{\bar{x}'}{\bar{x}}\right)^{k_1} \left(\frac{s_x^2}{s_x'^2}\right)^{k_2}$

(iii) $\hat{\sigma}_3^2 = s_y^2 + k_1(\bar{x}' - \bar{x}) + k_2(s_x'^2 - s_x^2)$

and (iv) $\hat{\sigma}_4^2 = (1 - k_1 - k_2)s_y^2 + k_1 s_y^2 \left(\frac{\bar{x}}{\bar{x}'}\right) + k_2 s_y^2 \left(\frac{s_x^2}{s_x'^2}\right)$

where k_1 and k_2 are the characterizing scalars to be chosen suitably. It may be easily checked that the regularity conditions mentioned for the generalized estimator $\hat{\sigma}_{gd}^2$ are satisfied for all the four estimators $\hat{\sigma}_1^2$, $\hat{\sigma}_2^2$, $\hat{\sigma}_3^2$ and $\hat{\sigma}_4^2$.

II. Bias And Mean Square Error

Let

$$\mu_{rs} = \frac{1}{N}\sum_{i=1}^{N}(Y_i - \bar{Y})^r (X_i - \bar{X})^s$$

where (Y_i, X_i) are the values on the characters (y, x) respectively for the $i^{th}(i = 1,2,...,N)$ unit of the population.

Further, let

$e_1 = \bar{x} - \bar{X}$, $e_1' = \bar{x}' - \bar{X}$, $e_2 = s_y^2 - S_y^2$, $e_3 = s_x^2 - S_x^2$, $e_3' = s_x'^2 - S_x^2$ so that(for large N)

$E(e_1) = E(e_1') = E(e_2) = E(e_3) = E(e_3') = 0$ and

$$E(e_1^2) = \frac{\mu_{02}}{n},\ E(e_1'^2) = \frac{\mu_{02}}{n'},$$

$$E(e_2^2) = \frac{\mu_{20}^2}{n}\left(\frac{\mu_{40}}{\mu_{20}^2} - 1\right) = \frac{\mu_{20}^2}{n}(\beta_{2y} - 1),\ \beta_{2y} = \frac{\mu_{40}}{\mu_{20}^2}$$

$$E(e_3^2) = \frac{\mu_{02}^2}{n}\left(\frac{\mu_{04}}{\mu_{02}^2} - 1\right) = \frac{\mu_{02}^2}{n}(\beta_{2x} - 1),\ \beta_{2x} = \frac{\mu_{04}}{\mu_{02}^2},$$

$$E(e_3'^2) = \frac{\mu_{02}^2}{n'}(\beta_{2x} - 1),\ E(e_1 e_1') = \frac{\mu_{02}}{n'},\ E(e_1 e_2) = \frac{\mu_{21}}{n},\ E(e_1' e_2) = \frac{\mu_{21}}{n'},$$

$$E(e_1 e_3) = \frac{\mu_{03}}{n},\ E(e_1 e_3') = \frac{\mu_{03}}{n'},\ E(e_1' e_3) = \frac{\mu_{03}}{n'},\ E(e_1' e_3') = \frac{\mu_{03}}{n'},$$

$$E(e_2 e_3) = \frac{1}{n}(\mu_{22} - \mu_{20}\mu_{02}),\ E(e_2 e_3') = \frac{1}{n'}(\mu_{22} - \mu_{20}\mu_{02}),$$

$$E(e_3 e_3') = \frac{\mu_{02}^2}{n'}(\beta_{2x} - 1)\ .$$

Expanding $g(s_y^2, \bar{x}, \bar{x}', s_x^2, s_x'^2)$ about the point $Q = (\sigma_y^2, \bar{X}, \bar{X}, S_x^2, S_x^2)$ in third order Taylor's series, we have

