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Differential forms and cohomology of Perm algebras

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Abstract: *In this paper, we construct the module of derivations of a perm algebra, and we de ne the space of differential n-forms. Studying abelian extensions, we get the cohomology of perm algebras as a symmetric Hochschild cohomology. Finally, we relate these constructions with a notion of smoothness in a nonunitary context.*
Keywords: *perm algebra, derivation, differential form, Hochschild cohomology, smoothness.*

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I. Introduction

Discovered by F. Chapoton, "*perm algebras*" are governed by a quadratic operad whose dual operad is that of "*pre-Lie algebras*" factorizing the classical Liezation functor, $As \to Lie$, (see [2] and [6]). Parallelely, these algebras arise when one tries to define the notion of "*extended Leibniz algebra*". Explicitly, a perm algebra is a non-unital binary algebra subjected to the identities: $(ab)c = a(bc) = a(cb)$. Most examples of perm algebras come from the following observation. Given a right A-module R over a commutative and associative algebra A, equipped with an A-module morphism $f : R \to A$, then the product defined by $(r, r') \mapsto r \cdot f(r')$ endows R with a perm algebra structure.

This article begins by the fundamental examples of perm algebras, namely the construction of the free perm algebra over a $\mathbb{K}$-module V; here $\mathbb{K}$ the ground field over which we are working. We define the enveloping algebra of a perm algebra R and we define R-representations. This leads to the construction of the module of derivations and its representation. This yields a suitable definition of the space of differential forms as a "*perm differential graded algebra*". The study of abelian extensions of a perm algebra allows us to define the cohomology of a perm algebra out of the classical Hochschild cohomology. At the end, we axiomatize a notion of "*almost-freeness*" for perm algebras relatively to homomorphism lifting. A conjectural theorem asserts that we expect an Hochschild-Konstant-Rosenberg type isomorphism

$$H^*_{perm}(R) \cong \mathrm{Hom}_R(\Omega^*_{\mathbb{K}}(R), R),$$

when R is an almost-free perm algebra.

As pointed out by M. Markl, this work seems to be a "*Perm-geometry*" where Perm is the operad of perm algebras, as it is well-known for the operads Comm (usual geometry) and Ass (non-commutative geometry), see [10], Section 5. These gepmetries make sense since the three operads are defined in the category of sets.

II. Perm algebras

Definition 2.1. A perm algebra is a $\mathbb{K}$-module R equipped with a bilinear map $(-,-) : R \otimes R \longrightarrow R$ satisfying the (right commutative and associative) identities

$$(ab)c = a(bc) = a(cb) \qquad \forall\, a, b, c \in R. \tag{2.1}$$

Example 2.2.
1. Any associative and commutative algebra is a perm algebra. Conversely, a perm algebra with a unit-element is nothing but a unital associative and commutative algebra. But in sequel, we shall deal only with non-unital perm algebra.
2. If (A, d) is a commutative differential algebra [i.e. $d : A \longrightarrow A$ satisfies $d \circ d = 0$ and $d(ab) = ad(b) + bd(a)$] then the product given by $(a, b) \longmapsto ad(b)$, satisfies identities (2.1) of perm algebras.

More generally, a procedure for constructing perm algebras is the following:

Proposition 2.3. *Let A be an associative and commutative algebra. Let R be a right A-module with A-module morphism $f : R \longmapsto A$. Then the product given by $(r, r') \longmapsto rf(r')$ endows the $\mathbb{K}$-module R with a perm algebra structure that we denote by R_f.*

Proof. Indeed, for any r, r' and r'' in R, we have successively:

$$\begin{aligned}(rr')r'' &= (rf(r'))f(r'') = r(f(r')f(r''))\\ &= rf(r'f(r'')) = r(f(r'r'')) = r(r'r'')\\ &= rf(r''f(r')) = r(f(r''r')) = r(r''r').\end{aligned}$$

□

Assume now that V is a $\mathbb{K}$-module and let $S(V) := \oplus_{n\geq 0} S_n(V)$ be the "*free symmetric algebra*" on V. Then we have

Proposition 2.4. *The "free perm algebra" on V is the $\mathbb{K}$-module* $\text{Perm}(V) := V \otimes S(V)$ *equipped with the product given by*

$$(u \otimes a) \cdot (v \otimes b) := u \otimes (avb), \qquad u, v \in V,\ a, b \in S(V). \tag{2.2}$$

Proof. Ones easily checks that formula (2.2) defines a product satisfying identities (2.1). Now, given a $\mathbb{K}$-linear map $\varphi : V \to R$ where R is any perm algebra, then the $\mathbb{K}$-linear map $\phi : \text{Perm}(V) \longrightarrow R$ defined on generators by $\phi(v) = \varphi(v)$ and $\phi(v_0 \otimes v_1 \cdots \otimes v_n) := \varphi(v_0)\varphi(v_1)\cdots\varphi(v_n)$, $v, v_i \in V$ is a perm algebra morphism; the unique one such that $\phi \circ i = \varphi$ where $i : V(\cong V \otimes \mathbb{K}) \hookrightarrow V \otimes S(V)$ is the natural inclusion. This proves the universality of the perm algebra $\text{Perm}(V)$. □

Observe that the free perm algebra $\text{Perm}(V) = V \otimes S(V)$ is a special case of Proposition 2.2 where $A = S(V)$ and $R = V \otimes S(V)$ with the obvious right A-module structure, and $f : R \longrightarrow A$ is the *fusion* map $f(v) = v$ and

$$f(v_0 \otimes v_1 \cdots \otimes v_n) := f(v_0)f(v_1)\cdots f(v_n),\ v,\ v_i \in V.$$

Moreover it is shown that the category of perm algebras is governed by a quadratic operad whose dual operad is that of Pre-Lie algebras characterized by the identity:

$$(ab)c - a(bc) = (ac)b - a(cb).$$

These operads are both Koszul after [2] and [6].

