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The binomial coefficients in the Riemann Wave background. A possible proof of the Riemann Hypothesis

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Abstract: *We consider the series with positive summands given in [1] for the first Li-Keiper coefficient λ_1. We first carry out a numerical experiment to characterize the speed at which the above series contains (in memory) big amounts of the zeros on the critical line. Then we look at the truth of the RH as an extremal possible grow of λ_1 by means of all sets of zeros in the critical strip.*
Later, in the last section, we formulate the linear Equation for the coefficients φ_n related to the Li-Keiper coefficients λ_n.
Then, we conclude with a possible proof of the correctness of the Riemann Wave background and thus of a (possible) proof of the RH.
Key words: *The first increment of the Li-Keiper coefficients, nontrivial zeros, Series for the first coefficient λ_1, "Maxi-Min" criterion for the Riemann Hypothesis, Binomials coefficients, Partition Function, Free Energy, Riemann wave background, Riemann Hypothesis (RH).*

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I. Introduction

In this work, we first consider the original series obtained by Matiyasevich in his pioneering work [1], i.e. the expression for the first Li-Keiper coefficient $\lambda_1 = \gamma/2+1-(1/2)\cdot\log(4\cdot\pi) \sim 0.0230957...$
as a sum of positive summands [1] (See also[2, 3] for contributions in this direction) and related to the binary system; it is given by[1]:

$$\lambda_1 = \sum_{\rho}\frac{1}{\rho} = \left(\frac{1}{2}\right)\cdot\sum_{n=3}^{\infty}\frac{(2\cdot N_1(n)+3)}{[(2\cdot n)\cdot(2\cdot n+1)\cdot(2\cdot n+2)]} = \sum_{n=3}^{\infty} f(n) \tag{1}$$

where $N_1(n)$ is the number of units in the binary representation of n. (Notice that if N_0 is the number of zeros in the representation of n, $N_1(n) + N_0(n) = [\log_2(2\cdot n)]$, where the symbol [ ] denotes the ceiling.
Eq.(1) is a sum of positive summands thus increasing with n. It may also be written as a decreasing sequence using N_0 in the above Formula. With the following relation [1]

$$\gamma = \frac{1}{2} + \sum_{n=1}^{\infty}\frac{[log_2(2\cdot n)]}{[(2\cdot n)\cdot(2\cdot n+1)\cdot(2\cdot n+2)]}$$

then we have:

$$\gamma - \frac{67}{127} = \sum_{n=3}^{\infty}\frac{[log_2(2\cdot n)]}{[(2\cdot n)\cdot(2\cdot n+1)\cdot(2\cdot n+2)]}$$

i.e.

$$\lambda_1 = \gamma - \frac{67}{127} + \left(\frac{3}{2}\right)\cdot\sum_{n=3}^{\infty}\frac{1}{[(2\cdot n)\cdot(2\cdot n+1)\cdot(2\cdot n+2)]} +$$

$$+\sum_{n=3}^{\infty}(-N_0\,(n))\frac{1}{[(2\cdot n)\cdot(2\cdot n+1)\cdot(2\cdot n+2)]}=$$

$$=\ 0.0188823316+0.010279229+\sum_{n=3}^{\infty}(-N_0\,(n))\frac{1}{[(2\cdot n)\cdot(2\cdot n+1)\cdot(2\cdot n+2)]}=$$

$$=0.0291615606\ +\sum_{n=3}^{\infty}(-N_0\,(n))\frac{1}{[(2\cdot n)\cdot(2\cdot n+1)\cdot(2\cdot n+2)]} \qquad (2)$$

i.e. a constant (the first term) plus a sum of negative summands analogous to Eq. (1).
In the following numerical heuristic experiment, we consider Eq.(1) and compare each term in the summation with the amount given by the set of the reciprocal zeros (on the critical line) of the Zeta function; the values of the zeros are available from the Tables of Odlyzko [4]. We know that if $\lambda_1$exhausts exactly the sum of the reciprocals zeros on the critical line, then this is equivalent to the truth of the Riemann Hypothesis. Here, we analyze the "speed" at which the above series subsums such reciprocal nontrivial zeros.
We define:
M_1.
For n=3 in Eq.(1), $N_1(3) = 2$, amount:$M_1 = f(n=3)=(1/2)\cdot (7/(6\cdot7\cdot8)) = (1/2)\cdot(1/48) = 1/96 = =0.010416666667$.
For the zeros on the critical line we define$:= S_m = \ \sum_{k=1}^{m}\left(\frac{1}{\left(\frac{1}{4}+t_k^2\right)}\right)$
For m=4 we obtain:
$S_4 = 1/(0.25+14.134725142^2) +$
$+ 1/(0.25+21.022039639^2) +$
$+ 1/(0.25+25.010857580^2) +$
$+ 1/(0.25+30.424876126^2) = 0.00993850678 < 0.010416666...$

For n=3 $\rightarrow M_1= f(3) = 0.01041666..$; $S_4 = 0.00993850678 < M_1$.

$M_2 = f(3)+f(4) = (1/96) +(1/2)\cdot(2+3)/(8\cdot9\cdot10) =(1/96 +1/288) =1/72 = 0.013888888888..$

For m=11 (levels up to $t_{11} = 52.970….$) we have $S_{11} = 0.0138443286..$

For n=3 and n=4 $\rightarrow M_2 = 0.0138888..$; $S_{11} = 0.0138443286 < M_2$.

$M_3=f(3)+f(4)+f(5) =1/72+7/2\cdot(10\cdot11\cdot12) =262/(1320\cdot12) = 0.0165404040...$

For m=42 (levels up to t_{42}= 127.516...) we have $S_{42} = 0.0165404040..$

For n=3, n=4 and n=5: $M_3 = 0.01654040..$; $S_{42} = 0.0165\ 4040 \sim M_3$.

Thus, three terms in the above Matiyasevich series comprises the contribution of the reciprocal values of the first 42 zeros on the critical line. We continue the analysis for an additional information on this point, and compute M_4.
Below, the Table 1 of the values of $:=M_n = f(3)+f(4) +...f(n+2)$,n=1...up to n=11.

M_4	0.018142968
M_5	0.019482253
M_6	0.019992874
M_7	0.020504570
M_8	0.020883358
M_9	0.021253911
M_{10}	0.021478270

M_{11}	0.021707208
M_{12}	0.021891937
M_{13}	0.022076749

Table 1.

