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Some Fascinating Observations on Harshad Numbers and Amicable Pairs

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Abstract: *The purpose of this article is to present some fascinating observations on Harshad numbers and amicable pairs with examples. A Harshad number (or Niven Number) is an integer that is divisible by the sum of its digits. The Ramanujan number 1729 is a Harshad number. Two numbers are amicable (or friendly) if each number is equal to the sum of the proper divisors of the other. Proper divisors of a number are divisors excluding the number itself. Then, the pair of these two numbers is called amicable pair. Harshad amicable pairs, happy amicable pairs, quasi-amicable pairs and some other types of amicable pairs were also mentioned in this article with examples.*

Key words: *Harshad numbers, Amicable pairs, Ramanujan number 1729.*

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I. Introduction

Harshad numbers were defined by Dattathreya Ramachandra Kaprekar (1905 – 1986). D.R.Kaprekar was an Indian recreational mathematician. He worked as a school teacher in Devlali, a town very close to Nasik in Maharashtra, from 1930 to1962. ‘Harshad’ means ‘joy giving’. Harshad numbers are also known as Niven numbers in the name of the Mathematician Ivan M. Niven.

Definition: A *Harshad number* is an integer that is divisible by the sum of its digits.

For example, 24 is a Harshad number. The sum of its digits 2 and 4 is 6, which is the divisor of 24. Few other examples of Harshad numbers are 21, 108, 153, 378, 2620, 2924, 6804, etc. There are 50 Harshad numbers in the first 200 integers, excluding the trivial 1-digit numbers. 6804 is said to be a multiple Harshad number (MHN), because $6804 \div 18 = 378$, $378 \div 18 = 21$, $21 \div 3 = 7$ & $7 \div 7 = 1$. Another example of multiple Harshad number is 378, since $378 \div 18 = 21, 21 \div 3 = 7$ & $7 \div 7 = 1$.

Definition of Amicable Numbers: Two numbers are called amicable (or friendly), if the first number is equal to the sum of the proper divisors of the second, and if the second number is equal to the sum of the proper divisors of the first. Then, the pair of these two numbers is called *amicable pair*.

A pair of numbers (m, n) is amicable, when

$$\sigma(m) = n \text{ and } \sigma(n) = m \qquad (1)$$

where, $\sigma(m)$ and $\sigma(n)$ denote the sum the proper divisors of m & n respectively. Proper divisors means all the divisors excluding the number itself. The numbers 220 and 284 are the smallest pair of amicable numbers known to Greek mathematician Pythagoras (570 BC to 475 BC). Let $m = 220$ & $n = 284$. The proper divisors of number 220 are 1,2,4,5,10,11,20,22,44,55 and 110. The proper divisors of number 284 are 1, 2,4,71 and 142. Using (1), we can write

$$\sigma(m) = \sigma(220) = 1 + 2 + 4 + 5 + 10 + 11 + 20 + 22 + 44 + 55 + 110 = 284 = n$$
$$\sigma(n) = \sigma(284) = 1 + 2 + 4 + 71 + 142 = 220 = m$$

In 1636 Pierre de Fermat discovered another pair of amicable numbers (17296, 18416). This pair is called Fermat’s pair. In 1638 Descartes gave the third pair of amicable numbers (9363584, 9437056), known as Descartes’ pair. These results were actually rediscoveries of numbers known to Arab mathematicians. In 18th century, Swiss mathematician Leonhard Euler (1707-1783) gave a list of 64 amicable pairs (two of which later shown to be not amicable). In 1866, B.N.I. Paganini, a 16 years Italian, showed that the pair (1184, 1210) is amicable. It was the second lowest pair and had been completely overlooked until then. As on 28 September, 2007, about 11994387 pairs of amicable numbers are known. In January 2018, there are 1221159849 known amicable pairs. The following is the list of first fifteen amicable pairs.

(220,284), (1184,1210), (2620,2924), (5020,5564), (6232,6368), (10744,10856), (12285,14595), (17296, 18416), (63020,76084), (66928,66992), (67095,71145), (69615,87633), (79750,88730), (100485,124155) and (122265,139815).