III. Derivations Of A Perm Algebra

In this paragraph, R is denote a perm algebra, and we evolve an idea by D. Husemoller.

Definition 3.1. A representation of R (or R-representation) is a $\mathbb{K}$-module M equipped with two actions of R, say $R \times M \longrightarrow M$ and $M \times R \longrightarrow M$, satisfying the axioms

$$\begin{cases}(mb)c \overset{1}{=} m(bc) \overset{2}{=} m(cb)\\ (am)b \overset{3}{=} a(mb) \overset{4}{=} a(bm) \overset{5}{=} (ab)m,\end{cases} \tag{3.1}$$

for any $a, b, c \in R$ and $m \in M$.

In fact, relations 1, 3 and 5 say that M is a (non-unital) bimodule over the associative algebra R; relations 2 and 4 express the right-commutativity of M with respect to the actions of R.

For example, any perm algebra R is an R-representation. Other examples will come together with following notion of "*enveloping algebra*".

	$a'\otimes b'$	a'_l	a'_r
$a\otimes b$	$(aa')\otimes(bb')$	$(aa')\otimes b$	$a\otimes(ba')$
a_l	$(aa')\otimes b'$	$(aa')_l$	$a\otimes a'$
a_r	$a'\otimes(ab')$	$a'\otimes a$	$(aa')_r$

3.1 Enveloping algebra of perm algebra

Let R_l and R_r be two copies of R, seen as $R_l = R\otimes\mathbb{K}$ and $R_r = \mathbb{K}\otimes R$. We consider the $\mathbb{K}$-module

$$E(R) := R\otimes R \quad \oplus \quad R_l \quad \oplus R_r$$

equipped with associative product defined by the following table:

Proposition 3.2. *Any R-representation is equivalent to a left $E(R)$-module.*

Proof. Ones can easily see an R-representation via the operations

$$(a\otimes a')\cdot m := (am)a' = a(ma'),$$

$$a_l\cdot m := am, \qquad a'_r\cdot m := ma'.$$

Conversely, and left $E(R)$-module M is a R-representation by $a\cdot m := a_l\cdot m$ and $m\cdot a := a_r\cdot m$. □

3.2 Universal derivation of perm algebra

Definition 3.3. A "derivation" from R to a R-representation M is a map $\delta : R\to M$ such that

$$\delta(ab) = a\delta(b) + \delta(a)b, \qquad \forall\, a,b\in R.$$

For example, given a fixed element $u\in M$, the map

$$[u,-] : R\to R, \quad a\mapsto [u,a] = ua - au,$$

is a derivation from R to M, called "*inner derivation*".

We denote by $\mathrm{Der}(R,M)$ the set of all derivations from R to M; it a right R-module with the action:

$$(\delta x)(a) := \delta(a)x, \quad \forall\, a,x\in R.$$

Definition 3.4. A derivation $d : R\to M$ is said to be "universal" if for any derivation $\delta : R\to N$, there exists a unique R-linear map $\phi : M\to N$ such that $\delta = \phi\circ d$.

Consider the R-linear map

$$\mu : E(R)\to R, \quad \omega = \sum a\otimes b + a'_l + a''_r \mapsto \sum ab + a' + a''.$$

An element ω belongs to $\ker\mu$ iff $\sum ab + a' + a'' = 0$ i.e $a'' = -\sum ab - a'$. So one can write

$$\begin{aligned}\omega &= \sum a\otimes b + a'_l - (\sum ab + a')_r\\ &= \sum (a\otimes b - (ab)_r) + a'_l - a_r\\ &= \sum (a_l - a_r)\cdot b + (a'_l - a'_r).\end{aligned}$$

Therefore $I := \ker\mu$ is genetated by the symboles $da + (da')\cdot b$ where $da := a_l - a_r$. We denote by

$$\Omega^1_{\mathbb{K}}(R) := I/I^2.$$

It is a R-representation and we have

$$\begin{aligned}a(db) + (da)b &= a\,[b_l - b_r] + [a_l - a_r]\,b\\ &= [(ab)_l - a\otimes b] + [a\otimes b - (ab)_r]\\ &= d(ab).\end{aligned}$$

Thus, the map

$$d : R \to \Omega^1_{\mathbb{K}}(R), \quad a \mapsto da := a_l - a_r,$$

is a derivation. It is universal since, for any other derivation $\delta : R \to N$, the map $\phi : \Omega^1_{\mathbb{K}}(R) \to N$ given by

$$\phi(da) := \phi(a_l - a_r) = \delta(a) \quad \text{and} \quad \phi(da \cdot b) := \phi(da)b = \delta(a)b$$

is the unique one satisfying $\phi \circ d$. Now, it is obvious that we have a representability interpretation

$$\mathrm{Der}(R, M) \cong \mathrm{Hom}_R\left(\Omega^1_{\mathbb{K}}(R), M\right).$$

Proposition 3.5. *The $\mathbb{K}$-module $\Omega^1_{\mathbb{K}}(R)$ is in fact a right-symmetric bimodule over R that is*

$$\alpha \cdot a \cdot db = \alpha \cdot db \cdot a, \qquad \forall\, \alpha, a, b \in R.$$

Proof. Indeed we have

$$\begin{aligned} \alpha(adb) - dba) &= \alpha\left[a(b_l - b_r) - (b_l - b_r)\, a\right] \\ &= (\alpha ab)_l - \alpha a \otimes b - \alpha b \otimes a + \alpha \otimes ba \\ &= \alpha\left[(ba)_l - b \otimes a - a \otimes b + (ba)_r\right] \\ &= \alpha \cdot (b_l - b_r) \cdot (a_l - a_r) \in I^2. \end{aligned}$$

Thus, we have finished. □

3.3 Differential forms

Since the underlying $\mathbb{K}$-module of the free perm algebra over V is $V \otimes S(V)$, one can put

$$\Omega^{n+1}_{\mathbb{K}}(R) := \Omega^1_{\mathbb{K}}(R) \otimes \Lambda^n_R\left(\Omega^1_{\mathbb{K}}(R)\right).$$

Each n-forms space $\Omega^n_{\mathbb{K}}(R)$ is generated by the symboles

$$\omega := a_0 da_1 \otimes da_2 \wedge \cdots \wedge da_n + db_1 \otimes db_2 \wedge \cdots \wedge db_n, \quad a_i, b_j \in R,$$

and we have a map

$$d : \Omega^n_{\mathbb{K}}(R) \to \Omega^{n+1}_{\mathbb{K}}(R), \qquad \omega \mapsto da_0 \otimes da_1 \wedge da_2 \wedge \cdots \wedge da_n.$$

Here we have put $\Omega^0_{\mathbb{K}}(R) := R$ and $d(a) = a_l - a_r$, $a \in R$.