Remark: From t_{43} = 129.578704200,we may compute the sum of nontrivial reciprocal zeros (as an approximation) using the weight

$dN(t) = (1/2\,\pi)\cdot\log(t/2\pi)\cdot dt$ for the function$(1/(1/4+t^2))$, i.e.

$$\int_{129.578}^{216.169} dN(t)\cdot\left(\frac{1}{\left(\frac{1}{4}+t^2\right)}\right) = 0.001602134..$$

S_{89} = 0.016540404+0.001602134 = 0.018142538 < 0.0181429681= $M_{4.}$

M_4 = 0.018142968$\cdot$ S_{89} = 0.018142358 < M_4..

M_4 comprises (~) the amount of the first 89 reciprocal zeros with the series.
Below as an illustration we construct the plot of the levels M_N for very small values of N and the corresponding calculated amount of the subset of the reciprocal zeros (S_m) using the Tables.

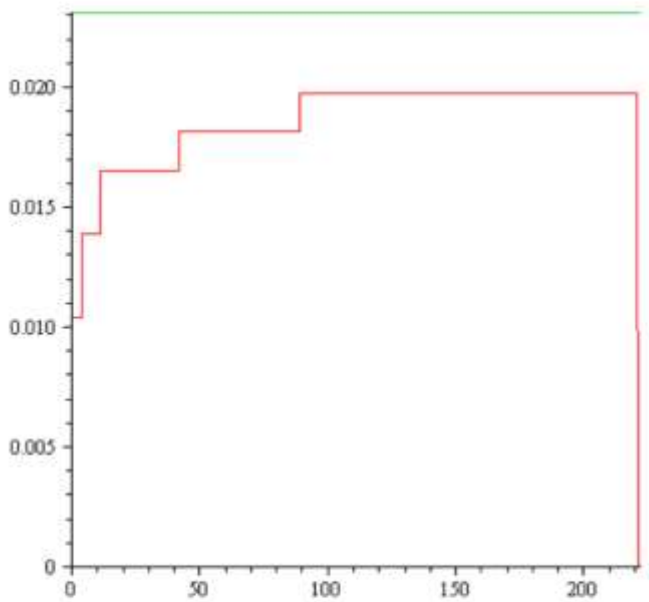

Fig. 1. The first 5 levels M_1..M_5 and the levelλ_1 = 0.023957...

Below in Fig. 2 we present the plot of M_n here as a function of (1/N) where N is the number of zeros corresponding to M_n.

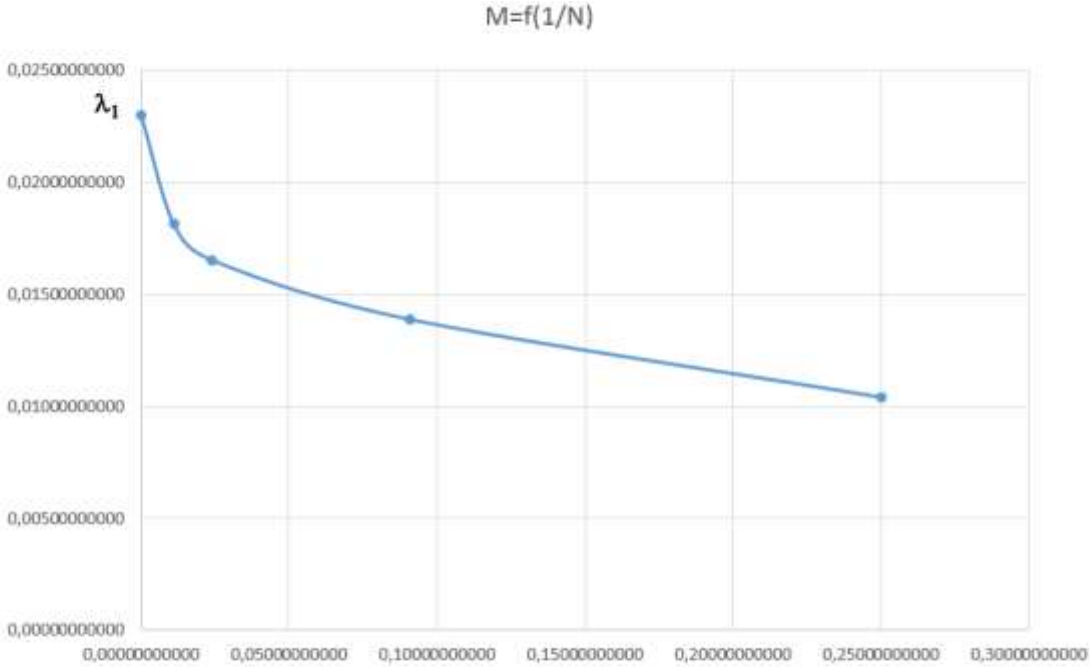

Fig. 2. M_n as a function of (1/N)where N is the number of zeros corresponding to M_n

II. A criterion for the truth of the R.H. assuming that the zeros are simple

With the above small numerical experiment on the series M_n of positive summands we was looking at the "speed" of the series and this suggests a complement in our study of λ_1 alone i.e. – the following first definition of an increment- Δ_1. Let

$$\lambda_0 = \sum_{\rho}\left(1-\left(1-\frac{1}{\rho}\right)^0\right) = 0$$

where ρ runs on all non trivial zeros.

We now consider the first "increment" [8] which we define as:

$$\Delta_1 := \sum_{\rho}\left(1-\left(1-\frac{1}{\rho}\right)^1\right) - \sum_{\rho}\left(1-\left(1-\frac{1}{\rho}\right)^0\right) \tag{3}$$

The sum is on all the zeros of the Zeta function in the critical strip, i.e. (without any further assumptions) of the general form s= $\sigma_k \pm i\cdot t_k$ and s=(1- σ_k) $\pm i\cdot t_k$ ($1/2 \le \sigma_k \le 1$, $t_k > 0$ for all k =1,2...).

With this definition we then have:

$$\Delta_1 = \sum_{(\sigma_k,t_k)}\left[\frac{2\cdot\sigma_k}{(\sigma_k^2+t_k^2)}+\frac{2\cdot(1-\sigma_k)}{((1-\sigma_k)^2+t_k^2)}\right] = \sum_k R_k\,(\sigma_k,t_k) \tag{4}$$

We notice that R_k has an extremum (a maximum) for all k at σ_k=1/2.
In fact the Taylor expansion around σ_k=1/2 is given by:
$R_k = R_k(\sigma_k=1/2, t_k) - A(t_k)\cdot(\sigma_k-1/2)^2) + O((\sigma_k-1/2)^3)$, A > 0 for all t_k.

