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Periodic Solutions for Neutral Functional Differential Equations

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Abstract:
In this paper, we investigate the existence of periodic solutions for the first order neutral functional differential equation. Some new results are established by employing the Schauder fixed-point theorem and Compression mapping theorem.
***Key Word**: Functional differential equation; Schauder fixed-point theorem; Compression mapping theorem; Periodic solution.*

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I. Introduction

In this paper we consider the existence of periodic solutions for the following first order neutral functional differential equation(NFDE)

$$(x(t)-cx(t-\tau))' = -a(t)x(t) + f(t, x(t-\delta_1), \cdots x(t-\delta_m)) \qquad (1.1)$$

where $c, \tau, \delta_1, \cdots \delta_m \in R$, and

(H1) $|c| < 1/2$, $a \in C(R, R)$, $a \geq 0$ or $a \leq 0$ with a are not equivalent to zero.

(H2) $f \in C(R^{m+1}, R)$ with $f(\cdot, 0, \cdots 0)$ are not equivalent to zero.

(H3) $a(\cdot)$ and $f(\cdot, u_1, \cdots u_m)$ are $w-$ periodic, $w > 0$ is a constant.

In recent years, there has been a few papers written on the existence of periodic solutions of the neutral functional differential equation, which arise from a number of mathematical ecological models, economical and control models, population models and other models. In [1], by using Krasnoselskii fixed-point theorem, author(s) discussed the existence of positive periodic solutions of the following neutral functional differential equation

$$\left[x(t)-cx(t-\tau(t))\right]' = -b(t)x(t) + g(t, x(t-\tau(t)))$$

where $b \in C(R, (0, \infty)), \tau \in C(R, R), g \in C(R \times R, R)$, and $b(t)$, $\tau(t)$, $g(t, x)$ are $w-$ periodic functions in t and $|c| < 1$ are constants. In [2], existence of positive periodic solutions of neutral differential equations with variable coefficients

$$\left[x(t)-c(t)x(t-\tau)\right]' = -b(t)x(t) + g(t, x(t-\tau))$$

was discussed, where $b \in C(R, (0, \infty)), c \in C^1(R, R), g \in C(R \times R, R), \tau > 0$, and b, c are $w-$ periodic functions, g is $w-$ periodic with respect to first variable. In the above two papers, in order to apply the fixed point theorem, author(s) constructed two appropriate operators from the linear term $b(t)x(t)$, one is compact and the other is compressed, where prerequisite is that delay terms in neutral term and nonlinear term are same. Hence, this technique is invalid for (1.1). There are also some other papers dealing with periodic solutions of neutral functional differential equations , see[3-8,10,11]and references therein. For related books, we refer the reader to [9,12].

The aim of this paper is that according applying the Schauder fixed-point theorem and the Compression mapping theorem to established some sufficient conditions guaranteeing the existence of $w-$ periodic solutions of (1.1).

II. Preliminaries

Let $X=\{x\in C(R,R):x(t)=x(t+w),t\in R\}$ with the norm $\|x\|=\max_{t\in[0,w]}|x(t)|$, Then X is a Banach space.

Lemma 2.1 (Schauder fixed-point theorem). Let X be a Banach space and $\Omega\subset X$ a bounded closed and convex subset of X . If $F:\Omega\to\Omega$ is a compact operator, then $Fx=x$ has a solution in Ω.

Lemma 2.2 (Compression mapping theorem). Let X be a Banach space. $F:X\to X$ is a compression mapping, that is, there exists $0<\theta<1$ such that $d(Fx,Fy)\le\theta d(x,y)$ for every pair $x,y\in X$. Then $Fx=x$ has a unique solution in X .