IV. Perm Differential Graded Algebras

Definition 4.1. A "perm differential graded algebra" ("PDG algebra" for short) is a graded $\mathbb{K}$-module $P_* := \oplus_{n\geq 0} P_n$ equipped with a binary product $(-,-) : P_r \otimes P_s \to P_{r+s}$ and a map $d : P_r \to P_{r+1}$ satisfying the identities

$$\begin{aligned} (ab)c &= a(bc) = (-1)^{st} a(cb) \\ d(ab) &= (da)b + (-1)^r a(db) \\ d \circ d &= 0 \end{aligned} \tag{4.1}$$

for any $a \in P_r$, $b \in P_s$ and $c \in P_t$.

Remark 4.2. Any commutative differential graded algebra (A, d) is a PDG algebra. Furthermore, given a commutative differential algebra (A, d), then the product given by $(a, b) \mapsto adb$ satisfies axioms (4.1) of PDG algebra with the same map d.

Recall that $\Omega^*_{\mathbb{K}}(R)$ is the sum $\oplus_{n\geq 0}\Omega^n_{\mathbb{K}}(R)$ where each $\Omega^n_{\mathbb{K}}(R)$ is generated by symboles

$$\omega := a_0 da_1 \otimes da_2 \wedge \cdots \wedge da_n + db_1 \otimes db_2 \wedge \cdots \wedge db_n$$

together with the map $d(\omega) := da_0 \otimes da_1 \wedge da_2 \wedge \cdots \wedge da_n$. Moreover, P_* be a PDG algebra for any perm morphism $\varphi : R \to P_0$, we put

$$\phi(\omega) := \varphi(a_0) d\varphi(a_1) \otimes d\varphi(a_2) \wedge \cdots \wedge d\varphi(a_n) + d\varphi(b_1) \otimes d\varphi(b_2) \wedge \cdots \wedge d\varphi(b_n).$$

Then get a well-defined morphism $\phi : \Omega^n_{\mathbb{K}}(R) \to P_*$ such that $\phi \circ d = \varphi$. So we have

Theorem 4.3. *The algebra $(\Omega^n_{\mathbb{K}}(R), d)$ is the universal PDG algebra over R.*

V. Cohomology of perm algebras

5.1 Abelian extensions of a perm algebra

Here we try to define the cohomology of perm algebras by studying abelian extensions of a given perm algebra R by an R-representation M, fixed for this paragraph.

An *abelian extension* of R by M is short exact sequence of perm algebras

$$0 \longrightarrow M \overset{j}{\to} E \overset{p}{\to} R \longrightarrow 0$$

where M is seen as an abelian perm algebra (i.e., $mm' = 0$, $\forall\, m, m' \in M$) Two such extensions (E) and (E') are said "*equivalent*" if there exists a perm algebra morphism $\phi : E \to E'$ such that

$$p' \circ \phi = p \quad \text{and} \quad \phi \circ j = j'$$

that is , making commutative the diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \xrightarrow{j} & E & \xrightarrow{p} & R & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle \phi} & & \| & & \\ 0 & \longrightarrow & M & \xrightarrow{j'} & E' & \xrightarrow{p'} & R & \longrightarrow & 0 \end{array}$$

By the "*Five Lemma*", such morphism ϕ is necessarily bijective. For instance, the direct sum $R \oplus M$ in abelian extension of R by M (with the obvious inclusion and projection). Moreover this latter is trivially "*split*". In fact, one says that an abelian extension (E) is "*split*" if there exists a $\mathbb{K}$-linear map $s : R \to E$ such that $p \circ s = id_R$. Moreover, if the "*section*" s is a perm algebra morphism, then the extension (E) is said to be "*strongly split*" or "*inessential*".

Any abelian extension (E) with a section s yields another R-representation structure on M by

$$r \cdot m := s(r)j(m) \qquad \text{and} \qquad m \cdot r := j(m)s(r),$$

the last products being taken in E (this has a sense since $M = \ker p$). This new structure is naturally independant of the choice of the section s. Indeed, if s' is a another section of p, then we have

$$p(s'(r) - s(r)) = ps'(r) - ps(r) = r - r = 0$$

that is, $s'(r) - s(r) \in \ker p = \operatorname{im} j$. And since M abelian, we have

$$s'(r)j(m) = s(r)j(m) \quad \text{and} \quad j(m)s'(r) = j(m)s(r), \qquad \forall\, m \in M, \forall\, r \in R.$$

From now on, we are interested only in set of equivalence classes of split abelian extensions such that the R-representation structure of M is the prescribed one.