From above, explicitly:

$R_k' = 2/(1/4+t_k^2) - 6\cdot(1/(1/4+t_k^2)^3)\cdot(t_k^2-1/12)\cdot((\sigma_k-1/2)^2)+O((\sigma_k-1/2)^3))$ (5)

Neglecting the last term in Eq. (5) we represent below (for the first level t_1 =14.134725...) , R_1 and R_1' as a function of σ= x for $0 \le x \le 1$,using Eq.(4).

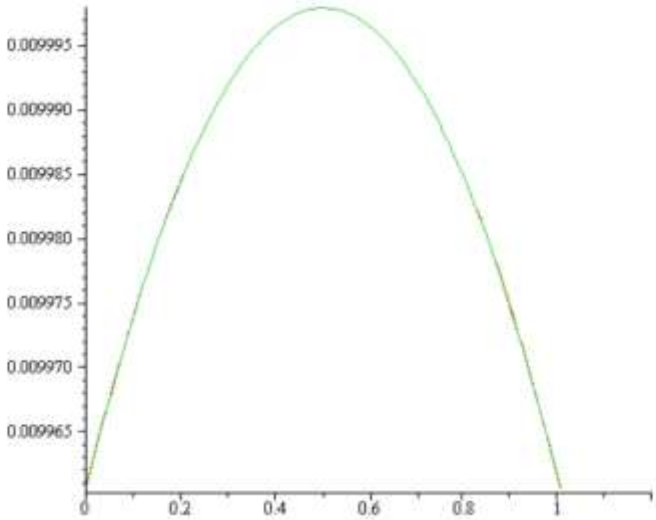

Fig. 3. R_1 and R_1' (the two plots are superimposed) and $R_1(0) = R_1(1) > 0$.

Moreover, R_k has the Riemann symmetry, i.e. $R_k(\sigma_k,t) = R_k(1-\sigma_k, t)$ and $R_k(\sigma_k=1/2,t_k) = 2/(1/4+t_k^2)$ $>R_k(\sigma_k=0,t_k) = 2/(1+t_k^2)$.In considering Δ_1 of Eq.(4) and without any assumptions we could characterize the truth of the RH using Δ_1 with the following:

Criterion 1

"RH is true if Δ_1 as defined by Eq.(3),Eq.(4) assume the maximum possible value"; but then $\Delta_1 \sim 2.\lambda_1$ =2.0.0230957..

Moreover, if a nontrivial zero is off the critical line, big fluctuations are known to occur in the lamda's (λ_n, at high values of n and t. See Appendix 2 for the general formula of Δ_n).

We then consider another definition related to Eq.(4).

We assume that the zeros are simple; then for σ_k=1/2, we should divide Δ_1 by 2, i.e.

$\Delta_1' := (1/2)\cdot\Delta_1$,

because these are the simple zeros of the Zeta function for σ_k=1/2 and

$\Delta_1' = \Sigma(1/2)\cdot R_k(\sigma_k=1/2,t_k) =$

$$\sum_{t_k}\left(\frac{1}{\left(\frac{1}{4}+t_k^2\right)}\right) < \sum_k R_k\,(\sigma_k = 0, t_k) = \sum_k \left(\frac{2}{(1+t_k^2)}\right) \quad (5)$$

since for all t_k, $1/(1/4+t_k^2) < 2/(1+t_k^2)$. From this, we have the following criterion:

Criterion 2

"RH is true if, assuming the zeros are simple, Δ_1 assume the minimum possible value", i.e.

$$f(\{\sigma_k, t_k\}) = \sum_{\rho}\left(1-\left(1-\frac{1}{\rho}\right)^1\right)$$

is an absolute minimum for any fixed distribution of energy levels $\{t_k\}$. This concludes our Maxi-Min criterion for the RH.

III. The first few Li-Keiper coefficients: the Binomial Coefficients in the Riemann Wave background.

The first one, contains the two fundamental constants γ (the Euler-Mascheroni constant and π).

$$\lambda_1 = 1+ \gamma/2-(1/2)\cdot\log(4\cdot\pi) =0.0230957... \quad (6)$$

As it is known, this value is independent of the truth of the RH.
For the second coefficients λ_2 we have the lower bound:

$$\lambda_2> 4\cdot \lambda_1 - \lambda_1^2 \quad (7)$$

In fact, $(1/2)\cdot(\lambda_2 + \lambda_1^2) - \lambda_1 > \lambda_1$ as was proved in [8].

Eq.(7) give us with Eq.(6) that $\lambda_2>0.091849\ldots$

We have

$$\varphi_2 - \varphi_1 = \frac{1}{2}\left(\frac{\xi''}{\xi}\right)\Big|_{S=1} \text{ and } \varphi_2>2\cdot\varphi_1 \quad (8)$$

with $\varphi_1 = \lambda_1, \varphi_2= (1/2)\cdot(\lambda_2 + \lambda_1^2)$.

For the third coefficient with the definition of it i.e.

$\lambda_3 = (1/2!)\cdot d^3/ds^3(s^2\cdot\log(\xi (s))|_{s=1}$ where the ξ function is given by

$$\xi (s) = (1/2)\cdot s\cdot(s-1)\cdot\pi^{-s/2}\cdot\Gamma(s/2)\cdot \zeta (s) \quad (9)$$

we have:

$\lambda_3 = (1/2) \cdot d^3/ds^3(s^2 \cdot\log(\xi (s)) |_{s=1} = 3\cdot(\xi '/\xi)|_{s=1} +3\cdot(\xi '/\xi)'|_{s=1} +(\xi '/\xi)''|_{s=1}$

using the same for $\lambda_2 = 2\cdot(\xi '/\xi)|_{s=1} + (\xi '/\xi)'|_{s=1}$, with $\lambda_1 = (\xi '/\xi)|_{s=1}$.

We find

$\lambda_3 = (3-(3/2)\cdot \lambda_1)\cdot\lambda_2 - 3\cdot\lambda_1 +3\cdot \lambda_1^2-(1/2) \lambda_1^3 +(1/2)\cdot(\xi '''/\xi)|_{s=1}$

We now apply our inequality (Eq. (7)) and we obtain:

$\lambda_3> (4\cdot\lambda_1 - \lambda_1^2)\cdot(3-(3/2)\cdot \lambda_1) -3\cdot\lambda_1 +3\cdot\lambda_1^2-(1/2) \lambda_1^3 +(1/2)\cdot(\xi '''/\xi)|_{s=1}$

that is

$$\lambda_3> 9\cdot \lambda_1 -6\cdot \lambda_1^2 + \lambda_1^3 + (1/2)\cdot(\xi '''/\xi)|_{s=1} > 9\cdot \lambda_1 -6\cdot \lambda_1^2 + \lambda_1^3 \quad (10)$$

Since it is known that $d^n/ds^n \xi (s) |_{s=1}> 0$ [9].