II. Ramanujan Number 1729 as Harshad Number

Srinivasa Ramanujan (22 December 1887 - 26 April 1920) was an Indian mathematician who lived during the British rule in India. The number 1729 is the number of a taxi which the British mathematician Prof.G.H. Hardy had hired when he came to see ailing Ramanujan in hospital. When Prof. Hardy remarked to Ramanujan that the taxi had a dull number 1729, Ramanujan immediately responded that it is a very interesting smallest number which can be expressed as the sum of cubes of two numbers in two different ways. Since then 1729 is called Ramanujan number. The number 1729 is also known as Hardy – Ramanujan number.

$$1729 = 12^3 + 1^3 = 10^3 + 9^3 \quad (2)$$
$$12^3 + 1^3 = 1728 + 1 = 1729$$
$$10^3 + 9^3 = 1000 + 729 = 1729$$

Harshad numbers are those numbers which are divisible by the sum of their digits. The number 1729 is divisible by 1+ 7 +2 + 9 = 19 ($i.e. 1729 \div 19 = 91$). So, the Ramanujan number 1729 is a Harshad number.

III. Rules for Finding Amicable Pairs

The following are two rules for finding amicable pairs. But, both these two rules do not help us to find all possible amicable pairs.

(a) *Thabit Ibn Qurra's Rule*:

In 9th century, the brilliant Islamic mathematician Thabit Ibn Qurra (826 - 900) gave the following rule for finding amicable pairs. This rule is also known as Thabit Ibn Qurra theorem.
If the prime numbers h, t & s are defined by the following relations

$$h = 3 \times 2^n - 1 \quad (3)$$
$$t = 3 \times 2^{n-1} - 1 \quad (4)$$
$$s = 9 \times 2^{2n-1} - 1 \quad (for\, n > 1) \quad (5)$$

then, the pair of amicable numbers is $(2^n ht, 2^n s)$.

Examples:

(i) For $n = 2$, we obtain $h = 3 \times 2^2 - 1 = 11, t = 3 \times 2 - 1 = 5$ and $s = 9 \times 2^3 - 1 = 71$. The numbers 5,11 and 71 are prime numbers. Then, the amicable pair is (220, 284), as $2^n ht = 2^2 \times 11 \times 5 = 220$ and $2^n s = 2^2 \times 71 = 284$.

(ii) When $n = 4$, we get $h = 3 \times 2^4 - 1 = 47, t = 3 \times 2^3 - 1 = 23$ and $s = 9 \times 2^7 - 1 = 1151$. The numbers 23, 47 and 1151 are prime numbers and the amicable pair is (17296,18416), because $2^n ht = 2^4 \times 47 \times 23 = 17296$ and $2^n s = 2^4 \times 1151 = 18416$.

(iii) For $n = 7$, we obtain $h = 3 \times 2^7 - 1 = 383, t = 3 \times 2^6 - 1 = 191$ and $s = 9 \times 2^{13} - 1 = 73727$. The numbers 383,191and 73727 are prime numbers and the amicable pair is (9363584, 9437056), since $2^n ht = 2^7 \times 383 \times 191 = 9363584$ and $2^n s = 2^7 \times 73727 = 9437056$.

This rule will not give amicable numbers for all values of $n > 1$. For $n = 3$, we get $s = 287$, which is not a prime number. Hence, this rule will not give amicable pair for $n = 3$. Similarly, when $n = 5$, we obtain $h = 95$, which is not a prime number. So, we cannot find amicable pair for $n = 5$ using this rule. Also, this rule will not give amicable pair for $n = 6$, because in this case $t = 95$, which is not a prime number. Thus, this 9th century rule gives amicable pairs for certain values of n.

(b) *Euler's Rule*: It states that if

$$h = (2^{n-m} + 1)2^n - 1 \quad (6)$$
$$t = (2^{n-m} + 1)2^m - 1 \quad (7)$$
$$s = (2^{n-m} + 1)^2 2^{n+m} - 1 \quad (n > m > 0) \quad (8)$$

where n and m are integers and h, t and s are prime numbers, then $2^n ht$ and $2^n s$ are a pair of amicable numbers. Tabit Ibn Qurra theorem is obtained from Euler's rule by substituting $m = n - 1$. Euler's rule gives additional amicable pairs for $(m, n) = (1,8)\&(29,40)$. For these two cases, $m \neq (n - 1)$.

IV. Distribution of Amicable Pairs

There are 1427 pairs of amicable numbers below 10^{10} and these were compiled and published by H.J.J. te. Riele [1]. The distribution of amicable pairs up to 10^{19} is given in Table-1.