$$\hat{\sigma}_{gd}^2 = g\left(\sigma_y^2, \overline{X}, \overline{X}, S_x^2, S_x^2\right) + \left(s_y^2 - S_y^2\right)g_0 + \left(\overline{x} - \overline{X}\right)g_1 + \left(\overline{x}' - \overline{X}\right)g_2$$
$$+ \left(s_x^2 - S_x^2\right)g_3 + \left(s_x'^2 - S_x^2\right)g_4 + \frac{1}{2!}\Big\{ \left(s_y^2 - S_y^2\right)^2 g_{00}$$
$$+ \left(\overline{x} - \overline{X}\right)^2 g_{11} + \left(\overline{x}' - \overline{X}\right)^2 g_{22} + \left(s_x^2 - S_x^2\right)^2 g_{33} + \left(s_x'^2 - S_x^2\right)^2 g_{44}$$
$$+ 2\left(s_y^2 - S_y^2\right)\left(\overline{x} - \overline{X}\right)g_{01} + 2\left(s_y^2 - S_y^2\right)\left(\overline{x}' - \overline{X}\right)g_{02}$$
$$+ 2\left(s_y^2 - S_y^2\right)\left(s_x^2 - S_x^2\right)g_{03} + 2\left(s_y^2 - S_y^2\right)\left(s_x'^2 - S_x^2\right)g_{04}$$
$$+ 2\left(\overline{x} - \overline{X}\right)\left(\overline{x}' - \overline{X}\right)g_{12} + 2\left(\overline{x} - \overline{X}\right)\left(s_x^2 - S_x^2\right)g_{13}$$
$$+ 2\left(\overline{x} - \overline{X}\right)\left(s_x'^2 - S_x^2\right)g_{14} + 2\left(\overline{x}' - \overline{X}\right)\left(s_x^2 - S_x^2\right)g_{23}$$
$$+ 2\left(\overline{x}' - \overline{X}\right)\left(s_x'^2 - S_x^2\right)g_{24} + 2\left(s_x^2 - S_x^2\right)\left(s_x'^2 - S_x^2\right)g_{34} \Big\}$$
$$+ \frac{1}{3!}\Big\{ \left(s_y^2 - S_y^2\right)\frac{\partial}{\partial s_y^2} + \left(\overline{x} - \overline{X}\right)\frac{\partial}{\partial \overline{x}} + \left(\overline{x}' - \overline{X}\right)\frac{\partial}{\partial \overline{x}'}$$
$$+ \left(s_x^2 - S_x^2\right)\frac{\partial}{\partial s_x^2} + \left(s_x'^2 - S_x^2\right)\frac{\partial}{\partial s_x'^2} \Big\}^3 g\left(s_{y*}^2, \overline{x}_*, \overline{x}'_*, s_{x*}^2, s_{x*}'^2\right) \quad (2.1)$$

where second order partial derivatives g_{11}, g_{22}, g_{33}, g_{44}, g_{12}, g_{13}, g_{14}, g_{23}, g_{24} and g_{34} are given by

$$g_{11} = \left.\frac{\partial^2 g\left(s_y^2, \overline{x}, \overline{x}', s_x^2, s_x'^2\right)}{\partial \overline{x}^2}\right]_Q, \quad g_{22} = \left.\frac{\partial^2 g\left(s_y^2, \overline{x}, \overline{x}', s_x^2, s_x'^2\right)}{\partial \overline{x}'^2}\right]_Q,$$

$$g_{33} = \left.\frac{\partial^2 g\left(s_y^2, \overline{x}, \overline{x}', s_x^2, s_x'^2\right)}{\partial \left(s_x^2\right)^2}\right]_Q, \quad g_{44} = \left.\frac{\partial^2 g\left(s_y^2, \overline{x}, \overline{x}', s_x^2, s_x'^2\right)}{\partial \left(s_x'^2\right)^2}\right]_Q,$$

$$g_{12} = \left.\frac{\partial^2 g\left(s_y^2, \overline{x}, \overline{x}', s_x^2, s_x'^2\right)}{\partial \overline{x} \partial \overline{x}'}\right]_Q, \quad g_{13} = \left.\frac{\partial^2 g\left(s_y^2, \overline{x}, \overline{x}', s_x^2, s_x'^2\right)}{\partial \overline{x} \partial s_x^2}\right]_Q,$$

$$g_{14} = \left.\frac{\partial^2 g\left(s_y^2, \overline{x}, \overline{x}', s_x^2, s_x'^2\right)}{\partial \overline{x} \partial s_x'^2}\right]_Q, \quad g_{23} = \left.\frac{\partial^2 g\left(s_y^2, \overline{x}, \overline{x}', s_x^2, s_x'^2\right)}{\partial \overline{x}' \partial s_x^2}\right]_Q,$$

$$g_{24} = \left.\frac{\partial^2 g\left(s_y^2, \overline{x}, \overline{x}', s_x^2, s_x'^2\right)}{\partial \overline{x}' \partial s_x'^2}\right]_Q, \quad g_{34} = \left.\frac{\partial^2 g\left(s_y^2, \overline{x}, \overline{x}', s_x^2, s_x'^2\right)}{\partial s_x^2 \partial s_x'^2}\right]_Q;$$

g_0, g_1, g_2, g_3, g_4, g_{00}, g_{01}, g_{02}, g_{03} and g_{04} are already defined and

$$s_{y*}^2 = \sigma_y^2 + \theta\left(s_y^2 - S_y^2\right), \qquad \overline{x}_* = \overline{X} + \theta\left(\overline{x} - \overline{X}\right),$$

$$\bar{x}'_* = \bar{X} + \theta(\bar{x}' - \bar{X}) \quad , \quad s^2_{x*} = S^2_x + \theta(s^2_x - S^2_x),$$
$$s'^2_{x*} = S^2_x + \theta(s'^2_x - S^2_x) \quad \text{for } 0 < \theta < 1.$$

