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On a Generalization of Fibonacci and Lucas Quaternions

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Abstract:
In this study, we define a type of generalized quaternions whose coefficients are a generalization of Fibonacci and Lucas quaternions. We give Binet – like formulas and generating functions for these kind of quaternions. By using Binet – like formulas, we obtain generalizations of some well – known identities such as, Vajda's, Catalan's, Cassini's and d'Ocagne's identities.
***Key Word**: Generalized Fibonacci numbers; Generalized Lucas numbers; Generating function; Binet – like formula.*

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I. Introduction

Fibonacci and Lucas sequences are two well – known sequences among integers sequences. Fibonacci sequence is the sequence of the numbers which satisfy the following second order recurrence relation

$$F_n = F_{n-1} + F_{n-2} \quad (n \geq 2)$$

where the initial conditions $F_0 = 0$ and $F_1 = 1$. The numbers which form the Lucas sequence satisfy the same recurrence relation except the initial conditions. Lucas numbers satisfy

$$L_n = L_{n-1} + L_{n-2} \quad (n \geq 2)$$

where the initial conditions $L_0 = 2$ and $L_1 = 1$. Generating functions for the Fibonacci sequence $\{F_n\}_{n=0}^{\infty}$ and Lucas sequence $\{L_n\}_{n=0}^{\infty}$ are

$$\sum_{n=0}^{\infty} F_n x^n = \frac{x}{1-x-x^2} \text{ and } \sum_{n=0}^{\infty} L_n x^n = \frac{2-x}{1-x-x^2}.$$

respectively. Binet formulas for the Fibonacci and Lucas numbers are, respectively

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \text{ and} L_n = \alpha^n + \beta^n$$

where $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$ are the roots of the characteristic equation $x^2 - x - 1 = 0$ and the positive root α is known as "golden ratio". See [8] for details.

Another pair of integer sequences are Pell and Pell – Lucas sequences. Whereas Pell sequence consists of the numbers that satisfy another second order recurrence relation

$$P_n = 2P_{n-1} + P_{n-2} \quad (n \geq 2)$$

where the initial conditions $P_0 = 0$ and $P_1 = 1$, Pell – Lucas sequence consists of the numbers that satisfy the same recurrence relation

$$Q_n = 2Q_{n-1} + Q_{n-2} \quad (n \geq 2)$$

where the initial conditions $Q_0 = 1$ and $Q_1 = 1$,Generating functions for the Fibonacci sequence $\{P_n\}_{n=0}^{\infty}$ and Pell – Lucas sequence $\{Q_n\}_{n=0}^{\infty}$ are

$$\sum_{n=0}^{\infty} P_n x^n = \frac{x}{1-2x-x^2} \text{ and } \sum_{n=0}^{\infty} Q_n x^n = \frac{2-2x}{1-2x-x^2}.$$

respectively. Binet formulas for the Pell and Pell – Lucas numbers are, respectively

$$P_n = \frac{\delta^n - \gamma^n}{\delta - \gamma} \text{ and} Q_n = \frac{\delta^n + \gamma^n}{2}$$

where $\delta = 1 + \sqrt{2}$ and $\gamma = 1 - \sqrt{2}$ are the roots of the characteristic equation $x^2 - 2x - 1 = 0$ and the positive root α is known as "silver ratio".See [9] for details.

The last sequences we mention, satisfying a second order recurrence relation, are Jacobsthal and Jacobsthal – Lucas sequences. Jacobsthal numbers satisfy the recurrence relation

$$J_n = J_{n-1} + J_{n-2} \quad (n \geq 2)$$

where the initial conditions $J_0 = 0$ and $J_1 = 1$. Jacobsthal – Lucas numbers satisfy the same recurrence relation

$$j_n = j_{n-1} + 2j_{n-2} \quad (n \geq 2)$$

where the initial conditions $j_0 = 2$ and $j_1 = 1$. Generating functions for the Jacobsthal sequence $\{J_n\}_{n=0}^{\infty}$ and Jaconsthal – Lucas sequence$\{j_n\}_{n=0}^{\infty}$ are

$$\sum_{n=0}^{\infty} J_n x^n = \frac{x}{1-x-2x^2} \text{and} \sum_{n=0}^{\infty} Q_n x^n = \frac{2-x}{1-x-2x^2}.$$

respectively. Binet formulas for the Jacobsthal and Jacobsthal –Lucas numbers are

$$J_n = \frac{\theta^n - \nu^n}{\theta - \nu} \text{and} j_n = \theta^n + \nu^n$$

respectively.