In fact if we consider the Riemann series around s=1/2 [9], given by:

$$\xi(s) = \sum_{n=0}^{\infty} a_{2n} \cdot \left(s - \frac{1}{2}\right)^{2\cdot n}$$

In the expansion, all coefficients are positive, i.e. $a_{2n} > 0$ for all n. Thus for n=1, 2, 3 we have:

$$\xi'(1) = \sum_{n=0}^{\infty} a_{2n} \cdot (2 \cdot n) \left(\frac{1}{2}\right)^{2 \cdot n - 1} = a_2 + \left(\frac{1}{2}\right) \cdot a_4 + \left(\frac{3}{16}\right) \cdot a_6 + \cdots > 0$$

$$\xi''(1) = \sum_{n=0}^{\infty} a_{2n} \cdot (2 \cdot n) \cdot (2 \cdot n - 1) \cdot \left(\frac{1}{2}\right)^{2 \cdot n - 2} = a_2^2 + 3 \cdot a_4 + \left(\frac{15}{8}\right) \cdot a_6 + \cdots > 0$$

$$\xi'''(1) = \sum_{n=0}^{\infty} a_{2n} \cdot (2 \cdot n) \cdot (2 \cdot n - 1) \cdot (2 \cdot n - 2) \cdot \left(\frac{1}{2}\right)^{2 \cdot n - 3} = 12 \cdot a_4 + 15 \cdot a_6 + \cdots > 0$$

$$\left(\frac{d^n}{ds^n} \xi(s)\right)\Big|_{s=1} > 0 \quad \forall n. \qquad (10')$$

(Notice that a_0 =0.4977120.. (~0.5!), a_2 = 0.011485..and...a_4= 0.00012345 , a_6= 8.10^{-7}....)

Moreover, with the first three terms we obtain:

$\xi'(1)/\xi(1) = 2\cdot(a_2 + (1/2)\cdot a_4 + (3/16)\cdot a_6 + ...) =$

$= 2\cdot(0.011485)+(1/2)\cdot 0.00012345\sim$

$\sim 0.023033... < \lambda_1 = 0.0230957...$

Here for symmetry we write the inequalities using the three functions φ_1, φ_2, φ_3 (emerging for every n as φ_n) in the expansion of the associated partition function [8]. We have for n =1, 2, 3, 4 …

$\varphi_1 = (1)\cdot \lambda_1 + 0.$

$\varphi_2 = (1/2)\cdot (\lambda_2 + \lambda_1^2)$

$\varphi_3 = (1/3)\cdot(\lambda_3+(3/2)\cdot \lambda_1\cdot\lambda_2 +(1/2)\cdot\lambda_1^3)$

$\varphi_4= (1/4)\cdot(\lambda_4 + (4/3)\cdot\lambda_1\cdot\lambda_3 +(1/2)\cdot\lambda_2^2 +\lambda_1^2\cdot\lambda_2 +(1/6)\cdot\lambda_1^4)$

Now, for n=3 we have obtained from the definition, i.e. $\lambda 3 = d^3/ds^3 (s^2\cdot\log(\xi (s))|_{s=1}$:

$3\cdot\varphi_3 -6\cdot\varphi_2 + 3\cdot\varphi_1 = (1/2)\cdot(\xi'''/ \xi) (1).$ (11)

For example,

$$\varphi_3 - 2\varphi_2 + \varphi_1 = \frac{1}{3!}\left(\frac{\xi'''}{\xi}(s)\right)\Big|_{s=1}$$

Equality i.e. Eq.(11) may be pursued for n >3. Notice that if we apply the inequality above, i.e. $\lambda_2 > 2\cdot\lambda_1 - \lambda_1^2$, we obtain $\lambda_3 > 3\cdot\lambda_1 - 3\cdot\lambda_1^2 + \lambda_1^3 = 0.067699$ (the true value is 0.020763..).
With $\lambda_2 > 2\cdot\lambda_1 - \lambda_1^2$, $\lambda_2 > 0.045657$ (the true value is 0.0923457..)
If we apply our bound of Eq.(8) we obtain $\lambda_3 > 0.204673$. The true value is $\lambda_3 = 0.20763…$
Remark:
The right hand side of Eq.(10) is the third term (in z=1-1/s) of our lower bound i.e. of the Riemann wave background, with the appearance of the Koebe function K(z) as argument of the $\log(1+\lambda_1\cdot K(z))$ [8].
In fact, the first three terms of the above expansion of the logarithm at z=0, are given by:
$\log(1+\lambda_1\cdot K(z)) = \log(1+\lambda_1\cdot z/(1-z)^2) = \lambda_1\cdot z +(1/2) (4\cdot\lambda_1-\lambda_1^2) \cdot z^2+ (1/3)\cdot(9\cdot\lambda_1- 6\cdot\lambda_1^2 + \lambda_1^3)\cdot z^3 +O(z^4)$.

Equality by (11) may be pursued for n>3. In fact, we have calculated
$\lambda_4 = (1/3!)\cdot(d^4/ds^4(s^3\cdot\log(\xi(s))))$ and we obtain:
$4\cdot\varphi_4 - 12\cdot\varphi_3 + 12\cdot\varphi_2 - 4\cdot\varphi_1 = (1/6)\cdot\xi''''(1)$
i.e. $\varphi_4 = 3\cdot\varphi_3 + 1\cdot\varphi_1 - 3\cdot\varphi_2 + 1\cdot\varphi_1 + (1/4!)\cdot(\xi''''/\xi)(1)$
Here, to start -with the true values we verify that - $\varphi_4 = 4\cdot\varphi_1 + 0.0022384 > 4\cdot\varphi_1$.
Thus: $\varphi_4 > 4\cdot\lambda_1$, i.e. $\lambda_4 > 16\lambda_1 - 20\lambda_1^2 + 8\lambda_1^3 - \lambda_1^4 = 0.358961\ldots$ (the true value is 0.368…).
In fact, the first four terms of the expansion of the above logarithm at z= 0, are given by:
$\log(1+\lambda_1\cdot K(z)) = \log(1+\lambda_1\cdot z/(1-z)^2) = \lambda_1\cdot z + (1/2)(4\cdot\lambda_1 - \lambda_1^2)\cdot z^2 + (1/3)\cdot(9\cdot\lambda_1 - 6\cdot\lambda_1^2 + \lambda_1^3)\cdot z^3 + +1/4(16\lambda_1 - 20\lambda_1^2 + +8\lambda_1^3 - \lambda_1^4) + O(z^5)$.