Let us consider a $\mathbb{K}$-bilinear map $f : R \times R \to M$ and let $R \oplus_f M$ be the $\mathbb{K}$-module $R \oplus M$ equipped with the product given by

$$(r, m) \cdot (r', m') := (rr', rm' + mr' + f(r, r')). \tag{5.1}$$

Then it is straightforward to check that the product (5.1) is associative iff

$$f(r, r')r'' + f(rr', r'') = rf(r, r'') + f(r, r'r'') \qquad \forall\, r, r', r'' \in R \tag{5.2}$$

and is right-commutative iff

$$rf(r', r'') + f(r, r'r'') = rf(r'', r') + f(r, r''r') \qquad \forall\, r, r', r'' \in R. \tag{5.3}$$

In fact, relation (5.2) can be rewritten

$$rf(r', r'') - f(rr', r'') + f(r, r'r'') - f(r, r')r'') = 0$$

which is nothing but the 2-cocyclicity condition for the Hochschild coboundary of (unital) associative algebras, see [8]. On the other hand, the relation (5.3) expresses the fact that

$$r\left(f(r', r'') - f(r'', r')\right) = f(r, r'r'') - f(r, r''r')$$

that is, the double dual version (linearly speaking) to the right-commutativity. It can be seen as the conjonction of the following generalizations:

$$af(a_{\sigma(1)},\dots,a_{\sigma(n+1)}) = af(a_1,\dots,a_{n+1}), \quad \forall\,\sigma\in S_{n+1} \tag{5.4}$$

and

$$f(a_0,a_1,\dots,a_{j-1},ba_j,a_{j+1},\dots,a_n) = f(a_0,a_1,\dots,a_{j-1},a_jb,a_{j+1},\dots,a_n) \tag{5.5}$$

where $f: R^{\otimes n+1}\to M$ and $j=1,\dots,n$. We need some other relations

$$af(ba_0,a_1,\dots,\dots,a_n) = af(a_0b,a_1,\dots,a_n), \tag{5.6}$$

$$aa'_jf(a_0,a_1,\dots,a_n) = aa_jf(a_0,a_1,\dots,a_{j-1},a'_j,a_{j+1},\dots,a_n), \quad j=0,\dots,n, \tag{5.7}$$

$$af(a_0,\dots,a_{j-1},a_jb,a_{j+1},\dots,a_n) = af(a_0b,a_1,\dots,a_n), \quad j=1,\dots,n. \tag{5.8}$$

Proposition 5.1. *If $f: R^{\otimes n+1}\to M$ satisfies (5.4), (5.7), (5.5), (5.6) and (5.8), so does the Hochschild coboundary $\delta(f)$.*

We shall this proposition in following subsection.

5.2 Cohomology of a perm algebra

According to Proposition 5.1, we define the "*cohomology of perm algebra R with values in an R-representation M*" to be the cohomology of the complex $\mathbb{B}^*(R,M) = (\oplus_{n\geq 0}\mathbb{B}^n(R,M),\delta)$ where

$$\mathbb{B}^0(R,M) := M, \quad \mathbb{B}^1(R,M) := \{f: R\to M : af(bc) = af(cb), \forall\, a,b,c\in R\},$$

and

$$\mathbb{B}^n(R,M) := \left\{f\in \mathrm{Hom}(R^{\otimes n},M)\,\text{satisfying (5.4), (5.7), (5.5), (5.6) and (5.8)}\right\},$$

for $n\geq 2$, where the Hochschild coboundary δ acts usual by

$$\delta(f)(a_0,\dots,a_n) := a_0f(a_1,\dots,a_n) + \sum_{i=0}^{n-1}(-1)^{i+1}f(a_0,\dots,a_{i-1},a_i,a_{i+1},a_{i+2},\dots,a_n) + (-1)^{n+1}f(a_0,\dots,a_{n-1}).$$

We denote it by $H^*_{\mathbf{perm}}(R,M)$ or simply by $H^*_{\mathbf{perm}}(R) := H^*_{\mathbf{perm}}(R,R)$.
For $n=0$, $H^0_{\mathbf{perm}}(R,M)$ is the submodule of "*invariants*" of M that is,

$$H^0_{\mathbf{perm}}(R,M) = M^R := \{m\in M : rm = mr, \forall\, r\in R\}.$$

For $n=1$, a 1-cocycle is a $\mathbb{K}$-linear map $D: R\to M$ such that

$$D(ab) = D(a)b + aD(b), \quad \forall\, a,b\in R$$

that is, a "*derivation*" from R to M Observe that the additional relation $aD(b) = aD(cb)$ is fulfilled by any derivation thanks to right-commutativity in M. The set of all derivations from R to M is denoted by $\mathrm{Der}(R,M)$ and we have

$$H^1_{\mathbf{perm}}(R,M) = \mathrm{Der}(R,M)/\left\{\text{inner derivations}\right\}$$

where "{inner derivation}" is the subset of derivation of the form $ad_m(r) = [m,r] = mr - rm$, for a fixed element $m\in M$.
For $n=2$, we have the classical classification theorem

Theorem 5.2. *Let R be perm algebra and let M be an R-representation. Then, there is a canonical bijection*

$$H^2_{\mathbf{perm}}(R,M)\cong \mathbf{Ext}(R,M)$$

that is, the set of equivalence classes of split abelian extension of R by M.

Proof. By construction of the cohomology $H^*_{\mathbf{perm}}(R, M)$, any element $f \in \mathbb{B}^2(R, M)$ is a 2-cocycle iff the algebra $R \oplus_f M$ is a perm algebra for the product (μ). To be more precise, any 2-cocycle $f \in \mathbb{B}^2(R, M)$ determines a split abelian extension of R by M and any split abelian extension defines a 2-cocycle $f \in \mathbb{B}^2(R, M)$ given by

$$f(a, b) := s(ab) - s(a)s(b), \quad \forall\, a, b \in R \tag{5.9}$$

where s is a section splitting the extension. In fact, the cocycle f measures the obstruction for s to be an algebra morphism that is, the obstruction to this extension to be inessential.