Employing regularity conditions $g(\sigma^2_y, \bar{X}, \bar{X}, S^2_x, S^2_x) = \sigma^2_y$, $g_0 = 1$, $g_{00} = 0$, $g_1 = -g_2$, $g_{01} = -g_{02}$, $g_3 = -g_4$ and $g_{03} = -g_{04}$ from (1.12) to (1.18) in (2.1), wehave

$$\begin{aligned}\hat{\sigma}^2_{gd} &= \sigma^2_y + (s^2_y - S^2_y) + (\bar{x} - \bar{X})g_1 - (\bar{x}' - \bar{X})g_1 \\ &+ (s^2_x - S^2_x)g_3 - (s'^2_x - S^2_x)g_3 + \frac{1}{2!}\Big\{ (\bar{x} - \bar{X})^2 g_{11} \\ &+ (\bar{x}' - \bar{X})^2 g_{22} + (s^2_x - S^2_x)^2 g_{33} + (s'^2_x - S^2_x)^2 g_{44} \\ &+ 2(s^2_y - S^2_y)(\bar{x} - \bar{X})g_{01} - 2(s^2_y - S^2_y)(\bar{x}' - \bar{X})g_{01} \\ &+ 2(s^2_y - S^2_y)(s^2_x - S^2_x)g_{03} - 2(s^2_y - S^2_y)(s'^2_x - S^2_x)g_{03} \\ &+ 2(\bar{x} - \bar{X})(\bar{x}' - \bar{X})g_{12} + 2(\bar{x} - \bar{X})(s^2_x - S^2_x)g_{13} \\ &+ 2(\bar{x} - \bar{X})(s'^2_x - S^2_x)g_{14} + 2(\bar{x}' - \bar{X})(s^2_x - S^2_x)g_{23} \\ &+ 2(\bar{x}' - \bar{X})(s'^2_x - S^2_x)g_{24} + 2(s^2_x - S^2_x)(s'^2_x - S^2_x)g_{34} \Big\} \\ &+ \frac{1}{3!}\Big\{ (s^2_y - S^2_y)\frac{\partial}{\partial s^2_y} + (\bar{x} - \bar{X})\frac{\partial}{\partial \bar{x}} + (\bar{x}' - \bar{X})\frac{\partial}{\partial \bar{x}'} \\ &+ (s^2_x - S^2_x)\frac{\partial}{\partial s^2_x} + (s'^2_x - S^2_x)\frac{\partial}{\partial s'^2_x} \Big\}^3 g(s^2_{y*}, \bar{x}_*, \bar{x}'_*, s^2_{x*}, s'^2_{x*})\end{aligned}$$

or

$$\begin{aligned}\hat{\sigma}^2_{gd} - \sigma^2_y &= e_2 + (e_1 - e'_1)g_1 + (e_3 - e'_3)g_3 + \frac{1}{2!}\Big\{ e_1^2 g_{11} + e'^2_1 g_{22} \\ &+ e_3^2 g_{33} + e'^2_3 g_{44} + 2(e_1 e_2 - e'_1 e_2)g_{01} \\ &+ 2(e_2 e_3 - e_2 e'_3)g_{03} + 2e_1 e'_1 g_{12} + 2e_1 e_3 g_{13} \\ &+ 2e_1 e'_3 g_{14} + 2e'_1 e_3 g_{23} + 2e'_1 e'_3 g_{24} + 2e_3 e'_3 g_{34} \Big\} \\ &+ \frac{1}{3!}\Big\{ (s^2_y - S^2_y)\frac{\partial}{\partial s^2_y} + (\bar{x} - \bar{X})\frac{\partial}{\partial \bar{x}} + (\bar{x}' - \bar{X})\frac{\partial}{\partial \bar{x}'} \\ &+ (s^2_x - S^2_x)\frac{\partial}{\partial s^2_x} + (s'^2_x - S^2_x)\frac{\partial}{\partial s'^2_x} \Big\}^3 g(s^2_{y*}, \bar{x}_*, \bar{x}'_*, s^2_{x*}, s'^2_{x*}). \quad (2.2)\end{aligned}$$