IV. Concluding Remark: A possible proof of The RH

In this work we have analyzed the Li-Keiper coefficients both in a numerical as well as in an analytical context.
For the first, i.e. λ_1, we have started with the series of positive summands and carried out a numerical experiment looking at the "speed" at which such a series (connected with the binary system) contains amounts of the reciprocal values of the nontrivial zeros on the critical line, and we have obtained the behavior given in the plot of M_n as a function of 1/N, where N is the number of zeros related to M_n.
That analysis suggested to us a definition of the first increment connected with λ_1 with the proposal of a criterion for the truth of the RH using λ_1 alone.
For λ_2 we have used our established lower bound to obtain a lower bound on λ_3, using also the positivity property of the n-ten derivative of the Riemann ξ function at the border of the critical strip, i.e. at s=1 [9].
Finally, we have:

- **Binomial structure into the Riemann wave background;**
- **A possible proof of the Riemann Hypothesis.**

We now write: $1\cdot\varphi_1 = (1)\cdot\lambda_1$

$$2\cdot(1\cdot\varphi_2 - 1\cdot\varphi_1) = \frac{1}{2}\cdot\frac{\xi''}{\xi}(1)$$

$$3\cdot(1\cdot\varphi_3 - 2\cdot\varphi_2 + 1\cdot\varphi_1) = \frac{1}{6}\cdot\frac{\xi'''}{\xi}(1)$$

$$4\cdot(1\cdot\varphi_4 - 3\cdot\varphi_3 + 3\cdot\varphi_2 - 1\cdot\varphi_1) = \frac{1}{24}\cdot\frac{\xi^{IV}}{\xi}(1) \qquad (12)$$

We recognize in the parenthesis the binomial coefficients, where:

$$(1-1)^n = \sum_{k=0}^{n} 1^n\cdot(-1)^{n-k}\cdot\binom{n}{k} = 0$$

a signal, coming from the structure above of the binomials for the concrete appearance of the Riemann wave background [8].
For n=3:

$$\varphi_3 = 2\cdot\varphi_2 - \varphi_1 + \frac{1}{3!}\cdot\left(\frac{\xi'''}{\xi}\right)(1) > 2\cdot\varphi_2 - \varphi_1 > 3\cdot\varphi_1 \qquad (13)$$

from Eq.(12), $\varphi_4 = 3\cdot\varphi_3 - 3\cdot\varphi_2 + \varphi_1 + (1/(4!))\cdot(\xi^{IV}/\xi)(1)$ and from above.
$\varphi_4 > 3\cdot(2\cdot\varphi_2 - \varphi_1) - 3\cdot\varphi_2 + \varphi_1 + (1/(4!))\cdot(\xi^{IV}/\xi)(1) + (1/(3!))\cdot(\xi'''/\xi)(1)$
$\varphi_4 = 3\cdot\varphi_2 - 2\cdot\varphi_1 + \delta,\ \delta > 0,\quad \delta = \frac{1}{4!}\left(\frac{\xi^{IV}}{\xi}\right)(1) + \frac{1}{3!}\left(\frac{\xi'''}{\xi}\right)(1) > 0$

$$\rightarrow \varphi_4 > 4\cdot\varphi_1 \qquad (14)$$

$$\varphi_n > n\cdot\varphi_1 \qquad (15)$$

To conclude, the general structure is as follows:

$$\sum_{k=0}^{n-1}(-1)^{2n-k}\cdot\binom{n-1}{k}\cdot\varphi_{n-k}=\frac{1}{\Gamma(n+1)}\cdot\left(\frac{\frac{d^n}{ds^n}\xi}{\xi}\right)\Bigg|_{s=1} \tag{16}$$

and this gives $\varphi_n > n.\ \varphi_1$ (15),for all n.
The binomial structure, and Eq.(15) ensures the correctness of our Riemann Wave background [8], and then the truth of the RH by means of $\log(1+\lambda_1(\gamma,\pi)\cdot K(z))$.
Moreover, if we sum the right hand sides of Eq.(16) over n, we obtain the number:

$$\frac{\pi}{3}\left(\frac{\xi'}{\xi}(2)\right)=\frac{\pi}{3}\left[\frac{1}{2}(\gamma+\log(4\pi)+3)-12logA\right]=0.072325988$$

Where A is the Glaisher-Kinkelin constant.
Since:

$$\zeta'(-1)=\frac{1}{12}-lnA=-0.1654211937$$

Then, with $N_0(n)$ of Eq. (2) [1]:

$$\frac{\pi}{3}\cdot\left(1+\frac{1}{2}\ln 2+12\,\zeta'(-1)+\sum_{n=1}^{\infty}\frac{N_0(n)}{2n(2n+1)(2n+2)}\right)$$

which gives the constant: 0.0723…
The same number is obtained with the primes $\left(\frac{\xi'}{\xi}(2)\right)$.

$$\frac{\pi}{3}\cdot\left(\frac{3}{2}-\frac{1}{2}ln\,\pi-\frac{\gamma}{2}-\sum_{p=2}^{\infty}\frac{lnp}{p^2-1}\right)$$

Where the sum is on all the primes p.
Then, for λ_2 we obtain the upper bound:
$\lambda_2 < 2\cdot 0.072325988 - (\lambda_1)^2 = 0.14411856$. The true value is: $\lambda_2 = 0.092\ldots \sim 0.10\ldots$
Summing the left hand sides of Eq.(16) over n, we have:

$$\sum_{n=1}^{\infty}\varphi_n-\sum_{n=1}^{\infty}\varphi_n\cdot n+\sum_{n=1}^{\infty}\varphi_n\cdot\frac{1}{2}\cdot n\cdot(n-1)-\frac{1}{6}\sum_{n=1}^{\infty}\varphi_n\cdot n\cdot(n+1)(n+2)+\cdots$$

A very strong cancellation.
Moreover:

$$\frac{\pi}{3}\left(\frac{\xi'}{\xi}(s)\right)\Bigg|_{s=2}=\frac{\pi}{3}\left[\left(\frac{d}{dz}ln\xi(z)\right)\cdot\frac{dz}{ds}\right]\Bigg|_{z=\frac{1}{2}}=\frac{\pi}{12}\sum_{n}\lambda_n\cdot\left(\frac{1}{2}\right)^{n-1}$$