Table-1: Distribution of amicable pairs below 10^{19}

X	No. of amicable pairs whose smaller number is less than X
$!0^3$	1
10^4	5
10^5	13
10^6	42
10^7	108
10^8	236
10^9	586
10^{10}	1427
10^{11}	3340
10^{12}	7642
10^{13}	17519
10^{14}	39374
10^{15}	87102
10^{16}	190775
10^{17}	415523
10^{18}	901312
10^{19}	1947667

It was observed in [3] that if the number of amicable pairs whose smaller number is less than 10^n is $y,$ then the number of amicable pairs whose smaller number is less than 10^{n+1} shall be at least $2y$, where n is any positive integer greater than 2. For example, the number of amicable pairs below 10^7 is 108. So, the number amicable pairs below 10^8 shall be at least $2 \times 108 = 216$, which is less than the actual number 236. This approximately proves the said observation.

V. Some Other Observations on Amicable Pairs

(a) There is no amicable pair in which one of the two numbers is a square.

(b) There are some amicable pairs (m, n) in which the sum of the digits of m and n is equal. There are 427 such amicable pairs in the first 5000 amicable pairs. Three examples are given below.

- Considering the pair (69615,87633),

Sum of the digits of 69615 = 6+9+6+1+5 = 27
Sum of the digits of 87633 = 8+7+6+3+3 = 27

- For the pair (100485, 124155),

Sum of the digits of 100485 = 1+0+0+4+8+5 = 18
Sum of the digits of 124155 = 1+2+4+1+5+5 = 18

- For the pair (1358595,1486845),

Sum of the digits of 135895 = 1+3+5+8+5+9+5 = 36
Sum of the digits of 1486845 = 1+4+8+6+8+4+5 = 36

(c) For an amicable pair (m, n), where $m < n$, the minimum value of m/n is 0.6979 and maximum value of m/n is 0.999858. The minimum value corresponds to the amicable pair (938304290, 1344480478) and the maximum value corresponds to the amicable pair (4000783984, 4001351168).

(d) It is not known that there exist an infinite number of amicable pairs. However, if this is the case, the list of amicable pairs shows that

$$\lim_{m\to\infty}\left(\frac{m}{n}\right) \to 1 \quad (9)$$

(e) Using (1), we have

$$\sigma(n) = m \text{ or } \sigma\big(\sigma(m)\big) = n \ \Rightarrow \sigma^{(2)}(m) = m$$

After making successive substitutions for m & n, we can write

$$\sigma^{(2)}(m) = \sigma^{(4)}(m) = \sigma^{(6)}(m) = \cdots = m \quad (10)$$

Similarly, we can find out the following relation.

$$\sigma^{(2)}(n) = \sigma^{(4)}(n) = \sigma^{(6)}(n) = \cdots = n \quad (11)$$

(f) Let in the amicable pair (m, n), both the numbers m and n end in the same digit p. The following Table-2 shows the smallest amicable pairs for $p = 0$ to 9.

Table-2: Smallest amicable pairs

p	Smallest amicable pair ending in the same digit p
0	79750, 88730
1	1558818261, 1596205611
2	106930732, 1142071892
3	664747083, 673747893
4	196724, 202444
5	12285, 14595
6	17296, 18416
7	290142314847, 292821792417
8	469028, 486178
9	68606181189, 70516785339

(g) In 1986 te Riele found 37 pairs of amicable numbers having the same pair sum 1296000. That is,

$$m + n = \sigma(m) + \sigma(n) = 129600 \quad (12)$$

Two examples of such amicable pairs are (609928, 686072) and (643336, 652664).

(h) In November 1997, six amicable pairs having the same pair sum 4169926656000 were discovered. i.e., for these pairs,

$$m + n = \sigma(m) + \sigma(n) = 4169926656000 \quad (13)$$

This sextuple is (1953433861918, 2216492794082), (1968039941816, 2201886714184), (1981957651366, 2187969004634),(1993501042130, 2176425613870),(2046897812505, 2123028843495) and (2068113162038, 2101813493962).

(i) The earliest known odd amicable pairs, such as (12285, 14595), (67095, 71145), (100485,124155) etc., were all divisible by 3.

(j) Two complex numbers also form an amicable pair, known as *Gaussian amicable pair*. Two examples are $(4232 - 8280\,i, 8008 + 3960\,i)$ and $(-1105 + 1020\,i,\ -2639 - 1228\,i)$. That is,

$$\sigma(8008 + 3960\,i) = 4232 - 8280\,i \quad (14)$$

$$\sigma(4232 - 8280\,i) = 8008 + 3960\,i \quad (15)$$

and

$$\sigma(-2639 - 1228\,i) = -1105 + 1020\,i \quad (16)$$

$$\sigma(-1105 + 1020\,i) = -2639 - 1228\,i \quad (17)$$

(k) The numbers of the amicable pairs can be expressed by prime factorization. The following is the prime factorization of the numbers of first ten amicable pairs.