Therefore, it is left to us to show that its equivalence class, characterized by the morphism ϕ, only involves coboundaries of $\mathbb{B}^1(R, M)$. To this end, let $\phi : R \oplus_f M \to E$ be an equivalent abelian extension to $R \oplus_f M$ that is, a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \xrightarrow{i} & R \oplus_f M & \xrightarrow{\pi} & R & \longrightarrow & 0 \\ & & \| & & \downarrow \phi & & \| & & \\ 0 & \longrightarrow & M & \xrightarrow{j} & E & \xrightarrow{p} & R & \longrightarrow & 0 \end{array}$$

Denoting by $\sigma : R \to R \oplus_f M$, $r \mapsto (r, 0)$, the trivial section of $\pi : R \oplus_f M \to R$, $(r, m) \mapsto r$, we have

$$p \circ (\phi \circ \sigma) = (p \circ \phi) \circ \sigma = \pi \circ \sigma = id_R.$$

Therefore the map $s := \phi \circ \sigma$ is also a section of p. It corresponds to s a 2-cocycle $f' \in \mathbb{B}^2(R, M)$ related to the extension (E). Since the map f' is given by the formula (5.9), one easily checks that the difference $f - f'$ is nothing but the coboundary $\delta(g)$ where $g : R \to M$ is the map $a \mapsto g(a) := \sigma(a) - s(a)$. In fact, a priori, the map g takes its values in the direct sum $R \oplus M$. But since the initial R-representation structure of M coincide with the structure induced by the sections, we have $\pi \circ g = p \circ g = 0$. So $\mathrm{im}(g) \subset M$; from whence the Theorem. □

VI. Conjectural smoothness

From now, we refer to the paper by Cuntz-Quillen ([3]). Let R be a perm algebra with an abelian ideal M and let $p : R \to A$ be a surjective morphism such that the sequence $0 \to M \xrightarrow{j} R \xrightarrow{p} A \to 0$ is exact. We are looking for a perm algebra morphism $l : A \to R$ such that $p \circ l = id_A$. Then we have an isomorphism $R \cong A \oplus M$ relative to which l becomes an inclusion of A. Therefore $H^2_{\mathbf{perm}}(A, M) \cong \mathbf{Ext}(A, M) = 0$ for all R-representation in M.

Definition 6.1. A perm algebra A is called "almost-free" when for any abelian extension R of A, there exists a lifting homomorphism $A \to R$.

We expect an interpretation of the cohomology theory $H^*_{\mathbf{perm}}(A, M)$ as $\mathbf{Ext}^*_{E(A)}(A, M)$, and the exact sequence

$$0 \longrightarrow \Omega^1_{\mathbb{K}}(A) \longrightarrow E(A) \longrightarrow A \longrightarrow 0$$

could yield the isomophism

$$H^{n+1}_{\mathbf{perm}}(A, M) \cong \mathbf{Ext}^{n+1}_{E(A)}(A, M) \cong \mathbf{Ext}^{n}_{E(A)}(\Omega^1_{\mathbb{K}}(A), M).$$

These suppose the construction of derived functors in a non-unitary context.

Theorem 6.2. *The following conditions are equivalent:*

1. *The perm algebra A is almost-free;*
2. *The A-bimodule $\Omega^1_{\mathbb{K}}(A)$ is projective;*
3. *The perm algebra A has cohomology dimension ≤ 1 with respect to the symmetric Hochschild cohomology $H^*_{\mathbf{perm}}(A)$.*

We can see that any classical smooth algebra A (commutative and unital) is almost-free. Also, any free perm algebra is almost-free.

Theorem 6.3. *The following properties are equivalent:*

1. *The perm algebra A has cohomological dimension zero with respect to symmetric Hochschild cohomology.*
2. *The A-module A is projective.*
3. *Any derivation $D : A \to M$ is inner.*

Definition 6.4. We call "separable" a perm algebra A when it has the above equivalent properties.

VII. Proof of the Proposition

We separate the proof into two lemmas which help understanding the reason of these five relations over which the cohomology of perm algebra built.

Lemma 7.1. *If $f : R^{\oplus(n+1)} \longrightarrow M$ satisfies (5.6) and (5.5), so does the Hochschild coboundary $\delta(f)$.*

Proof. If $j = n+1$, then by assumption and perm axioms we have

$$\begin{aligned}\delta(f)(a_0,\dots,a_n,ba_{n+1}) &= a_0 f(a_1,\dots,a_n,ba_{n+1})\\ &+\sum_{i=0}^{n-1}(-1)^{i+1}f(a_0,\dots,a_{i-1},a_ia_{i+1},a_{i+2},\dots,a_n,ba_{n+1})\\ &+(-1)^{n+1}f(a_0,\dots,a_{n-1},a_n(ba_{n+1}))+(-1)^n f(a_0,\dots,a_n)(ba_{n+1})\\ &:= a_0 f(a_1,\dots,a_{n+1}b)\\ &+\sum_{i=0}^{n-1}(-1)^{i+1}f(a_0,\dots,a_{i-1},a_ia_{i+1},a_{i+2},\dots,a_n,a_{n+1}b)\\ &+(-1)^{n+1}f(a_0,\dots,a_{n-1},a_n(a_{n+1}b))+(-1)^n f(a_0,\dots,a_n)(a_{n+1}b)\\ &=\delta(f)(a_0,\dots,a_n,a_{n+1}b).\end{aligned}$$

If $2 \le j \le n$, then we have

$$\delta(f)(a_0,\dots,a_{j-1},ba_j,a_{j+1},\dots,a_{n+1}) = \delta(f)(a_0,\dots,a_{j-1},a_jb,a_{j+1},\dots,a_{n+1})$$

by assumption and because

$$\begin{aligned}&f(a_0,\dots,a_{j-1},(ba_j)a_{j+1},a_{j+2},\dots,a_{n+1})\\ &=f(a_0,\dots,a_{j-1},b(a_{j+1}a_j),a_{j+2},\dots,a_{n+1}) &&\text{by perm axioms}\\ &=f(a_0,\dots,a_{j-1},(a_{j+1}a_j)b,a_{j+2},\dots,a_{n+1}) &&\text{by assumption}\\ &=f(a_0,\dots,a_{j-1},a_{j+1}(a_jb),a_{j+2},\dots,a_{n+1}) &&\text{by associativity}\\ &=f(a_0,\dots,a_{j-1},(a_jb)a_{j+1},a_{j+2},\dots,a_{n+1}) &&\text{by assumption}\end{aligned}$$

