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On Beal's Conjecture, the Case (x,y,z)=(n,2n-1,n)

Marouane RHAFLI

Abstract
This study presents a special case for Beal's conjecture by giving the exact analytic expression, as well as a solution suggestion to the general case

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I. Introduction

Beal's conjecture is a conjecture in number theory, it states that

if $A^x + B^y = C^z$

where A, B, C, x, y, and z are positive integers with x, y, $z > 2$, then A, B, and C have a common prime factor. Equivalently, There are no solutions to the above equation in positive integers A, B, C, x, y and z with A, B and C being pairwise co-prime and all of x, y and z being greater than 2. Billionaire banker Andrew Beal formulated this conjecture in 1993 while investigating generalizations of Fermat's last theorem [2]. Several cases have been proved for $n \geq 2$, such as $(x, y, z) = (5, 2n, 2n)$ [3], $(x, y, z) = (3, n, n)$ [4]... and many other ones

In this paper, we'll approve the Beal's conjecture for the special case $(x, y, z) = (n, 2n-1, z)$ by giving a formula for it, we'll also propose a suggestion why A, B and C can never be coprimes and their $gcd(A, B, C) \geq p_n$

II. Development

2.1 Theorem

Every odd composite number $N \geq 9$ can be written as [1]

$$p_n^2 + 2p_n c = N \qquad (p_n \in P | p_n \geq 3, c \in Z^+) \tag{1}$$

<u>Proof of theorem</u>

Let $p_n \in P$ such as $p_n \geq 3$, q and odd number ≥ 3 and $k \in Z^+$, every odd number can be written as

$$p_n + 2k = q$$

By multiplying both sides of the above equation with p_n, we get

$$p_n^2 + 2p_nc = N \qquad (N = p_nN^*)$$

2.2 The case $(x, y, z) = (n, 2n-1, n)$

Let take the case when the odd composite N is a multiple of the prime p_n, and let N^* be another odd composite such as

$$N = p_nN^*$$

Then (1) can be written as

$$p_n^2 + 2p_nc = p_nN^*$$

Let the following be

$$\begin{bmatrix} p_n^2 + 2p_nc = p_nN^* \\ (p_n^2 + 2p_nc)^2 = (p_nN^*)^2 \\ (p_n^2 + 2p_nc)^3 = (p_nN^*)^3 \\ \ldots \\ (p_n^2 + 2p_nc)^\alpha = (p_nN^*)^\alpha \end{bmatrix} = \begin{bmatrix} p_n^2 + 2p_nc = p_nN^* \\ (p_n^2)^2 + 2(2p_nc)p_n^2 + (2p_nc)^2 = (p_nN^*)^2 \\ (p_n^2)^3 + 3(2p_nc)(p_n^2)^2 + 3(2p_nc)^2(p_n^2) + (2p_nc)^3 = (p_nN^*)^3 \\ \ldots \\ \sum_{k=\circ}^{\alpha} \binom{\alpha}{k}(p_n^2)^k(2p_nc)^{\alpha-k} = (p_n^2 + 2p_nc)^\alpha = (p_nN^*)^\alpha \end{bmatrix}$$

From the Newton binomial theorem we have

$$(p_n^2)^2 + 2(2p_nc)p_n^2 + (2p_nc)^2 = (p_nN^*)^2$$

By rearranging and factorizing we get

$$(p_n^2)(2^2c^2) + p_n^3(4c + p_n) = (p_nN^*)^2$$

or as

$$((p_n)\sqrt{(2^2c^2)})^2 + (p_n\sqrt[3]{(4c + p_n)})^3 = (p_nN^*)^2 \tag{2}$$

Equation (2) is of the form $A^x + B^y = C^z$ with $(A = (p_n)\sqrt{(2^2c^2)};\ B = p_n\sqrt[3]{(4c + p_n)};\ C = p_nN^*)$ and $(x = z = 2; y = 3)$

By expanding to a general formula we conclude the following theorem 1

3 Theorem 1 (case $(x, y, z) = (n, 2n-1, n)$)

if $A^x + B^y = C^z$

where x, y, z and C are positive integers and A, B are either positive integers or fractional numbers with x, y, $z > 2$; $(x = z; y = 2x - 1)$;$(a, b, c, .., \epsilon)$ are the newton binomial coefficients of $(x_1 + y_1)^z$ with the number of coefficient $H = z - 2$ and $x - \alpha = 2$, then A, B, and C have a common prime factor defined by

$$(p_n \sqrt[x]{(2^x m^x + 2^{x-1} a.p_n m^{x-1} + 2^{x-2} b.p_n^2 m^{x-2} + ...2^{x-\alpha} \epsilon.p_n^{\alpha} m^{x-\alpha})})^x + (p_n \sqrt[y]{(2mz + p_n)})^y = (p_n N^*)^z \quad (3)$$