Taking expectation on both the sides of (2.2), to the first degree of approximation, we have

$$E(\hat{\sigma}^2_{gd} - \sigma^2_y) = E\Big\{ e_2 + (e_1 - e'_1)g_1 + (e_3 - e'_3)g_3 + \frac{e_1^2}{2} g_{11} + \frac{e'^2_1}{2} g_{22}$$

$$+\frac{e_3^{\,2}}{2}g_{33}+\frac{e_3'^{\,2}}{2}g_{44}+\left(e_1e_2-e_1'e_2\right)g_{01}+\left(e_2e_3-e_2e_3'\right)g_{03}$$

$$\left.+e_1e_1'g_{12}+e_1e_3g_{13}+e_1e_3'g_{14}+e_1'e_3g_{23}+e_1'e_3'g_{24}+e_3e_3'g_{34}\right\}$$

or
$$Bias\left(\hat{\sigma}_{gd}^2\right)=\frac{\mu_{02}}{2n}g_{11}+\frac{\mu_{02}}{2n'}g_{22}+\frac{\mu_{02}^2}{2n}(\beta_{2x}-1)g_{33}+\frac{\mu_{02}^2}{2n'}(\beta_{2x}-1)g_{44}$$
$$+\left(\frac{1}{n}-\frac{1}{n'}\right)\mu_{21}g_{01}+\left(\frac{1}{n}-\frac{1}{n'}\right)(\mu_{22}-\mu_{20}\mu_{02})g_{03}+\frac{\mu_{02}}{n'}g_{12}$$
$$+\frac{\mu_{03}}{n}g_{13}+\frac{\mu_{03}}{n'}g_{14}+\frac{\mu_{03}}{n'}g_{23}+\frac{\mu_{03}}{n'}g_{24}$$
$$+\frac{\mu_{02}^2}{n'}(\beta_{2x}-1)\quad . \qquad (2.3)$$

Squaring both the sides of (2.2) and taking expectation to the first degree of approximation, we have

$$E\left(\hat{\sigma}_{gd}^2-\sigma_y^2\right)^2=E\left\{e_2^{\,2}+\left(e_1^{\,2}+e_1'^{\,2}-2e_1e_1'\right)g_1^{\,2}+\left(e_3^{\,2}+e_3'^{\,2}-2e_3e_3'\right)g_3^{\,2}\right.$$
$$+2\left(e_1e_2-e_1'e_2\right)g_1+2\left(e_2e_3-e_2e_3'\right)g_3$$
$$\left.+2\left(e_1e_3-e_1e_3'-e_1'e_3+e_1'e_3'\right)g_1g_3\right\}$$

or
$$MSE\left(\hat{\sigma}_{gd}^2\right)=\frac{\mu_{20}^2}{n}\left(\beta_{2y}-1\right)+\left(\frac{\mu_{02}}{n}+\frac{\mu_{02}}{n'}-2\frac{\mu_{02}}{n'}\right)g_1^{\,2}$$
$$+\left\{\frac{\mu_{02}^2}{n}\left(\beta_{2x}-1\right)+\frac{\mu_{02}^2}{n'}\left(\beta_{2x}-1\right)-2\frac{\mu_{02}^2}{n'}\left(\beta_{2x}-1\right)\right\}g_3^{\,2}$$
$$+2\left(\frac{\mu_{03}}{n}-\frac{\mu_{03}}{n'}\right)g_1g_3+2\left(\frac{\mu_{21}}{n}-\frac{\mu_{21}}{n'}\right)g_1$$
$$+2\left\{\frac{(\mu_{22}-\mu_{20}\mu_{02})}{n}-\frac{(\mu_{22}-\mu_{20}\mu_{02})}{n'}\right\}g_3$$

or
$$MSE\left(\hat{\sigma}_{gd}^2\right)=MSE\left(s_y^2\right)+\left(\frac{1}{n}-\frac{1}{n'}\right)\left\{\ \mu_{02}g_1^{\,2}+\mu_{02}^2(\beta_{2x}-1)g_3^{\,2}\right.$$
$$\left.+2\mu_{03}g_1g_3+2\mu_{21}g_1+2(\mu_{22}-\mu_{20}\mu_{02})g_3\ \right\}. \qquad (2.4)$$