Define $A:X\to X$ by

$$(Ax)(t)=x(t)-cx(t-\tau)$$

Lemma 2.3 If $|c|\ne1$, then A has a bounded inverse A^{-1} on X and for all $x\in X$

$$(A^{-1}x)(t)=\begin{cases}\sum\limits_{j\ge0}c^{j}x(t-\tau j), & |c|<1,\\ -\sum\limits_{j\ge1}c^{-j}x(t+\tau j), & |c|>1.\end{cases}$$

Consider the equation

$$(x(t)-cx(t-\tau))'=-a(t)x(t)+p(t), \quad (2.1)$$

where $c,\tau\in R,\ a,p\in X$.

Lemma 2.4 Assume that (H1) is satisfied,then (2.1) has a unique periodic solution $\tilde{x}_p$ with

$$\|\tilde{x}_p\|\le\frac{1}{1-2|c|}\max_{t\in[0,\omega]}h(t). \quad (2.2)$$

where

$$h(t)=\int_t^{t+w}|G(t,s)p(s)|ds,\qquad G(t,s)=\frac{e^{\int_t^s a(r)dr}}{e^{\int_0^w a(r)dr}-1}.$$

Further, if there are constants $\alpha<\beta$ such that for all $t\in R$,

$$\left[|c|\beta+(1-|c|)\alpha\right]|a(t)|\le p(t)\operatorname{sgn}a\le\left[|c|\alpha+(1-|c|)\beta\right]|a(t)|. \quad (2.3)$$

Then $\alpha\le\tilde{x}_p\le\beta$ for all $t\in R.$

Proof : Define the operator B on X by

$$(Bx)(t)=cx(t-\tau)+\int_t^{t+w}G(t,s)(p(s)-ca(s)x(t-\tau))ds.$$

It is easy to check that the fixed point of T on X is the periodic solution of (2.1) .

For $x,y\in X$, we have

$$\begin{aligned}&|(Bx)(t)-(By)(t)|\\ &=\left|c(x(t-\tau)-y(t-\tau))+\int_t^{t+w}G(t,s)\left[(p(s)-ca(s)x(s-\tau))-(p(s)-ca(s)y(s-\tau))\right]ds\right|\\ &\le\left|c(x(t-\tau)-y(t-\tau))+\int_t^{t+w}G(t,s)(-ca(s)x(s-\tau)+ca(s)y(s-\tau))ds\right|\\ &\le|c|\cdot|x(t-\tau)-y(t-\tau)|+|c|\cdot\left|\int_t^{t+w}G(t,s)a(s)(x(s-\tau)-y(s-\tau))ds\right|\\ &\le|c|\cdot\|x-y\|+|c|\cdot\int_t^{t+w}G(t,s)a(s)ds\cdot\|x-y\|\\ &\le2|c|\cdot\|x-y\|,\end{aligned}$$

where we use the fact that $\int_t^{t+w} G(t,s)a(s)ds=1.$

If $c \neq 0,$ by **Lemma 2.2,** we obtain that (2.1) has a unique w- periodic solution since $0 < 2|c| < 1.$

If $c = 0$, (2.1) has a unique w- periodic solution $\tilde{x}_p(t) = \int_t^{t+w} G(t,s)p(s)ds.$

On the other hand,

$$\begin{aligned}|\tilde{x}_p(t)| &= \left| c\tilde{x}_p(t-\tau) + \int_t^{t+w} G(t,s)(p(s) - ca(s)\tilde{x}_p(s-\tau))ds \right| \\ &\leq |c| \cdot |\tilde{x}_p(t-\tau)| + \int_t^{t+w} |G(t,s)p(s)|ds + |c| \cdot \int_t^{t+w} |G(t,s)a(s)\tilde{x}_p(s-\tau)|ds \\ &\leq |c| \cdot \|\tilde{x}_p\| + \max_{t\in[0,w]} h(t) + |c| \cdot \int_t^{t+w} |G(t,s)a(s)|ds \cdot \|\tilde{x}_p\| \\ &\leq 2|c| \cdot \|\tilde{x}_p\| + \max_{t\in[0,w]} h(t),\end{aligned}$$

which implies that (2.2) holds.