Where $N*$ is an odd composite generated by a prime $p_n \geq 3$ using the equation $p_n^2 + 2p_n m = p_n N^*$

4 Testing examples

4.1 A and B are non-integers

- Announcement:

By using the theorem 1, find the Beal's equation $A^x + B^y = C^z$
where A, B and C have the same prime factor $p_n = 7$ for the odd composite $N^* = 9$ when $z = 7$

- Solution:
The integers x, y, z are:
From the theorem 3

$$x = z = 7$$

and

$$y = 2x - 1 = 13$$

The constant m is :
By using the equation $p_n^2 + 2p_n m = p_n N^*$ and solving for m we get

$$m = 1$$

The constant α is :

$$\alpha = x - 2 = 5$$

The binomial coefficients H and $a, b, ..., \epsilon$ are :

$$H = z - 2 = 5$$

The 5^{th} consecutive Newton binomial coefficients for $(x_1 + y_1)^7$ are

$$(x_1 + y_1)^7 = x^7 + 7x^6y + 21x^5y^2 + 35x^4y^3 + 35x^3y^4 + 21x^2y^5 + 7xy^6 + y^7$$

then

$$a = 7; b = 21; c = 35; d = 35; e = 21$$

As we found all the unknown values, let's substitute them accordingly with theorem 3

$$(p_n \sqrt[x]{(2^x m^x + 2^{x-1} a.p_n m^{x-1} + 2^{x-2} b.p_n^2 m^{x-2} + 2^{x-3} c.p_n^3 m^{x-3}) + 2^{x-4} d.p_n^4 m^{x-4}) + 2^{x-5} e.p_n^5 m^{x-5}))^x$$

$+$

$$(p_n \sqrt[y]{(2mz + p_n)})^y = (p_n N^*)^z$$

After numerical application, the both right and hand side equals to $(7 \times 9)^7 =$ 3938980639167

But the terms with the nth roots are fractional numbers (non-integers)

4.2 A is non-integer, B is integer

- Announcement:

By using the theorem 1, find the Beal's equation $A^x + B^y = C^z$

where A, B and C have the same prime factor $p_n = 161039$ for, $m = 2$ and $z = 3$

By following the same process as the previous example, we found B as an integer

$$B = p_n \times 11 = 161039 \times 11$$

5 Theorem 2 (case $(x, y, z) = (n, 2n-1, n)$)

if $A^x + B^y = C^z$

Where x, y, z and C are positive integers and A, B are either positive integers or fractional numbers with x, y, $z > 2$; $(x = z; y = 2x - 1)$;$(a, b, c, .., \epsilon)$ are the newton binomial coefficients of $(x_1 + y_1)^z$ with the number of coefficient $H = z-2$ and $x-\alpha = 2$, then A, B and C will never have different primes factors.

5.1 Proof of theorem 2

It exists a connection between two different primes p_n and p_j, when they generate the same odd composite, its general formula is defined by the periodic equation

$$p_n^2 p_j + 2p_n p_j k = N \qquad k \in Z \tag{4}$$

The proof of the above equation

Start from the trivial odd numbers equation ($p_n + 2k = N^*, p_n \geq 3, k \geq 0$) and multiply both sides with $p_n p_j$

E.g.: for the primes $3, 5$, they always generate the same odd composite

$$45 + 30k = 45 \qquad k = 0$$
$$45 + 30k = 75 \qquad k = 1$$
$$45 + 30k = 105 \qquad k = 2$$
$$\ldots$$