If $j = 1$, then we have

$$\delta(f)(a_0,ba_1,a_2,\dots,a_{n+1}) = \delta(f)(a_0,a_1b,a_2,\dots,a_{n+1})$$

by assumption and because

$$\begin{aligned}f(a_0,(ba_1)a_2,a_3,\dots,a_{n+1}) &= f(a_0,b(a_2a_1),a_3,\dots,a_{n+1}) &&\text{by perm axioms}\\ &= f(a_0,(a_2a_1)b,a_3,\dots,a_{n+1}) &&\text{by assumption}\\ &= f(a_0,a_2(a_1b),a_3,\dots,a_{n+1}) &&\text{by associativity}\\ &= f(a_0,(a_1b)a_2,a_3,\dots,a_{n+1}) &&\text{by assumption}\end{aligned}$$

One the other hand, we have

$$\begin{aligned}a\delta(f)(ba_0,a_1,\dots,a_{n+1}) =&a(ba_0)f(a_1,\dots,a_{n+1}) - af((ba_0)a_1,a_2,\dots,a_{n+1})\\ &+\sum_{i=1}^{n}(-1)^{i+1}af(ba_0,a_1,\dots,a_{i-1},a_ia_{i+1},a_{i+2},\dots,a_{n+1})\\ &+(-1)^n af(ba_0,a_1,\dots,a_n)a_{n+1}\\ =&a(a_0b)f(a_1,\dots,a_{n+1}) - af((a_0b)a_1,a_2,\dots,a_{n+1})\\ &+\sum_{i=1}^{n}(-1)^{i+1}af(a_0b,a_1,\dots,a_{i-1},a_ia_{i+1},a_{i+2},\dots,a_{n+1})\\ &+(-1)^n af(a_0b,a_1,\dots,a_n)a_{n+1}\\ =&a\delta(f)(a_0b,a_1,\dots,a_{n+1}),\end{aligned}$$

by assumption and because we have

$$\begin{aligned} af((ba_0,)a_1,a_2,\ldots,a_{n+1}) &= af(b(a_1a_0),a_2,\ldots,a_{n+1}) && \text{by perm axioms}\\ &= af((a_1a_0)b,a_2,\ldots,a_{n+1}) && \text{by assumption}\\ &= af(a_1(a_0b),a_2,\ldots,a_{n+1}) && \text{by associativity}\\ &= af((a_0b)a_1,a_2,\ldots,a_{n+1}) && \text{by assumption.} \end{aligned}$$

This achieves proving the Lemma. □

Lemma 7.2. *If $f: R^{\oplus(n+1)} \longrightarrow M$ satisfies (5.4), (5.7), (5.6) and (5.8), so does the Hochschild coboundary $\delta(f)$.*

Proof. Let us first show the stability of relation (5.8) with respect to the coboundary δ. If $2 \le j \le n$, then observe first that we have

$$\begin{aligned} aa_0f((a_1b,a_2,\ldots,a_{n+1}) &= a(a_1b)f(a_0,a_2,\ldots,a_{n+1}) && \text{by (5.7)}\\ &= (ab)a_1f(a_0,a_2,\ldots,a_{n+1}) && \text{by perm axioms}\\ &= (ab)a_0f(a_1,a_2,\ldots,a_{n+1}) && \text{by (5.7)}\\ &= a(a_0b)f(a_1,a_2,\ldots,a_{n+1}) && \text{by perm axioms.} \end{aligned}$$

Therefore, by assumption and perm axioms, we have

$$\begin{aligned} &a\delta(f)(a_0,\ldots,a_{j-1},a_jb,a_{j+1}\ldots,a_{n+1})\\ =&aa_0f(a_1,\ldots,a_{j-1},a_jb,a_{j+1},\ldots,a_{n+1}) - af(a_0a_1,a_2,\ldots,a_{j-1},a_jb,a_{j+1},\ldots,a_{n+1})\\ &+\sum_{i=1}^{j-2}(-1)^{i+1}af(a_0,\ldots,a_{i-1},a_ia_{i+1},a_{i+2},\ldots,a_{j-1},a_jb,a_{j+1},\ldots,a_{n+1})\\ &+(-1)^jaf(a_0,\ldots,a_{j-2},a_{j-1}(a_jb),a_{j+1},\ldots,a_{n+1})\\ &+(-1)^{j+1}af(a_0,\ldots,a_{j-1}(a_jb),a_{j+1},a_{j+2},\ldots,a_{n+1})\\ &+\sum_{i=j+1}^{n}(-1)^{i+1}af(a_0,\ldots,a_{i-1},a_jb,a_{j+1},\ldots,a_{i-1},a_ia_{i+1},a_{i+2},\ldots,a_{n+1})\\ &+(-1)^naf(a_0,\ldots,a_{j-1},a_jb,a_{j+1},\ldots,a_n)a_{n+1}\\ =&aa_0f(a_1b,a_2,\ldots,a_{n+1}) - af((a_0a_1)b,a_2,\ldots,a_{n+1})\\ &+\sum_{i=1}^{j-2}(-1)^{i+1}af(a_0b,a_1,\ldots,a_{i-1},a_ia_{i+1},a_{i+2},\ldots,a_{n+1})\\ &+(-1)^jaf(a_0b,a_1,\ldots,a_{j-2},a_{j-1}a_j,a_{j+1},\ldots,a_{n+1})\\ &+(-1)^{j+1}af(a_0b,a_1,\ldots,a_{j-1},a_ja_{j+1},a_{j+2},\ldots,a_{n+1})\\ &+\sum_{i=j+1}^{n}(-1)^{i+1}af(a_0b,a_1,\ldots,a_{i-1},a_ia_{i+1},a_{i+2},\ldots,a_{n+1})\\ &+(-1)^naf(a_0b,a_1,\ldots,a_n)a_{n+1}\\ =&a\delta(f)(a_0b,a_1,\ldots,a_{n+1}). \end{aligned}$$

The cases $j=1$ and $j=n+1$ are obvious thanks to relation (5.4) for $\sigma=(1,2)$ and $\sigma=(2,n+1)$ respectivily.