Finally, let (2.3) hold. Define the operator K, S on X by

$$(Kx)(t) = cx(t-\tau), \qquad (Sx)(t) = \int_t^{t+w} G(t,s)(p(s) - ca(s)x(s-\tau))ds.$$

Then $B = K + S,\ K : X \to X$ is a compression mapping and $S : X \to X$ is a compact operator.

Let $\Omega = \{x \in X : \alpha \leq x \leq \beta\}$. For any $x, y \in X$,

$$c\alpha \leq Kx \leq c\beta \quad if \quad c \geq 0,$$
$$c\beta \leq Kx \leq c\alpha \quad if \quad c < 0,$$
$$-c\beta \leq -c\int_t^{t+\omega} G(t,s)a(s)y(s-\tau)ds \leq -c\alpha \quad if \quad c \geq 0,$$
$$-c\alpha \leq -c\int_t^{t+\omega} G(t,s)a(s)y(s-\tau)ds \leq -c\beta \quad if \quad c < 0,$$
$$|c|\beta + (1-|c|)\alpha \leq \int_t^{t+\omega} G(t,s)p(s)ds \leq |c|\alpha + (1-|c|)\beta.$$

Hence, for any $x, y \in X, \alpha \leq Kx + Sy \leq \beta,$ that is, $K(\Omega) + S(\Omega) \subset \Omega$.

By the Krasnoselskii fixed-point theorem, (2.2) has a solution in Ω. Hence, $\alpha \leq \tilde{x}_p \leq \beta.$

If (H1) holds,for any $p \in X,$ (2.1) has a unique periodic solution $\tilde{x}_p$. Thus we can define an operator $T : \tilde{x}_p = Tp.$

Lemma 2.5 The operator $T : X \to X$ is compact if (H1) holds.

Proof. Assume $D \subset X$ is a bounded sunset, it is also known from (2.2) that $T(D)$ is bounded and uniformly bounded. Let $p \in D$ and $x = Tp$, then

$$(x(t) - cx(t-\tau))' = -a(t)x(t) + p(t)$$

Let $y = x(t) - cx(t-\tau)$, there is a constant $\gamma > 0$ only dependent of D such that

$$|x| \leq \gamma, \qquad |y'| \leq \gamma, \qquad |y| \leq \gamma.$$

Using**Lemma 2.3**, we have

$$x(t) = A^{-1}y = \sum_{j\geq 0} c^j y(t-\tau j).$$

Notice that the series $\sum_{j\geq 0} c^j y(t-\tau j)$ is uniform convergence.

$$|x'(t)| = |(A^{-1}y)'| = \left|(\sum_{j\ge 0}(c^j y(t-\tau j)))'\right| = \left|\sum_{j\ge 0} c^j y'(t-\tau j)\right| \le \gamma \sum_{j\ge 0}|c^j| = \frac{\gamma}{1-|c|}$$

For $\forall \varepsilon > 0$; $\exists \xi = (1-|c|)\varepsilon / \gamma > 0$ such that when $|t_1 - t_2| < \xi$, for every $x \in D$, it follows from the mean value of Lagrange theorem that

$$|x(t_1) - x(t_2)| \le \frac{\gamma}{1-|c|}|t_1 - t_2| \le \varepsilon.$$

Namely, T is equicontinuous. Therefore $T : X \to X$ is compact.

III. Results

Theorem 1. Let (H1)-(H3) hold. Further assume that $\exists M > 0$ such that for any $t \in R, u_i \in [-M, M](1 \le i \le m)$,

$$|f(t, u_1, \cdots, u_m)| \le (1-2|c|)|a(t)|M.$$

Then (1.1) has a nontrivial, w- periodic solution $u : |u| \le M$.