There are many generalizations of these three families of integer sequences. Bilgici [3] gave one of them by changing recurrence relations. The generalized Fibonacci and Lucas sequences are defined by the numbers that satisfy the recurrence relation

$$f_0 = 0, f_1 = 1, \ \ f_n = 2af_{n-1} + (b - a^2) f_{n-2} (n \geq 2)$$

and

$$l_0 = 2, f_1 = 2a, \ \ l_n = 2af_{n-1} + (b - a^2) l_{n-2} (n \geq 2)$$

where a and b are any real numbers. Taking $(a, b) \to \left(\frac{1}{2}, \frac{5}{4}\right)$, $(1,2)$and $\left(\frac{1}{2}, \frac{9}{4}\right)$generalized Fibonacci and Lucas sequences reduce to classical Fibonacci and Lucas sequences, Pell and Pell – Lucas sequences, Jacobsthal and Jacobsthal – Lucas sequences, respectively. Binet formulas for these generalized Fibonacci and Lucas numbers are

$$f_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \text{and} l_n = \alpha^n + \beta^n$$

where $\alpha = a + \sqrt{b}$and $\beta = a - \sqrt{b}$ are the roots of the characteristic equation $x^2 - 2ax - (b - a^2) = 0$.

Quaternions were introduced by Sir Hamilton in 1853 to extend the Complex numbers. They forms a skew fields and have three imaginary units. The set of all Hamilton quaternions is

$$\mathbf{H} \coloneqq \{a + bi + cj + dk: a, b, c, d \in \mathbf{R}, i^2 = j^2 = k^2 = ijk - 1, ij = -ji = k, jk = -kj = i, ki = -ik = j\}.$$

Here, we study on generalized two parameters quaternions which is a generalization of Hamilton quaternions. The set of these quaternions is

$$\mathbf{Q} \coloneqq \{a + bi + cj + dk: a, b, c, d \in \mathbf{R}\}$$

and the multiplication table of the set $\{1, i, j, k\}$ is given in the following table.

Table 1. Multiplication rules of the set $\{1, i, j, k\}$.

$\cdot$	1	i	j	k
1	1	i	j	k
i	i	$-\lambda$	k	$-\lambda j$
j	j	$-k$	$-\mu$	μi
k	k	λj	$-\mu i$	$-\lambda\mu$

This generalization gives Hamilton quaternions for $(\lambda, \mu) = (1,1)$ and split – quaternions for $(\lambda, \mu) = (1, -1)$. Let $q = a + bi + cj + dk$ be a generalized quaternions. Then the conjugate of q is $q^* = a - bi - cj - dk$ and the norm of q is $N(q) = \sqrt{|qq^*|} = \sqrt{|a^2 + b^2\lambda + c^2\mu + d^2\lambda\, \mu|}$.

Fibonacci quaternions were introduced by Horadam [6] as follows

$$Q_n = F_n + iF_{n+1} + jF_{n+2} + kF_{n+3}$$

and Lucas quaternions were introduced by Iyer [7] as follows

$$K_n = L_n + iL_{n+1} + jL_{n+2} + kL_{n+3}.$$

After them, there have been many studies on quaternions whose coefficients are members of any integer sequences. Halici [5] gave the following Binet – like formulas for Fibonacci and Lucas quaternions

$$Q_n = \frac{\hat{\alpha}\alpha^n - \hat{\beta}\beta^n}{\alpha - \beta} \text{and} K_n = \hat{\alpha}\alpha^n + \hat{\beta}\beta^n$$

where $\hat{\alpha} = 1 + i\alpha + j\alpha^2 + k\alpha^3$ and $\hat{\beta} = 1 + i\beta + j\beta^2 + k\beta^3$. Akyigit et al. [2] defined split Fibonacci and split Lucas quaternions and gave some properties of these kinds of numbers. Akyigit et al. [1]examined Fibonacci and Lucas quaternions over the generalized quaternion algebra $\mathbf{Q}$. There are some studies about Fibonacci and Lucas numbers or generalizations Fibonacci and Lucas numbers over the quaternion algebra. Similarly, some authors studied on quaternions whose coefficients are Pell and Pell – Lucas numbers [4,11], while some studied on quaternions whose coefficients are Jacobsthal and Jacobsthal – Lucas numbers [10,12].

We study on generalized quaternions whose coefficients are generalized Fibonacci numbers f_n and generalized Lucas numbers l_n.