III. Optimum And Estimated Optimum Values

From (2.4), we see that the values of g_1 and g_3 for which $MSE(\hat{\sigma}_{gd}^2)$ is minimized, are given by

$$g_{1*}=\frac{\sigma_y^2}{\sigma_x}\left\{\frac{\gamma_1(\alpha-1)-\delta(\beta_{2x}-1)}{(\beta_{2x}-\beta_{1x}-1)}\right\} \qquad (3.1)$$

$$= \frac{\sigma_y^2}{\sigma_x} A$$

and $$g_{3*} = \frac{\sigma_y^2}{\sigma_x^2}\left\{\frac{\gamma_1\delta - (\alpha - 1)}{(\beta_{2x} - \beta_{1x} - 1)}\right\} \qquad (3.2)$$

$$= \frac{\sigma_y^2}{\sigma_x^2} B$$

where $A = \left\{\frac{\gamma_1(\alpha-1) - \delta(\beta_{2x}-1)}{(\beta_{2x} - \beta_{1x} - 1)}\right\}$, $B = \left\{\frac{\gamma_1\delta - (\alpha - 1)}{(\beta_{2x} - \beta_{1x} - 1)}\right\}$, $\gamma_1 = \frac{\mu_{03}}{\mu_{02}^{3/2}}$,

$\alpha = \frac{\mu_{22}}{\mu_{20}\mu_{02}}$ and $\delta = \frac{\mu_{21}}{\mu_{20}\mu_{02}^{1/2}}$, and the minimum mean square error is

$$MSE(\hat{\sigma}_{gd}^2)_{min} = MSE(s_y^2) - \left(\frac{1}{n} - \frac{1}{n'}\right)\left[\frac{\mu_{21}^2}{\sigma_x^2} + \frac{\sigma_y^4\{\gamma_1\delta - (\alpha-1)\}^2}{(\beta_{2x} - \beta_{1x} - 1)}\right] \qquad (3.3)$$

$$= MSE(s_y^2) - \left(\frac{1}{n} - \frac{1}{n'}\right)\frac{\mu_{21}^2}{\sigma_x^2} - \left(\frac{1}{n} - \frac{1}{n'}\right)(\beta_{2x} - \beta_{1x} - 1)\sigma_y^4 B^2 . \qquad (3.4)$$

Practically, the optimum values g_{1*} and g_{3*} in (3.1) and (3.2) may not be available always, hence the alternative is to replace the parameters involved therein by their unbiased or consistent estimatorsand thus get the estimated optimum values. Defining $m_{rs} = \frac{1}{(n-1)}\sum_{i=1}^{n}(y_i - \bar{y})^r(x_i - \bar{x})^s$, replacing $\sigma_y^2,\ \gamma_{1,}\ \alpha,\ \delta,\ \beta_{2x}$ and β_{1x} by $s_y^2,\ \hat{\gamma}_1 = \frac{m_{03}}{m_{02}^{3/2}},\ \hat{\alpha} = \frac{m_{22}}{s_y^2 m_{02}}$, $\hat{\delta} = \frac{m_{21}}{s_y^2 m_{02}^{1/2}},\ \hat{\beta}_{2x} = \frac{m_{04}}{m_{02}^2}$ and $\hat{\beta}_{1x} = \frac{m_{03}^2}{m_{02}^3}$, we get the estimated optimum values $\hat{g}_1$ and $\hat{g}_3$ to be

$$\hat{g}_1 = \frac{s_y^2}{m_{02}^{1/2}}\hat{A} \qquad (3.5)$$

and $$\hat{g}_3 = \frac{s_y^2}{m_{02}}\hat{B} \qquad (3.6)$$

where $\hat{A} = \left\{\frac{\hat{\gamma}_1(\hat{\alpha}-1) - \hat{\delta}(\hat{\beta}_{2x}-1)}{(\hat{\beta}_{2x} - \hat{\beta}_{1x} - 1)}\right\}$, $\hat{B} = \left\{\frac{\hat{\gamma}_1\hat{\delta} - (\hat{\alpha}-1)}{(\hat{\beta}_{2x} - \hat{\beta}_{1x} - 1)}\right\}$.

The generalized double sampling estimator $\hat{\sigma}_{gd}^2$ attains the minimum mean square error in (3.4) if the conditions from (1.12) to (1.18), (3.1) and (3.2) are satisfied for the estimator $\hat{\sigma}_{gd}^2$.