Observe that the proof of the stability of relation (5.6) is independent of the other relations, so we can avoid it in this Lemma. Thanks to relation (5.4), it is sufficient to have relation (5.7) done for $j=0$:

$$\begin{aligned} aa_0'\delta(f)(a_0,\ldots,a_{n+1}) =&(aa_0')a_0f(a_1,\ldots,a_{n+1}) - aa_0'f(a_0a_1,a_2,\ldots,a_{n+1})\\ &+\sum_{i=1}^{n}(-1)^{i+1}aa_0'f(a_0,a_1,\ldots,a_{i-1},a_ia_{i+1},a_{i+2},\ldots,a_{n+1})\\ &+(-1)^naa_0'f(a_0,a_1,\ldots,a_n)a_{n+1}\\ =&(aa_0)a_0'f(a_1,\ldots,a_{n+1}) - aa_0f((a_0'a_1,a_2,\ldots,a_{n+1})\\ &+\sum_{i=1}^{n}(-1)^{i+1}aa_0f(a_0',a_1,\ldots,a_{i-1},a_ia_{i+1},a_{i+2},\ldots,a_{n+1})\\ &+(-1)^naa_0f(a_0',a_1,\ldots,a_n)a_{n+1} \end{aligned}$$

by assumption, perm axioms and because

$$\begin{aligned} aa_0'f(a_0a_1,a_2,\ldots,a_{n+1}) &= a(a_0a_1)f(a_0',a_2,\ldots,a_{n+1}) && \text{by assumption}\\ &= (aa_1)a_0f(a_0',a_2,\ldots,a_{n+1}) && \text{by perm axioms}\\ &= (aa_1)a_0'f(a_0,a_2,\ldots,a_{n+1}) && \text{by assumption}\\ &= a(a_0'a_1)f(a_0,a_2,\ldots,a_{n+1}) && \text{by perm axioms}\\ &= aa_0f(a_0'a_1,a_2,\ldots,a_{n+1}) && \text{by assumption.} \end{aligned}$$

Therefore $\delta(f)$ satisfies relation (5.7).

Since the symmetric group S_{n+2} is generated by the transpositions, it is enough to establish the stability of relation (5.4) for the transpositions $\tau=(1,n+2)$ and $\tau=(j,j+1)$ where $1\le j\le n+1$. For $\tau=(1,2)$, we have

$$\begin{aligned} a\delta(f)(a_{\tau(1)},\ldots,a_{\tau(n+2)}) =& a\delta(f)(a_2,a_1,a_3,\ldots,a_{n+2})\\ & aa_2f(a_1,a_3\ldots,a_{n+2})-af(a_2a_1,a_3,\ldots,a_{n+2})\\ & +af(a_2,a_1a_3,a_4,\ldots,a_{n+2})+(-1)^{n+1}af(a_2,a_1,a_3,\ldots,a_{n+1})a_{n+2}\\ & +\sum_{i=3}^{n+1}(-1)^{i+1}af(a_2,a_1,a_3,\ldots,a_{i-1},a_ia_{i+1},a_{i+2},\ldots,a_{n+2})\\ =& aa_1f(a_2,a_3,\ldots,a_{n+2})-af(a_1a_2,a_3,\ldots,a_{n+2})\\ & +af(a_1,a_2a_3,a_4,\ldots,a_{n+2})+(-1)^{n+1}af(a_1,a_2,\ldots,a_{n+1})a_{n+2}\\ & +\sum_{i=3}^{n+1}(-1)^{i+1}af(a_1,a_2,\ldots,a_{i-1},a_ia_{i+1},a_{i+2},\ldots,a_{n+2}) \quad \text{by assumption}\\ =& a\delta(f)(a_1,\ldots,a_{n+2}). \end{aligned}$$

For $2\le j\le n$ and $\tau=(j,j+1)$, we have

$$a\delta(f)(a_{\tau(1)},\ldots,a_{\tau(n+2)})=a\delta(f)(a_1,\ldots,a_{n+2})$$

by assumption and because we have

$$\begin{aligned} & af(a_1,\ldots,a_{j-2},a_{j-1}a_{j+1},a_j,a_{j+2},\ldots,a_{n+2})\\ &= af(a_j,a_2,\ldots,a_{j-2},a_{j-1}a_{j+1},a_1,a_{j+2},\ldots,a_{n+2}) && \text{by (5.4) with } \sigma=(1,j-1)\\ &= af(a_ja_{j+1},a_2,\ldots,a_{j-1},a_1,a_{j+2},\ldots,a_{n+2}) && \text{by (5.8)}\\ &= af(a_{j+1}a_j,a_2,\ldots,a_{j-1},a_1,a_{j+2},\ldots,a_{n+2}) && \text{by (5.6)}\\ &= af(a_{j+1},a_2,\ldots,a_{j-2},a_{j-1}a_j,a_1,a_{j+2},\ldots,a_{n+2}) && \text{by (5.8)}\\ &= af(a_1,\ldots,a_{j-2},a_{j-1}a_j,a_{j+1},a_{j+2},\ldots,a_{n+2}) && \text{by (5.4) with } \sigma=(1,j), \end{aligned}$$