II. Definitions, Generating Functions and Binet – Like Formulas

Definitions of generalized Fibonacci and Lucas quaternions are in the following.

Definition 2.1. For any non-negative integer r, the rth generalized Fibonacci and Lucas quaternions are

$$S_n = f_n + if_{n+1} + jf_{n+2} + kf_{n+3}$$

and

$$T_n = l_n + il_{n+1} + jl_{n+2} + kl_{n+3}$$

respectively, where f_n and l_n are as defined above, and the elements of $\{1, i, j, k\}$ satisfy multiplication rules in Table 1.

By using identities $f_{-n} = -\frac{1}{(a^2-b)^n} f_n$ and $l_{-n} = \frac{1}{(a^2-b)^n} l_n$, we obtain generalized Fibonacci and Lucas quaternions for negative indices

$$S_{-n} = -\frac{1}{(a^2-b)^n}[f_n - if_{n-1} + jf_{n-2} - kf_{n-3}]$$

and

$$T_{-n} = \frac{1}{(a^2-b)^n}[l_n - il_{n-1} + jl_{n-2} - kl_{n-3}].$$

We also obtain the following recurrence relation for generalized Fibonacci and Lucas quaternions

$$S_n = 2aS_{n-1} + (b-a^2)S_{n-2}$$

and

$$T_n = 2aT_{n-1} + (b-a^2)T_{n-2}.$$

We give generating functions for the sequences of generalized Fibonacci and Lucas quaternions without proof because of their straightforwardness.

Theorem 2.2. The generating functions for thegeneralized Fibonacci sequence $\{S_n\}_{n=0}^{\infty}$ and the generalized Lucas sequence $\{T_n\}_{n=0}^{\infty}$ are

$$\sum_{n=0}^{\infty} S_n x^n = \frac{i + 2aj + (3a^2+b)k + x[1 + (b-a^2)j + (-2a^3+2ab)k]}{1 - 2ax - (b-a^2)x^2}$$

and

$$\sum_{n=0}^{\infty} T_n x^n = 2\frac{1 + ai + (a^2+b)j + (a^3+3ab)k + x[-a + (b-a^2)i + (-a^3+ab)j + (-a^4+b^2)k]}{1 - 2ax - (b-a^2)x^2}$$

respectively.

Binet – like formulas for the generalized Fibonacci and Lucas quaternions are given in the following theorem.

Theorem 2.3. For any real numbers a, b and any integer r, the rth generalized Fibonacci and Lucas quaternion are

$$S_n = \frac{\tilde{\alpha}\alpha^n - \tilde{\beta}\beta^n}{\alpha - \beta}$$

and

$$T_n = \tilde{\alpha}\alpha^n + \tilde{\beta}\beta^n$$

where $\alpha = a + \sqrt{b}, \beta = a - \sqrt{b}, \tilde{\alpha} = 1 + i\alpha + j\alpha^2 + k\alpha^3$ and $\tilde{\beta} = 1 + i\beta + j\beta^2 + k\beta^3$.

Proof. From the definition of the generalized Fibonacci quaternions, we have

$$\begin{aligned} S_n &= f_n + if_{n+1} + jf_{n+2} + kf_{n+3} \\ &= \frac{\alpha^n - \beta^n}{\alpha - \beta} + i\frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta} + j\frac{\alpha^{n+2} - \beta^{n+2}}{\alpha - \beta} + k\frac{\alpha^{n+3} - \beta^{n+3}}{\alpha - \beta} \\ &= \frac{1}{\alpha - \beta}[\alpha^n(1 + i\alpha + j\alpha^2 + k\alpha^3) - \beta^n(1 + i\beta + j\beta^2 + k\beta^3)]. \end{aligned}$$

The last equation proves the theorem. Binet – like formula for the generalized Lucas quaternions can be obtained in a similar way. □□

We need the following identities with Binet -–like formulas to obtain the properties of generalized Fibonacci and Lucas quaternions.

Lemma 2.4. For any real numbers a and b, we have

$$\tilde{\alpha}\tilde{\beta} = M + 2(a^2 - b)\sqrt{b}N$$

and

$$\tilde{\beta}\tilde{\alpha} = M - 2(a^2 - b)\sqrt{b}N$$

where

$$M = T_0 - 1 - (a^2 - b)\lambda - (a^2 - b)^2\mu - \lambda\mu(a^2 - b)^3$$

and

$$N = (b - a^2)\mu i + 2\lambda a j - k.$$

We easily obtain this lemma by using the multiplication rules in Table 1. Now we can obtain some well – known identities by using Binet – like formulas and Lemma 2.4.