This means that the function $\hat{\sigma}_{gd}^2 = g\left(s_y^2, \bar{x}, \bar{x}', s_x^2, s_x'^2\right)$ as an estimator of σ_y^2 should not involve only $\left(s_y^2, \bar{x}, \bar{x}', s_x^2, s_x'^2\right)$ but also g_{1*} and g_{3*} for the conditions (3.1) and (3.2) to be satisfied. Thus, we get the resulting estimator as a function $g\left(s_y^2, \bar{x}, \bar{x}', s_x^2, s_x'^2, g_{1*}, g_{3*}\right)$ satisfying the conditions (1.12) to (1.18) along with the conditions (3.1) and (3.2) to attain the minimum mean square error in (3.4). Replacing unknowns g_{1*} and g_{3*} in $g\left(s_y^2, \bar{x}, \bar{x}', s_x^2, s_x'^2, g_{1*}, g_{3*}\right)$, we get the estimator as a function $\hat{\sigma}_{ge}^2 = g\left(s_y^2, \bar{x}, \bar{x}', s_x^2, s_x'^2, \hat{g}_1, \hat{g}_3\right)$ or equivalently the function $\hat{\sigma}_{ge}^2 = g^*\left(s_y^2, \bar{x}, \bar{x}', s_x^2, s_x'^2, \hat{A}, \hat{B}\right)$ as an estimator depending upon estimated optimum values.Now expanding $g^*\left(s_y^2, \bar{x}, \bar{x}', s_x^2, s_x'^2, \hat{A}, \hat{B}\right)$ about the point $Q^* = \left(\sigma_y^2, \bar{X}, \bar{X}', S_x^2, S_x'^2, A, B\right)$ in Taylor'sseries,wehave

$$g^*\left(s_y^2, \bar{x}, \bar{x}', s_x^2, s_x'^2, \hat{A}, \hat{B}\right) = g^*\left(Q^*\right) + \left(s_y^2 - \sigma_y^2\right)\left.\frac{\partial g^*}{\partial s_y^2}\right]_{Q^*} + \left(\bar{x} - \bar{X}\right)g_1^*$$
$$+ \left(\bar{x}' - \bar{X}\right)g_2^* + \left(s_x^2 - S_x^2\right)g_3^* + \left(s_x'^2 - S_x^2\right)g_4^* + \left(\hat{A} - A\right)g_5^* + \left(\hat{B} - B\right)g_6^* + \ldots \qquad (3.7)$$

where $g^*\left(Q^*\right) = \sigma_y^2$, $g_1^* = \left.\frac{\partial g^*}{\partial \bar{x}}\right]_{Q^*} = 1$, $g_2^* = \left.\frac{\partial g^*}{\partial \bar{x}'}\right]_{Q^*}$, $g_3^* = \left.\frac{\partial g^*}{\partial s_x^2}\right]_{Q^*}$,

$g_4^* = \left.\frac{\partial g^*}{\partial s_x'^2}\right]_{Q^*}$, $g_5^* = \left.\frac{\partial g^*}{\partial \hat{A}}\right]_{Q^*}$ and $g_6^* = \left.\frac{\partial g^*}{\partial \hat{B}}\right]_{Q^*}$

or
$$g^*\left(s_y^2, \bar{x}, \bar{x}', s_x^2, s_x'^2, \hat{A}, \hat{B}\right) - \sigma_y^2 = \left(s_y^2 - \sigma_y^2\right) + \left(\bar{x} - \bar{X}\right)g_1^*$$
$$+ \left(\bar{x}' - \bar{X}\right)g_2^* + \left(s_x^2 - S_x^2\right)g_3^* + \left(s_x'^2 - S_x^2\right)g_4^* + \left(\hat{A} - A\right)g_5^* + \left(\hat{B} - B\right)g_6^* + \ldots \qquad (3.8)$$

Squaring both the sides of (3.8) and taking expectation, we see that the mean square error $E\left[g^*\left(s_y^2, \bar{x}, \bar{x}', s_x^2, s_x'^2, \hat{A}, \hat{B}\right) - \sigma_y^2\right]^2$ to the first degree of approximation, becomes equal to $MSE\left(\hat{\sigma}_{gd}^2\right)_{min}$ given by (3.4) if $g_5^* = g_6^* = 0$, and thus the estimator taken as a function $\hat{\sigma}_{ge}^2 = g^*\left(s_y^2, \bar{x}, \bar{x}', s_x^2, s_x'^2, \hat{A}, \hat{B}\right)$ depending upon estimated optimum values attains the minimum mean square error given by (3.4) if