Proof. Define operator Λ, F on X by

$$(\Lambda x)(t) = [(T \circ F)x](t) = T(f(t, x(t-\delta_1), \cdots, x(t-\delta_m))),$$
$$(Fx)(t) = f(t, x(t-\delta_1), \cdots, x(t-\delta_m)).$$

It is not difficult to show that

$$((\Lambda x)(t) - c(\Lambda x)(t-\tau))' = -a(t)(\Lambda x)(t) + f(t, x(t-\delta_1), \cdots, x(t-\delta_m)),$$

which follows that the fixed point of Λ on X is the periodic solution of (1.1) . Since $T : X \to X$ is relatively compact and $F : X \to X$ is continuous, $\Lambda : X \to X$ is relatively compact.

Let $\Omega = \{x \in X : \|x\| \le M\}$, then Ω is a bounded closed and convex set. From Lemma 2.4, we have

$$\|\Lambda x\| \le \frac{1}{1-2|c|}\max_{t\in[0,\omega]} H(t),$$

where $H(t) = \int_t^{t+w}|G(t,s) f(s, x(s-\delta_1), \cdots, x(s-\delta_m))|ds.$ For any $x \in \Omega$,

$$\begin{aligned} 0 \le H(t) &= \int_t^{t+w}|G(t,s) f(s, x(s-\delta_1), \cdots, x(s-\delta_m))|ds \\ &\le (1-2|c|)M\int_t^{t+w}|G(t,s)a(s)|ds \\ &= (1-2|c|)M\int_t^{t+w}G(t,s)a(s)ds \\ &= (1-2|c|)M, \end{aligned}$$

which implies that for any $x \in \Omega$,

$$\|\Lambda x\| \le \frac{1}{1-2|c|}\max_{t\in[0,\omega]} H(t) \le M.$$

Hence, $\Lambda(\Omega) \subset \Omega$. By **Lemma 2.1,** we obtain that (1.1) has a w- periodic solution $u : |u| \le M$. Moreover, u is nontrivial since $f(t, 0, \cdots 0)$ is not equivalent to zero.

Theorem 2. Let (H1-H3) hold. Further assume that there exists constant $0 < \mu < 1-2|c|$ such that for any $t \in R, u_i, v_i \in R(1 \le i \le m)$,

$$|f(t, u_1, \cdots, u_m) - f(t, v_1, \cdots, v_m)| \le \mu|a(t)|\max\{|u_i - v_i|, 1 \le i \le m\}.$$

Then (1.1) has a unique, nontrivial, w- periodic solution.

Proof. Define Λ, F as in the proof of **Theorem 1.** For any $u, v \in X$, we obtain that

$$|\Lambda u - \Lambda v| = |(T \circ F)u - (T \circ F)v| = |T(Fu - Fv)|$$

where

$$\left|(Fu)(t)-(Fv)(t)\right|=\left|f(t,u(t-\delta_1),\cdots,u(t-\delta_m))-f(t,v(t-\delta_1),\cdots,v(t-\delta_m))\right|.$$

From Lemma 2.4, we have

$$\|\Lambda u-\Lambda v\|\le\frac{1}{1-2|c|}\max_{t\in[0,\omega]}J(t),$$

where

$$J(t)=\int_t^{t+w}|G(t,s)|\left|(Fu)(s)-(Fv)(s)\right|ds.$$

Using the assumption, we obtain that

$$\begin{aligned}0\le J(t)&=\int_t^{t+w}|G(t,s)|\left|(Fu)(s)-(Fv)(s)\right|ds\\&\le\mu\int_t^{t+w}|G(t,s)||a(t)|\max\left\{|u(s-\delta_i)-v(s-\delta_i)|,1\le i\le m\right\}ds\\&=\mu\int_t^{t+w}|G(t,s)||a(t)|ds\|u-v\|\\&=\mu\|u-v\|.\end{aligned}$$

Hence,

$$\|\Lambda u-\Lambda v\|\le\frac{\mu}{1-2|c|}\|u-v\|.$$

By **Lemma 2.2,** we obtain that (1.1) has a unique w- periodic solution.