Which gives the same constant 0.072323…
The sum of the left hand side of Eq.(16) (linear in the φ') gives linearity in the Li-Keiper coefficients, i.e. from above:

$$\sum_{n=1}^{\infty}\lambda_n\cdot\left(\frac{1}{2}\right)^{n+1}=\left(\frac{\xi'}{\xi}\right)\Bigg|_{s=2} \tag{17}$$

The binomial structure Eq.(16), and Eq.(15) ensures the correctness of our demonstration [8].
If we reconsider the nontrivial zeros we have from above:

$$\frac{\pi}{12}\sum_n\lambda_n\cdot\left(\frac{1}{2}\right)^{n+1}=\frac{\pi}{12}\frac{d}{dz}\sum_n\frac{\lambda_n}{n}z^n\Big|_{z=\frac{1}{2}}=$$

$$\frac{\pi}{12}\left\{\frac{d}{ds}\left[ln\left(\prod_\rho\left(1-\frac{s}{\rho}\right)\left(1-\frac{s}{1-\rho}\right)\right)\right]\right\}\cdot\left(\frac{ds}{dz}\right)\Big|_{s=2}=$$

$$\frac{4\pi}{12}\sum_\rho\left(\frac{1}{s-\rho}+\frac{1}{s-(1-\rho)}\right)\Big|_{s=2}=$$

$$\frac{\pi}{3}\sum_\rho\left(\frac{1}{s-\rho}+\frac{1}{s-(1-\rho)}\right)\Big|_{s=2}=\frac{\pi}{3}\sum_\rho\left(\frac{3}{(2-\rho)(1+\rho)}\right)\Big|_{s=2}<\pi\cdot\lambda_1=$$

= π· 0.0230957…= 0.07255730.
We indicated this value with C_0. Thus, the upper bound to the above constant:
$0.07232\ldots < C_0 = 0.07255\ldots$ (18)

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Appendix 1

An analogy with a model of Statistical Mechanics: example with the 2-dimensional ferromagnetic Ising model in zero field solved by Onsager in 1944. We compute the first coefficient for the 2-d Ising model in zero field already considered in [6].
The partition function of Statistical Mechanics is givenby:

Z = Trace exp(-β·H) and the free energy by:

$$-\beta\cdot f = \log\big(Z(\beta,J)\big) = \left(\tfrac{1}{2}\right)\left(\tfrac{1}{2\pi}\right)^2 \int_0^{2\pi} d\alpha_1 \int_0^{2\pi} d\alpha_2\, log(1+z^2 - z(\cos(\alpha_1)+\cos(\alpha_2))$$

where z = sinh(2k) = sinh(2·β·J) (β is the inverse temperature and -J the interaction strength between two neighboring spins on the square lattice).
The zeros in z are on the unit circle |z| = 1and the critical temperature is given by
z= sinh(2·β·J) = 1.
The critical line for this model is obtained by the usual transformation z=1-1/s and the argument of log(Z) is now given by:

$\log(1+(1-1/s)^2 - s\cdot c)$ where $c = \cos(\alpha_1)+\cos(\alpha_2)$

the zeros in s are on the critical line, now given by $s = \sigma \pm i\cdot t_k = 1/2 \pm i\cdot t_k$ where

$$t_k = t_n(c) = \sqrt{\frac{2+c}{4\cdot(2-c)}} \quad \text{(notice that } |c|\le 2).$$

We compute :

$$\lambda_1 = \sum_{t_k}\left(\frac{1}{\left(\frac{1}{2}\pm i\cdot t_k\right)}\right) = \left(\frac{1}{2}\right)\cdot\left(\frac{1}{2\pi}\right)^2\cdot\int_0^{2\pi} d\alpha_1\cdot\int_0^{2\pi} d\alpha_2\cdot\left(\frac{1}{4}+\frac{2+c}{4\cdot(2-c)}\right) =$$

$$= \left(\tfrac{1}{2}\right)\cdot\left(\tfrac{1}{2\pi}\right)^2\cdot\int_0^{2\pi} d\alpha_1\cdot\int_0^{2\pi} d\alpha_2\cdot(2-c(\alpha_1,\alpha_2)) = 1 \;\rightarrow \lambda_1 = 1.$$

The same value of λ_1 is obtained with $\lambda_1 = d/ds\,(\log(Z(s))\,|_{s=1} =$

$$= \left(\frac{1}{2}\right)\cdot\left(\frac{1}{2\pi}\right)^2\cdot\int_0^{2\pi} d\alpha_1\cdot\int_0^{2\pi} d\alpha_2\cdot\left.\frac{(2\cdot(2-c)\cdot s-(2-c))}{(1+s^2\cdot(2-c)-s\cdot(2-c))}\right|_{s=1} = 1$$

This conclude an analogy with the 2-d Ising model in zero field for the first coefficient λ_1.

Appendix 2

As started in [8], keeping in mind that for large n a Li-Keiper coefficient λ_n become negative if there is the presence of at least a nontrivial zero off the critical line -here- we merely look at the increments Δ_n which are defined as follows

$$\Delta_n := \sum_\rho \left(1-\frac{1}{\rho}\right)^{(n-1)} \cdot \left(\frac{1}{\rho}\right) \qquad \text{(A21)}$$

where the sum is on all nontrivial zeros $\rho = \sigma \pm i\cdot t$ and $1-\rho = 1-\sigma \pm i\cdot t$ (if we assume that the nontrivial zeros are simple, Δ_n, for $\sigma = 1/2$, should be divided by 2. (We omit here the indices k for $\rho = \sigma_k \pm i\cdot t_k$).

Then without any assumption the amount is given by:

$$\Delta_n = \sum_\rho \left(\frac{2\cdot A_2^{(n-1)}\cdot \sin((n)\cdot\alpha_1 + (n-1)\cdot\alpha_2)}{A_1^n} + \frac{2\cdot A_1^{(n-1)}\cdot \sin((n-1)\cdot\alpha_1 + (n)\cdot\alpha_2)}{A_2^n} \right) \qquad \text{(A22)}$$

where $A_1 = |\sigma \pm i\cdot t| = (\sigma^2+t^2)^{(1/2)}$, $A_2 = |1-\sigma + i\cdot t| = ((1-\sigma)^2 + t^2)^{(1/2)}$

$\alpha_1 = \arctan(\sigma/t)$ and $\alpha_2 = \arctan((1-\sigma)/t)$.