$$\left.\begin{aligned}
&g^*\left(s_y^2,\bar{x},\bar{x}',s_x^2,s_x'^2,\hat{A},\hat{B}\right)\Big]_{Q^*}=\sigma_y^2,\ \frac{\partial g^*}{\partial s_y^2}\bigg]_{Q^*}=1,\\
&g_1^*=\frac{\partial g^*}{\partial \bar{x}}\bigg]_{Q^*}=-\frac{\partial g^*}{\partial \bar{x}'}\bigg]_{Q^*}=-g_2^*,\\
&g_3^*=\frac{\partial g^*}{\partial s_x^2}\bigg]_{Q^*}=-\frac{\partial g^*}{\partial s_x'^2}\bigg]_{Q^*}=-g_4^*,\ \frac{\partial^2 g^*}{\partial\left(s_y^2\right)^2}\bigg]_{Q^*}=0,\\
&g_{01}^*=\frac{\partial^2 g^*}{\partial s_y^2\partial \bar{x}}\bigg]_{Q^*}=-\frac{\partial^2 g^*}{\partial s_y^2\partial \bar{x}'}\bigg]_{Q^*}=-g_{02}^*,\\
&g_{03}^*=\frac{\partial^2 g^*}{\partial s_y^2\partial s_x^2}\bigg]_{Q^*}=-\frac{\partial^2 g^*}{\partial s_y^2\partial s_x'^2}\bigg]_{Q^*}=-g_{04}^*,\\
&\frac{\partial g^*}{\partial \bar{x}}\bigg]_{Q^*}=A,\ \frac{\partial g^*}{\partial s_x^2}\bigg]_{Q^*}=B,\\
&g_5^*=0\quad and\quad g_6^*=0\ .
\end{aligned}\right\}\qquad(3.9)$$

Satisfying the conditions in (3.9), some particular estimators depending on estimated optimum values $\hat{A},\hat{B}$ and attaining the minimum mean square error in (3.4), are given in the following section 4.

IV. Concluding Remarks

(a) From (3.4), we have

$$MSE\left(\hat{\sigma}_{gd}^2\right)_{min}=MSE\left(s_y^2\right)-\left(\frac{1}{n}-\frac{1}{n'}\right)\frac{\mu_{21}^2}{\sigma_x^2}-\left(\frac{1}{n}-\frac{1}{n'}\right)(\beta_{2x}-\beta_{1x}-1)\sigma_y^4B^2$$

and from (1.8), the minimum mean square error of the generalized double sampling estimator proposed by Rizvi, S.A.M and Rizvi S.A.H. (2017) $\hat{\sigma}_d^2=g\left(s_y^2,\bar{x},\bar{x}'\right)$in (1.1) is given by

$$MSE\left(\hat{\sigma}_d^2\right)_{min}=MSE\left(s_y{}^2\right)-\left(\frac{1}{n}-\frac{1}{n'}\right)\frac{\mu_{21}^2}{\mu_{02}}\ ,$$

which gives

$$MSE\left(\hat{\sigma}_{gd}^2\right)_{min}=MSE\left(\hat{\sigma}_d^2\right)_{min}-\left(\frac{1}{n}-\frac{1}{n'}\right)(\beta_{2x}-\beta_{1x}-1)\sigma_y^4B^2\qquad(4.1)$$

which shows that the estimator $\hat{\sigma}_{gd}^2$ being an extended or wider class of estimators than the class $\hat{\sigma}_d^2$ in (1.1), contains more efficient estimators in the sense of having smaller mean square error.

(b) The estimators $\hat{\sigma}_2^2 = s_y^2 \left(\frac{\bar{x}'}{\bar{x}} \right)^{k_1} \left(\frac{s_x^2}{s_x'^2} \right)^{k_2}$ and $\hat{\sigma}_3^2 = s_y^2 + k_1(\bar{x}' - \bar{x}) + k_2\left(s_x'^2 - s_x^2\right)$,

belonging to the class $\hat{\sigma}_{gd}^2$ of estimators and having the values of $\left(g_1^*, g_2^*\right)$ to be

$$\left(k_1 \frac{\sigma_y^2}{\bar{X}},\ k_2 \frac{\sigma_y^2}{S_x^2} \right) \text{ and } \left(-k_1,\ -k_2\right) \qquad (4.2)$$

respectively, will attain the minimum mean square error given in (3.4) or (4.1) for the optimum values of (k_1, k_2) equal to $\left(\frac{A\bar{X}}{\sigma_x}, B \right)$ and $\left(-\frac{\sigma_y^2}{\sigma_x} A,\ -\frac{\sigma_y^2}{\sigma_x^2} B \right)$ obtained by equating each of (4.2) to the optimum values $\left(\frac{\sigma_y^2}{\sigma_x} A,\ \frac{\sigma_y^2}{\sigma_x^2} B \right)$, that is, the mean square error of the estimators