$220 = 2^2 \times 5 \times 11$ $\qquad$ $284 = 2^2 \times 71$

$1184 = 2^5 \times 37$ $\qquad$ $1210 = 2 \times 5 \times 11^2$

$2620 = 2^2 \times 5 \times 131$ $\qquad$ $2924 = 2^2 \times 17 \times 43$

$5020 = 2^2 \times 5 \times 251$ $\qquad$ $5564 = 2^2 \times 13 \times 107$

$6232 = 2^3 \times 19 \times 41$ $\qquad$ $6368 = 2^5 \times 199$

$10744 = 2^3 \times 17 \times 79$ $\qquad$ $10856 = 2^3 \times 23 \times 59$

$12285 = 3^3 \times 5 \times 7 \times 13$ $\qquad$ $14595 = 3 \times 5 \times 7 \times 139$

$17296 = 2^4 \times 23 \times 47$ $\qquad$ $18416 = 2^4 \times 1151$

$63020 = 2^2 \times 5 \times 23 \times 137$ $\qquad$ $76084 = 2^2 \times 23 \times 827$

$66928 = 2^4 \times 47 \times 89$ $\qquad$ $66992 = 2^4 \times 53 \times 79$

(l) In 2005, Jobling has discovered a largest amicable pair with each number having 24073 digits. This pair is obtained by defining

$$a = 2 \times 5 \times 11 \quad (18)$$

$$S = 37 \times 173 \times 409 \times 461 \times 2136109 \times 2578171801921099 \times 68340174428454377539 \quad (19)$$

$$p = 925616938247297545037380170207625962997960453645121 \quad (20)$$

$$q = 210958430218054117679018601985059107680988707437025081922673599999 \quad (21)$$

$$q_1 = (p + q)p^{235} - 1 \quad (22)$$

$$q_2 = (p - S)p^{235} - 1 \quad (23)$$

Then, p, q, q_1 and q_2 are all primes. The numbers of the amicable pair are given by

$$m = a\,S\,p^{235}\,q_1 \quad (24)$$

$$n = a\,S\,p^{235}\,q_2 \quad (25)$$

Each number of the above amicable pair has 24073 decimal digits. Some information about the biggest amicable pairs is given in Table-3.

Table-3: Information about biggest amicable pairs.

Digits in each number of amicable pair	Date of discovery	Reference
4829	4 October 1997	M. Garcia, 1997
8684	6 January 2003	Jobling and Walker, 2003
16563	12 May 2004	Walker et.al., 2004
17326	12 May 2004	Walker et.al., 2004
24073	10 March 2005	Jobling, 2005

VI. Harshad Amicable Pairs

We know that Harshad numbers are those numbers which are divisible by the sum of their digits.

Definition: Harshad amicable pair (m, n) is an amicable pair in which both the numbers m and n are Harshad numbers.

Examples:

- In the amicable pair (2620, 2924), the number 2620 is divisible by 2+6+2+0=10 (i.e. 2620 ÷ 10 =262) and 2924 is divisible by 2+9+2+4=17 (i.e., 2924 ÷ 17 = 172). So, both these numbers are Harshad numbers and this amicable pair is Harshad amicable pair.
- Consider the amicable pair (10634085, 14084763). The number 10634085 is divisible by 1+0+6+3+4+0+8+5=27 (i.e. 10634085 ÷27 = 393855) and 14084763 is divisible by 1+4+0+8+4+7+6+3 = 33 (i.e. 14084763 ÷ 33 = 426811). Hence, this amicable pair is Harshad amicable pair.
- Some other examples of Harshad amicable pairs are (10634085, 14084763), (23389695, 25132545) and (34256222, 35997346).

There are 192 Harshad amicable pairs in the first 5000 amicable pairs.

VII. Happy Amicable Pairs

Definition: If you iterate the process of summing the squares of the decimal digits of a number and if the process terminates in 1, then the original number is called a *happy number.* Happy amicable pair is an amicable pair (m, n) such that both m and n are happy numbers.

Consider the amicable pair (10572550, 10854650). The following are the iterations of these two numbers.