Notice that if $\sigma = 1/2$, $A_1 = A_2$, $\alpha_1 = \alpha_2$ and Δ_n reduces to

$$\Delta_n\left(\sigma = \frac{1}{2}\right) = \sum_\rho 4\cdot\left(\sin(2\cdot n - 1)\cdot arctan\left(\frac{1}{2\cdot t}\right)\right)\cdot\left(\frac{1}{\sqrt{\left(\frac{1}{4}+t^2\right)}}\right) \qquad \text{(A23)}$$

For n=1, the above Formula gives:

$$\Delta_1 = \frac{2}{\sqrt{(\sigma^2+t^2)}}\cdot \sin\left(arctan\left(\frac{\sigma}{t}\right)\right) + \frac{2}{\sqrt{((1-\sigma)^2+t^2)}}\cdot \sin\left(arctan\left(\frac{1-\sigma}{t}\right)\right)$$

Since

$$\sin(\alpha_1) = \frac{\sigma}{\sqrt{(\sigma^2+t^2)}} \text{ and } \sin(\alpha_2) = \frac{(1-\sigma)}{\sqrt{((1-\sigma)^2+t^2)}}$$

we obtain:

$$\Delta_1 = \sum_{(\sigma,t)} \left[\frac{2\cdot\sigma}{(\sigma^2+t^2)} + \frac{2\cdot(\sigma-1)}{((1-\sigma)^2+t^2)}\right] \qquad \text{(A24)}$$

which is the same as Eq.(4). In the next pages we give as illustration some plots concerning the fluctuations using Eq.(A22), (variable n, t, σ).

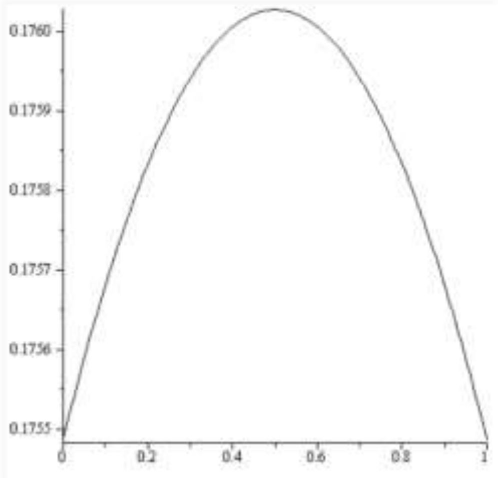

Fig. A_{21}.Plot of $\Delta_{10}(t_1=14.134725_{\dots}\ \sigma)$ as a function of σ(n=10)

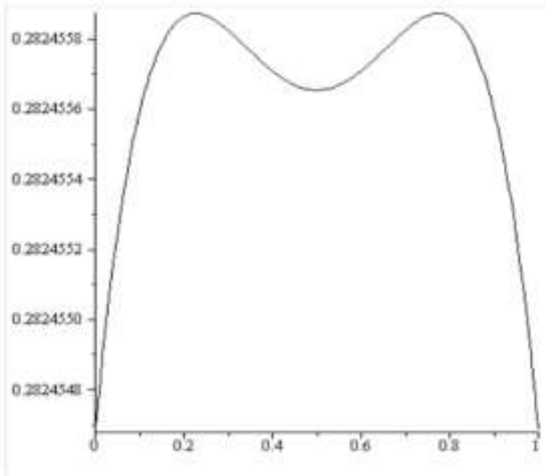

Fig.A_{22}. Plot of $\Delta_{22}(t_1=14.134725_{\dots}\ \sigma)$ as a function of σ showing a "borderline" for the onset of minimal fluctuations at the firs level t_1 .(n=22)

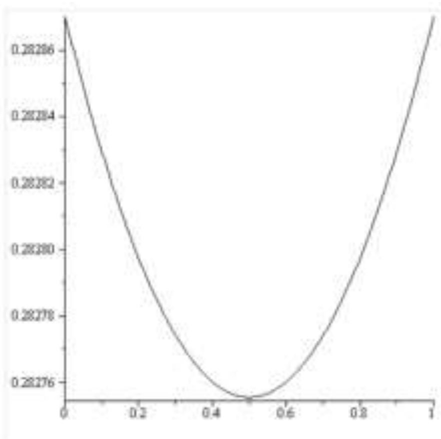

Fig.A_{23}. Plot of $\Delta_{23}(t_1=14.134725_{\dots}\ \sigma\ )$ as a function of σ(n=23)

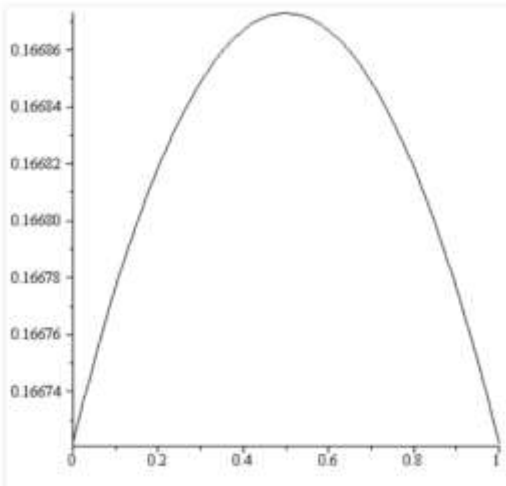

Fig.A_{24}.Plot of $\Delta_{23}(t_2 = 21.022039_{...}\ \sigma)$ as a function of $\sigma(n=23)$ (second level t_2)

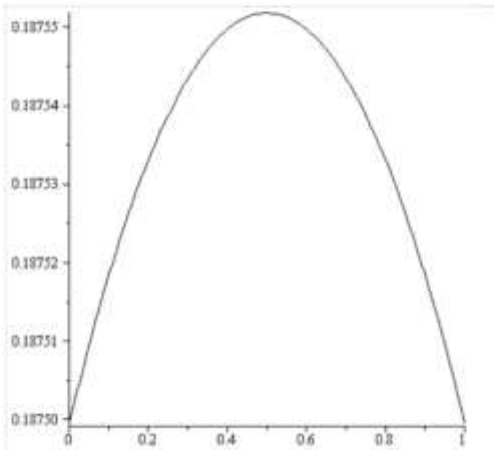

Fig.A_{25}.Plot of $\Delta_{23}(t_2 = 21.022039_{...}\ \sigma)$ as a function of $\sigma(n=30)$ (second level t_2)

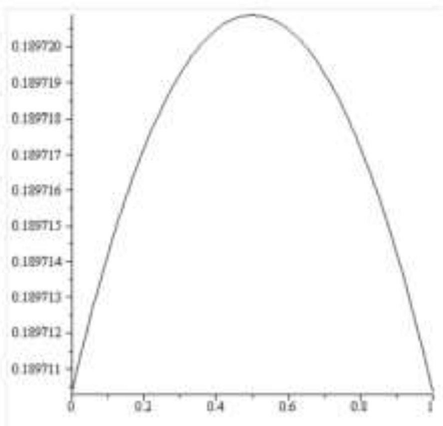

Fig.A_{26}.Plot of $\Delta_{23}(t_2 = 21.022039_{...}\ \sigma)$ as a function of $\sigma(n=32)$ (second level t_2) .