III. Results

First we begin with Vajda's identities which is given in the next theorem.

Theorem 3.1. Forany real numbers a, b, any integers p, q and r, we have

$$S_{p+q}S_{p+r} - S_pS_{p+q+r} = (a^2 - b)^p f_q(Mf_r - (a^2 - b)Nl_r)$$

and

$$T_{p+q}T_{p+r} - T_pT_{p+q+r} = -4b(a^2 - b)^p f_q(Mf_r - (a^2 - b)Nl_r).$$

Proof. From the Binet – like formula for the generalized Fibonacci quaternions in Theorem 2.3, we obtain

$$\begin{aligned}
&S_{p+q}S_{p+r} - S_pS_{p+q+r}\\
&= \frac{1}{4b}\left[\left(\tilde{\alpha}\alpha^{p+q} - \tilde{\beta}\beta^{p+q}\right)\left(\tilde{\alpha}\alpha^{p+r} - \tilde{\beta}\beta^{p+r}\right) - \left(\tilde{\alpha}\alpha^{p} - \tilde{\beta}\beta^{p}\right)\left(\tilde{\alpha}\alpha^{p+q+r} - \tilde{\beta}\beta^{p+q+r}\right)\right]\\
&= \frac{1}{4b}\left[-\tilde{\alpha}\tilde{\beta}\alpha^{p+q}\beta^{p+r} - \tilde{\beta}\tilde{\alpha}\alpha^{p+r}\beta^{p+q} + \tilde{\alpha}\tilde{\beta}\alpha^{p}\beta^{p+q+r} + \tilde{\beta}\tilde{\alpha}\alpha^{p+q+r}\beta^{p}\right]\\
&= \frac{(a^2 - b)^p}{4b}\left[-\tilde{\alpha}\tilde{\beta}\alpha^{q}\beta^{r} - \tilde{\beta}\tilde{\alpha}\alpha^{r}\beta^{q} + \tilde{\alpha}\tilde{\beta}\beta^{q+r} + \tilde{\beta}\tilde{\alpha}\alpha^{q+r}\right]\\
&= \frac{(a^2 - b)^p}{4b}\left[\tilde{\alpha}\tilde{\beta}\beta^{r}(\beta^q - \alpha^q) + \tilde{\beta}\tilde{\alpha}\alpha^{r}(\alpha^q - \beta^q)\right]\\
&= \frac{(a^2 - b)^p}{4b}\left[\left(M + 2(a^2 - b)\sqrt{b}N\right)\beta^{r}(\beta^q - \alpha^q) + \left(M - 2(a^2 - b)\sqrt{b}N\right)\alpha^{r}(\alpha^q - \beta^q)\right]\\
&= \frac{(a^2 - b)^p}{2\sqrt{b}} f_q\left[\left(-M - 2(a^2 - b)\sqrt{b}N\right)\beta^{r} + \left(M - 2(a^2 - b)\sqrt{b}N\right)\alpha^{r}\right]\\
&= (a^2 - b)^p f_q(Mf_r - (a^2 - b)Nl_r).
\end{aligned}$$

The second identity in theorem can be obtained similarly. □□

Taking $(a, b) \to \left(\frac{1}{2}, \frac{5}{4}\right), (1,2)$and $\left(\frac{1}{2}, \frac{9}{4}\right)$, Vajda's identitiesfor generalized Fibonacci and Lucas quaternions gives the following identities for the Fibonacci and Lucas, Pell and Pell – Lucas, Jacobsthal and Jacobsthal – Lucasquaternions

$$S_{p+q}S_{p+r} - S_pS_{p+q+r} = (-1)^p F_q(MF_r + NL_r), T_{p+q}T_{p+r} - T_pT_{p+q+r} = -5(-1)^p F_q(MF_r + NL_r).$$

$$S_{p+q}S_{p+r} - S_pS_{p+q+r} = (-1)^p P_q(MP_r + NQ_r), T_{p+q}T_{p+r} - T_pT_{p+q+r} = -8(-1)^p P_q(MP_r + NL_r)$$

and

$$S_{p+q}S_{p+r} - S_pS_{p+q+r} = (-2)^p J_q(MJ_r + Nj_r), T_{p+q}T_{p+r} - T_pT_{p+q+r} = -9(-2)^p J_q(MJ_r + Nj_r)$$

respectively.

If we take $r \to q$, Vajda's identitywith the identity $f_n l_n = f_{2n}$reduces to Catalan's identity which is given in the following theorem.