$s_y^2 \left(\frac{\bar{x}'}{\bar{x}} \right)^{\frac{A\bar{X}}{\sigma_x}} \left(\frac{s_x^2}{s_x'^2} \right)^{B}$ and $s_y^2 - \frac{\sigma_y^2}{\sigma_x} A(\bar{x}' - \bar{x}) - \frac{\sigma_y^2}{\sigma_x^2} B\left(s_x'^2 - s_x^2\right)$ to the first degree of approximation will be equal to that of (4.1). But $\frac{\sigma_y^2}{\sigma_x} A$ or $\frac{\sigma_y^2}{\sigma_x^2} B$ may be rarely known, hence replacing $\frac{\mu_{20}}{(\mu_{02})^{1/2}} A$ or $\frac{\mu_{20}}{\mu_{02}} B$ by consistent estimators from sample values, we get the estimators depending upon estimated optimum values to be

$$\hat{\sigma}_{2e}^2 = s_y^2 \left(\frac{\bar{x}'}{\bar{x}} \right)^{\frac{\hat{A}\bar{x}}{(\hat{\mu}_{02})^{1/2}}} \left(\frac{s_x^2}{s_x'^2} \right)^{\hat{B}} \quad \text{and} \quad \hat{\sigma}_{3e}^2 = s_y^2 \left\{ 1 - \frac{\hat{A}}{s_x}(\bar{x}' - \bar{x}) - \hat{B} \frac{\left(s_x'^2 - s_x^2\right)}{s_x^2} \right\},$$

which may belong to the class $\hat{\sigma}_{ge}^2$ and satisfy the conditions in (3.9), and also attain the minimum mean square error given by (3.4) or (4.1) to the first degree of approximation. The general result regarding $\hat{\sigma}_{ge}^2$ which attains the minimum mean square error (to the first degree of approximation) given in (3.4) or (4.1).

(c) Single sampling results may be easily found as the special cases of this study for $n' = N$.

References

[1]. **C**OCHRAN, **W.G.** (1977). Sampling Techniques (3rd Edition), John Wiley & Sons, Inc., New York.

[2]. **MURTHY, M. N.** (1967). Sampling Theory and Methods, Statistical Publishing Society, India.

[3]. **RIZVI, S. A. M., and RIZVI, S. A. H.**(2017).A Generalized Double sampling Estimator for Finite Population Variance, IOSR Journal of Mathematics (IOSR-JM), Vol. 13, Issue1 Ser.1 (Jan.-Feb.-2017), pp. 89-96, INDIA.

[4]. **RIZVI, S. A. H. and SINGH**, **R. KARAN** (2016).An Alternative Estimator of Finite Population Variance Using Auxiliary Information, Chapter 21, Recent Advances in Applied Statistics and its Applications. pp. 173-176 Lambert Academic Publishing, Republic of MOLDOVA, CHISINAU, EUROPE.

[5]. **RIZVI, S. A. H., and RIZVI, S. A. M.**(2009).Some Estimators of Population Variance Using Auxiliary Information, Journal of Applied Research, Vol. I, No.1, June-2009, pp. 54-61, INDIA.

[6]. **RIZVI, S. A. H., RIZVI, S. A. M., SINGH, R. and SINGH**, **R. KARAN** (2015).Difference and Regression Type Estimators for the Estimation of Finite Population Variance Using Auxiliary information (2015), International Journal of Statistics and Systems (IJSS), vol. 10, No. 1, pp. 111-115, INDIA.

[7]. **SINGH**, **R. KARAN,ZAIDI, S. M. H and RIZVI, S. A. H** (1995).Estimation of Finite Population Variance Using Auxiliary Information, Journal of Statistical Studies., Vol. 15, pp. 17-28, DHAKA.

[8]. **SINGH**, **R. KARAN,ZAIDI, S. M. H and RIZVI, S. A. H** (1996). Some Estimators of Finite Population Variance Using Information on Two Auxiliary Variables, Microelectron. Reliab., Vol. 36, No. 5, pp. 667-670, Elsevier Science Ltd., GREAT BRITAIN.

[9]. **SUKHATME, P. V., SUKHATME, B. V., SUKHATME, S. and ASOK, C.** (1984). Sampling theory of Surveys with Application, 3rd Ed., Ames, Iowa (USA) and Indian Society of Agricultural Statistics, New Delhi, (India).