$$\begin{aligned} & af(a_1,\ldots,a_{j-1},a_{j+1}a_{j+1}a_j,a_{j+2},\ldots,a_{n+2})\\ &= af(a_{j+1}a_j,a_2,\ldots,a_{j-1},a_1,a_{j+2},\ldots,a_{n+2}) && \text{by (5.4) with } \sigma=(1,j)\\ &= af(a_ja_{j+1},a_2,\ldots,a_{j-1},a_1,a_{j+2},\ldots,a_{n+2}) && \text{by (5.6)}\\ &= af(a_1,\ldots,a_{j-1},a_ja_{j+1},a_{j+2},\ldots,a_{n+2}) && \text{by (5.4) with } \sigma=(1,j), \end{aligned}$$

and

$$\begin{aligned} & af(a_1,\ldots,a_{j-1},a_{j+1},a_ja_{j+2},a_{j+3},\ldots,a_{n+2})\\ &= af(a_ja_{j+2},a_2,\ldots,a_{j+1},a_1,a_{j+3},\ldots,a_{n+2}) && \text{by (5.4) with } \sigma=(1,j+1)\\ &= af(a_j,a_2,\ldots,a_{j-1},a_{j+1}a_{j+2},a_1,a_{j+3},\ldots,a_{n+2}) && \text{by (5.8)}\\ &= af(a_1,\ldots,a_j,a_{j+1}a_{j+2},a_{j+3},\ldots,a_{n+2}) && \text{by (5.4) with } \sigma=(1,j+1), \end{aligned}$$

For $\tau=(n+1,n+2)$, we have

$$a\delta(f)(a_{\tau(1)},\ldots,a_{\tau(n+2)}) = a\delta(f)(a_1,\ldots,a_{n+2})$$

by assumption and because we have

$$\begin{aligned}
af(a_1,\ldots,a_{n-1},a_na_{n+2},\ldots,a_{n+2}) &= af(a_na_{n+2},a_2,\ldots,a_{n-1},a_1,a_{n+1}) && \text{by (5.4) with } \sigma=(1,n)\\
&= af(a_{n+2}a_n,a_2,\ldots,a_{n-1},a_1,a_{n+1}) && \text{by (5.6)}\\
&= af(a_{n+2},a_2,\ldots,a_{n-1},a_1,a_{n+1}a_n) && \text{by (5.8)}\\
&= af(a_{n+1}a_n,a_2,\ldots,a_{n-1},a_1,a_{n+2}) && \text{by (5.4) with } \sigma=(1,n)\\
&= af(a_na_{n+1},a_2,\ldots,a_{n-1},a_1,a_{n+2}) && \text{by (5.6)}\\
&= af(a_1,a_2,\ldots,a_{n-1},a_na_{n+1},a_{n+2}) && \text{by (5.4) with } \sigma=(1,n),
\end{aligned}$$

$$\begin{aligned}
af(a_1,\ldots,a_n,a_{n+2}a_{n+1}) &= af(a_{n+2}a_{n+1},a_2,\ldots,a_n,a_1) && \text{by (5.4) with } \sigma=(1,n+1)\\
&= af(a_{n+1}a_{n+2},a_2,\ldots,a_n,a_1) && \text{by (5.6)}\\
&= af(a_1,a_2,\ldots,a_n,a_{n+1}a_{n+2}) && \text{by (5.4) with } \sigma=(1,n),
\end{aligned}$$

and

$$\begin{aligned}
af(a_1,\ldots,a_n,a_{n+2})a_{n+1} &= aa_{n+1}f(a_{n+2},a_2,\ldots,a_n,a_1) && \text{by perm axioms and (5.4)}\\
&= aa_{n+2}f(a_{n+1},a_2,\ldots,a_n,a_1) && \text{by (5.7)}\\
&= af(a_1,a_2,\ldots,a_n,a_{n+1})a_{n+2} && \text{by perm axioms and (5.4).}
\end{aligned}$$

For $\tau=(1,n+2)$, we have

$$\begin{aligned}
a\delta(f)(a_{\tau(1)},\ldots,a_{\tau(n+2)}) =& a\delta(f)(a_{n+2},a_2,\ldots,a_{n+1},a_1)\\
=& aa_{n+2}f(a_2,\ldots,a_{n+1},a_1) - af(a_{n+2}a_2,a_3,\ldots,a_{n+1},a_1)\\
&+\sum_{i=2}^{n}(-1)^{i+1}af(a_{n+2},a_2,\ldots,a_{i-1},a_ia_{i+1},a_{i+2},\ldots,a_{n+1},a_1)\\
&+(-1)^n af(a_{n+2},a_2,\ldots,a_n,a_{n+1}a_1) + (-1)^{n+1}af(a_{n+2},a_2,\ldots,a_{n+1})a_1.
\end{aligned}$$

But we have

$$\begin{aligned}
aa_{n+2}f(a_2,\ldots,a_{n+1},a_1) &= aa_2f(a_{n+2},a_3,\ldots,a_{n+1},a_1) && \text{by (5.7)}\\
&= aa_2f(a_1,a_3,\ldots,a_{n+1},a_{n+2}) && \text{by assumption}\\
&= aa_1f(a_2,a_3,\ldots,a_{n+1},a_{n+2}) && \text{by (5.7),}
\end{aligned}$$

and

$$\begin{aligned}
af(a_{n+2}a_2,a_3,\ldots,a_{n+1},a_1) &= af(a_1a_2,a_3,\ldots,a_{n+1},a_{n+2}) && \text{by (5.6) and (5.8)}\\
&= af(a_{n+2},a_2,\ldots,a_n,a_{n+1}a_1) && \text{by (5.6)}\\
&= af(a_1,\ldots,a_n,a_{n+1}a_{n+2}) && \text{by (5.8)}\\
&= af(a_{n+2},a_2,\ldots,a_{n+1})a_1\\
&= aa_1f(a_2,a_3,\ldots,a_{n+2})
\end{aligned}$$

by assumption and perm axioms. Therefore, for $\tau=(1,n+2)$, we have

$$a\delta(f)(a_{\tau(1)},\ldots,a_{\tau(n+2)}) = a\delta(f)(a_1,\ldots,a_{n+2}),$$

which ends the proof of the Lemma, and that of Proposition 5.1. □

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