Rearanging and solving equation 4 for p_n we get

$$p_n = \frac{N}{2p_j k} - \frac{p_n^2}{2k}$$

Or as

$$p_n = p_j\left(\frac{N}{2p_j^2 k} - \frac{p_n^2}{2p_j k}\right) \tag{5}$$

Similarly for the same prime p_n and a different prime $p_i \neq p_j$, $k \neq k'$ we have

$$p_n = p_i\left(\frac{N}{2p_i^2 k'} - \frac{p_n^2}{2p_i k'}\right) \tag{6}$$

Replacing the equations 5 and 6 in the theorem 1 we get for B and C

$$B = (p_j(\frac{N}{2p_j^2 k} - \frac{p_n^2}{2p_j k}) \sqrt[y]{(2mz + p_n)})^y$$

$$C = (p_i(\frac{N}{2p_i^2 k'} - \frac{p_n^2}{2p_i k})N^*)^z$$

A, B and C seems to have now different prime factors, but actually, they are not, because

$$p_n = p_j\left(\frac{N}{2p_j^2 k} - \frac{p_n^2}{2p_j k}\right) = p_n\left(\frac{N}{2p_j k p_n} - \frac{p_n}{2k}\right)$$

It implies that A, B and C will always have the same prime factor and their $gdc(A, B, C) \geq p_n$ and the theorem 2 is then proved

6 Corollary (proposition to a generalized proof)

if $A^x + B^y = C^z$

Where A, B, C, x, y, and z are positive integers such as $x, y, z > 2$, then there are no solutions to the above equation if A, B and C are coprime

6.1 Proof of corollary

Assume that co-primes $p_n \neq p_j \neq p_i$ exists, and $(A = p_n A')$,$(B = p_n B')$ and $(C = p_n C')$ then $A^x + B^y = C^z$ becomes

$$(p_n A')^x + (p_n B')^y = (p_n C')^z$$

By using equations $1, 5$ and 6, the above equation becomes

$$(p_n A')^x + (p_j(\frac{N}{2p_j^2 k} - \frac{p_n^2}{2p_j k})B')^y = (p_i(\frac{N}{2p_i^2 k'} - \frac{p_n^2}{2p_i k'}C')^z \qquad (7)$$

Once again, the above equation seems to have different primes factors, but it is not, we can always factorize by p_n to get

$$(p_n A')^x + (p_n(\frac{N}{2p_j k p_n} - \frac{p_n}{2k})B')^y = (p_n(\frac{N}{2p_i k' p_n} - \frac{p_n}{2k'}C')^z$$

We can clearly see that, whenever we want to factorize by a different prime, we always get the same prime factor p_n. Moreover, the equation $p_n = p_j(\frac{N}{2p_j^2 k} - \frac{p_n^2}{2p_j k}) =$ is of the form

$$p_n = p_j \lambda \qquad (0 < \lambda < 1) \qquad (p_n < p_j)$$

As a prime number p is positive integer having exactly one positive divisor other than 1, then λ will be always a positive fractional number, that is to say, the existence of coprimes in Beal's equation requires their multiples to be fractional numbers, We conclude
A, B and C being positive integers will never be coprime and their $gcd(A, B, C) \geq p_n$

6.2 Testing example

In this numerical example, we'll see how the attempts to find a counterexample of Beal's conjecture always fail

Announcement

Fin and equation of the form $A^x + B^y = C^z$ with prime factors $p_n = 3$, $p_j = 5$, $p_i = 7$
where A, B, C, x, y, and z are positive integers such as $(x, y, z > 2)$.

Solution

By using equation 4, let's find the values of k, k' and the odd composite N for which $p_n^2 + 2p_n c$ generates the same odd composite

$$\begin{cases} p_n^2 p_j + 2p_n p_j k = N \\ p_n^2 p_i + 2p_n p_i k' = N \end{cases}$$

By solving the above system of equations and after numerical application, we get

$$k = \frac{18+42}{30}k'$$

We get then the odd composite $N = 105$ when $[k' = 1; k = 2]$
By replacing these values in equation 7

$$(3A')^x + (5(0,6)B')^y = (7(0,42857...)C')^z$$

The above equation has fractional numbers, the algorithms made to search for a Beal's conjecture counterexample will then always fail by taking in consideration just the positive integers, moreover even if these algorithms fall into an equation with positive integers the above equation will always switch back to the same factor $p_n = 3$

$$(3A')^x + (3B')^y = (3)C')^z$$

VII. Conclusion

There is a deep connection between prime numbers which yields to having only one prime factor, following this path,we gave a general formula for Beal's conjecture for the case $(x, y, z) = (n, 2n-1, n)$ and we proved that A, B and C can never be co-prime in this special case, we also gave a proposition in (Corollary) where we showed the connection between 2 primes, which is most likely the main reason why A, B, C can never be coprime

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