Theorem 3. Let (H1-H3) hold. Further assume that there are constants $\alpha<\beta$ such that for all $t\in R$, $u_i\in[\alpha,\beta](1\le i\le m)$,

$$\left[|c|\beta+(1-|c|)\alpha\right]|a(t)|\le f(t,u_1,\cdots,u_m)\operatorname{sgn}a\le\left[|c|\alpha+(1-|c|)\beta\right]|a(t)|.$$

Then (1.1) has a nontrivial, w- periodic solution $x:\alpha\le x\le\beta.$

Proof. Define operators Λ, F on X by

$$(\Lambda x)(t)=[(T\circ F)x](t)=T(f(t,x(t-\delta_1),\cdots,x(t-\delta_m))),$$
$$(Fx)(t)=f(t,x(t-\delta_1),\cdots,x(t-\delta_m)).$$

Let $\Omega=\{x\in X:\alpha\le x\le\beta\}$, then Ω is a bounded closed and convex set. From **Lemma 2.4**, we have

$$\alpha\le\Lambda x\le\beta.$$

Hence, $\Lambda(\Omega)\subset\Omega.$ By **Lemma 2.1,** we obtain that (1.1) has a w- periodic solution $u:\alpha\le u\le\beta.$

Example 1 Consider the equation

$$(x(t)-0.2x(t-\tau))'=x(t)+\frac{1}{4}\left(\cos t-x^2(t+\eta)x(t-\delta)\right)\qquad(3.1)$$

where $\tau,\eta,\delta\in R.$

In fact, $c=0.2,\ a\equiv-1,\ w=2\pi,\ f(t,u,v)=(\cos t-u^2v)/4.$ Let $M=1.$

For any $t\in R,u,v,\in[-M,M],$

$$|f(t,u,v)|\le0.5\le(1-2|c|)|a(t)|M=0.6.$$

Hence, by Theorem 1, (3.1) has a nontrivial, w- periodic solution $u:|u|\le1.$ Clearly, the results in [1,2] do not apply to (3.1).

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References

[1]. Y. Luo, W. Wang, J. Shen, Existence of positive periodic solutions for two kinds of neutral functional differential equations, Appl. Math. Leet.21(2008)581-587.
[2]. T. Candan, Existence of positive periodic solutions of first order neutral differential equations with variable coefficients, Appl. Math. Lett. 52(2016)142-148.
[3]. Abdelouaheb Ardjouni,Ali Rezaiguia,Ahcene Djoudi. Existence of positive periodic solutions for two types of second-order nonlinear neutral differential equations with infinite distributed delay. J. Journal of Applied Mathematics and Computing,2015,47(1-2).
[4]. A. Wan, D. Jiang, Existence of positive periodic solutions for functional differential equations, Kyushu J. Math. 56 (2002) 193–202.
[5]. D. Jiang, J.Wei, B. Zhang, Positive periodic solutions of functional differential equations and population models, Electron. J. DifferentialEquations 71 (2002) 1–13.
[6]. E. Serra, Periodic solutions for some nonlinear differential equations of neutral type, Nonlinear Anal. TMA 17(1991)139-151.
[7]. H.I. Freedman, J. Wu, periodic solutions of single-species models with periodic delay,SIAM Journal of Mathematical Analysis 23 (1992) 689-701.
[8]. H. Wang, Positive periodic solutions of functional differential equations, J. Differential Equations 202 (2004) 354–366.
[9]. J.K. Hale, Theory of Functional Differential Equations, Springer-Verlag, New York, 1977.
[10]. Muhammad N. Islam,Youssef N. Raffoul. Periodic solutions of neutral nonlinear system of differential equations with functional delay. J. Journal of Mathematical Analysis and Applications,2006,331(2).
[11]. W. Wang, Y. Luo, Periodic solutions for a class of second order differential equations, Electron. J. Qual. Theo.41(2013)1-10.
[12]. Y. Kuang, Delay Differential Equations with Applications in Population Dynamics, Academic Press, New York, 1993.

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