Theorem 3.2. Forany real numbers a, b, any integers p and q, we have

$$S_{p+q}S_{p-q} - \left[S_p\right]^2 = -(a^2 - b)^{p-q}\left(Mf_q^2 + (a^2 - b)Nf_{2q}\right)$$

and

$$T_{p+q}T_{p-q} - \left[T_p\right]^2 = 4b(a^2 - b)^{p-q}\left(Mf_q^2 + (a^2 - b)f_{2q}\right).$$

The most important result of the Catalan's identity is the Cassini's identity. Setting $q \to 1$ in Catalan's identity, we obtain Cassini's identity given in the following theorem.

Theorem 3.3. Forany real numbers a, b, any integer$p$, we have

$$S_{p+1}S_{p-1} - \left[S_p\right]^2 = -(a^2 - b)^{p-1}(M + 2a(a^2 - b)N)$$

and

$$T_{p+1}T_{p-1}-\left[T_p\right]^2=4b(a^2-b)^{p-1}(M+2a(a^2-b)N).$$

Another well – known identity is d'Ocagne's identity which can be found in the next theorem.

Theorem 3.4. Forany real numbers a, b, any integers p and q, we have

$$S_pS_{q+1}-S_{p+1}S_q=(a^2-b)^q\left[Mf_{p-q}+(a^2-b)Nl_{p-q}\right]$$

and

$$T_pT_{q+1}-T_{p+1}T_q=-4b(a^2-b)^q\left[Mf_{p-q}+(a^2-b)Nl_{p-q}\right].$$

Proof. From the Binet – like formula for the generalized Fibonacci quaternions in Theorem 2.3, we get

$$\begin{aligned}
&S_pS_{q+1}-S_{p+1}S_q\\
&=\frac{1}{4b}\left[\left(\tilde{\alpha}\alpha^p-\tilde{\beta}\beta^p\right)\left(\tilde{\alpha}\alpha^{q+1}-\tilde{\beta}\beta^{q+1}\right)-\left(\tilde{\alpha}\alpha^{p+1}-\tilde{\beta}\beta^{p+1}\right)\left(\tilde{\alpha}\alpha^q-\tilde{\beta}\beta^q\right)\right]\\
&=\frac{1}{4b}\left[-\tilde{\alpha}\tilde{\beta}\alpha^p\beta^{q+1}-\tilde{\beta}\tilde{\alpha}\alpha^{q+1}\beta^p+\tilde{\alpha}\tilde{\beta}\alpha^{p+1}\beta^q+\tilde{\beta}\tilde{\alpha}\alpha^q\beta^{p+1}\right]\\
&=\frac{1}{4b}\left[\tilde{\alpha}\tilde{\beta}\alpha^p\beta^q(\alpha-\beta)-\tilde{\beta}\tilde{\alpha}\alpha^q\beta^p(\alpha-\beta)\right]\\
&=\frac{(a^2-b)^q}{2\sqrt{b}}\left[\left(M+2(a^2-b)\sqrt{b}N\right)\alpha^p\beta^q-\left(M-2(a^2-b)\sqrt{b}N\right)\alpha^q\beta^p\right].
\end{aligned}$$

The last equation gives the first identity in the theorem. The second identity can be obtained similarly. □□

Again taking $(a,b)\to\left(\frac{1}{2},\frac{5}{4}\right)$, $(1,2)$and $\left(\frac{1}{2},\frac{9}{4}\right)$ we obtain the following d'Ocagne's identities for the Fibonacci and Lucas, Pell and Pell – Lucas, Jacobsthal and Jacobsthal – Lucas quaternions

$$S_pS_{q+1}-S_{p+1}S_q=(-1)^q\left[Mf_{p-q}-Nl_{p-q}\right], T_pT_{q+1}-T_{p+1}T_q=5(-1)^{q+1}\left[Mf_{p-q}-Nl_{p-q}\right].$$

$$S_pS_{q+1}-S_{p+1}S_q=(-1)^q\left[Mf_{p-q}-Nl_{p-q}\right], T_pT_{q+1}-T_{p+1}T_q=8(-1)^{q+1}\left[Mf_{p-q}-Nl_{p-q}\right]$$

and

$$S_pS_{q+1}-S_{p+1}S_q=(-2)^q\left[Mf_{p-q}-2Nl_{p-q}\right], T_pT_{q+1}-T_{p+1}T_q=9(-2)^{q+1}\left[Mf_{p-q}-2Nl_{p-q}\right]$$

respectively.

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