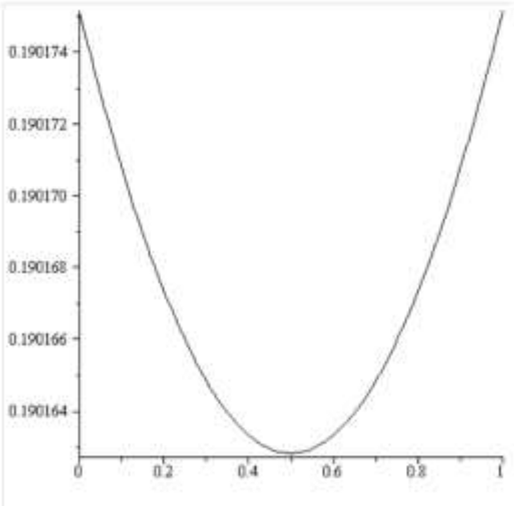

Fig.A27.Plot of $\Delta_{23}(t_2 = 21.022039_{...}\ \sigma)$ as a function of $\sigma(n=33)$ (second level t_2) .

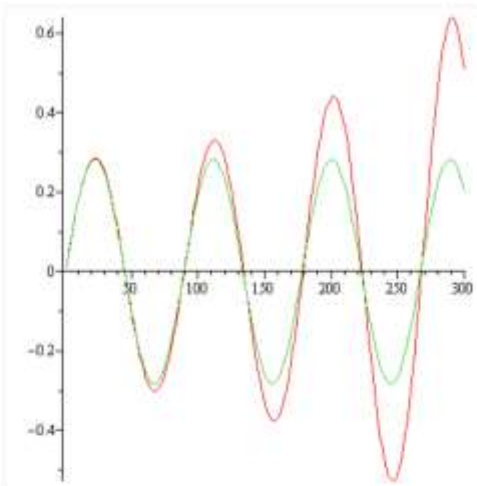

Fig.A28. $\Delta_n(14.134725,_{..}\sigma=1/2)$ in green and $\Delta_n(14.134725,_{...}\ \sigma=0$ or $\sigma=1)$ in red. (Onset of strong fluctuations at t_1 for $n \sim 40$ at $\sigma=0$).

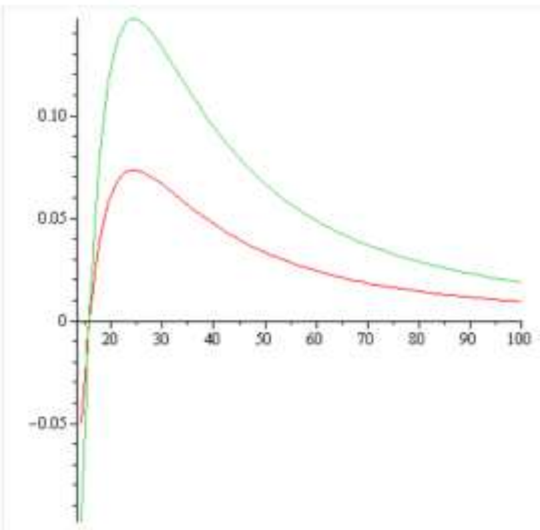

Fig.A29.$\Delta_n(14.134725,_{..}\sigma=1/2)$ in red and $\Delta_n(14.134725,_{..}\sigma=0$ or $\sigma=1)$ in green (border of the critical strip): here we suppose that the zeros are simple

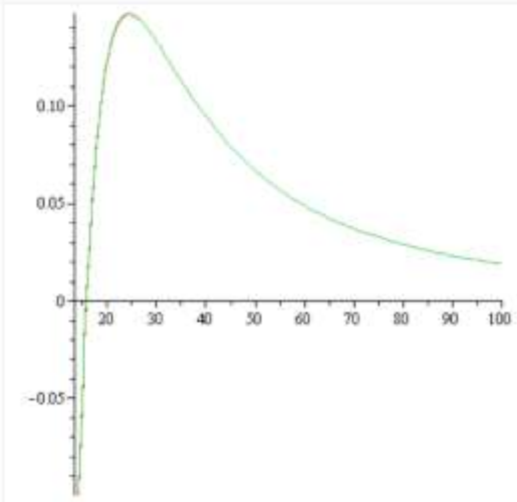

Fig.$A_{30.}$ Δ_n(14.134725,..σ=1/2) in red and Δ_n(14.134725,..σ=0 or σ=1) in red (border of the critical strip: here without assuming that the zeros are simple: the curves are superimposed).

In the last Figure, we present the difference i.e. Δ_n(14.134725,..σ=1/2) - Δ_n(14.134725,..σ=0).

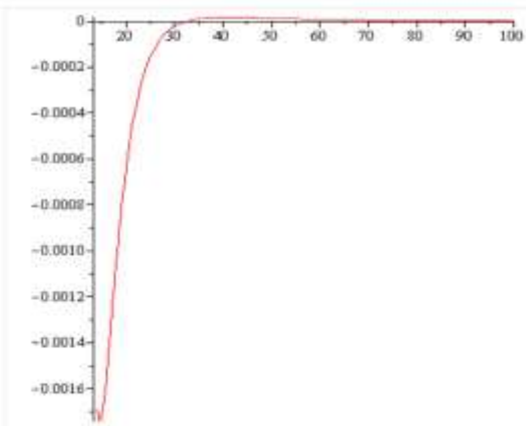

Fig.A_{31}.The difference indicated as above, at t>30 is positive, and this is more in favour of the content of Fig.A_{29} where the simplicity of the zeros was assumed with the indication of the minimal fluctuations at σ=1/2.

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