Iteration of 10572550:

$10572550 \rightarrow 1^2+0^2+5^2+7^2+2^2+5^2+5^2+0^2=129$

$129 \rightarrow 1^2+2^2+9^2=86$

$86 \rightarrow 8^2+6^2= 100$

$100 \rightarrow 1^2+0^2+0^2 =1$

As this iteration is ending in 1, the number 10572550 is a happy number.

Iteration of 10854650:

$10854650 \rightarrow 1^2+0^2+8^2+5^2+4^2+6^2+5^2+0^2 = 167$

$167 \rightarrow 1^2+6^2+7^2= 86$

$86 \rightarrow 8^2+6^2 = 100$

$100 \rightarrow 1^2+0^2+0^2 =1$

Since this iteration is terminating in 1, the number 10854650 is a happy number. Thus, the amicable pair (10572550, 10854650) is a happy amicable pair. Some other examples of happy amicable pairs are (32685250, 34538270), (35361326, 40117714) and (35390008, 39259592).

VIII. Quasi-Amicable Pairs

The theory of quasi-amicable pairs is mentioned in reference [4].

Definition: Two positive integers m and n are called quasi-amicable numbers, if

$$m = \sigma(n) - n - 1 \quad (26)$$

and

$$n = \sigma(m) - m - 1 \quad (27)$$

Where, $\sigma(N)$ denote the sum of the positive divisors of the integer $N (N > 1)$.The pair of the quasi-amicable numbers (m, n) is called quasi-amicable pair.

Eqns. (26) and (27) can also be written as

$$\sigma(m) = \sigma(n) = m + n + 1 \quad (28)$$

The following is the list of first ten quasi-amicable pairs.

(48, 75),(140, 195), (1050, 1925), (1575, 1638), (2024, 2295), (5775, 6128), (8892, 16587), (9504, 20735), (62744, 75495) and (186615, 206504).

It is noticed that each quasi-amicable pair is of opposite parity. That is, one number of the pair is even and the other one is odd. Now, let us prove that the quasi-amicable pairs (48, 75) and (140, 195) satisfy the expressions (26) & (27).

(i) For the pair (48,75), $m = 48$ and $n = 75$.

Divisors of 48 are 1, 2, 3, 4,6,8,12,16,24 and 48.

Divisors of 75 are 1, 3, 5, 15, 25 and 75.

$\sigma(m) = 1 + 2 + 3 + 4 + 6 + 8 + 12 + 16 + 24 + 48 = 124$

$\sigma(n) = 1 + 3 + 5 + 15 + 25 + 75 = 124$

Since, $m = 48 = 124 - 75 - 1 = \sigma(n) - n - 1$ and $n = 75 = 124 - 48 - 1 = \sigma(m) - m - 1$,the relations (26) & (27) are proved.

(ii) For the pair (140,195), $m = 140$ and $n = 195$.

Divisors of 140 are 1, 2, 4, 5, 7,10,14,20,28,35,70 and 140.

Divisors of 195 are 1, 3, 5, 13, 15, 39,65 and 195.

$\sigma(m) = 1 + 2 + 4 + 5 + 7 + 10 + 14 + 20 + 28 + 35 + 70 + 140 = 336$

$\sigma(n) = 1 + 3 + 5 + 13 + 15 + 39 + 65 + 195 = 336$

As, $m = 140 = 336 - 195 - 1 = \sigma(n) - n - 1$ and $n = 195 = 336 - 140 - 1 = \sigma(m) - m - 1$, the relations (26) & (27) are proved.

IX. Some Other Types of Amicable Pairs

Some other type of amicable pairs are given in reference [6]. The original definition of amicable pairs given in introduction section is not applicable for this type of amicable pairs. The following are the different ways of forming three other types of amicable pairs.

(1) In the first type of amicable pairs, each number of the pair is separated into two parts. Then, each number of the pair is expressed as the sum of the squares of two parts of the other number. For example, let us consider two numbers 3869 and 6205. The number 3869 is divided into two parts 38 and 69. The other number 6205 is divided into two parts 62 and 05. It was noticed that

$$3869 = 62^2 + 05^2 = 3844 + 25 \text{ and } 6205 = 38^2 + 69^2 = 1444 + 4761$$

Hence, the given two numbers (3869, 6205) form an amicable pair. Some other examples of this type of amicable pairs are (5965, 7706), (43354, 127165), (137461, 231290), (1261485, 2221101), (1528804, 2981200) and (7414650, 22171581). These amicable pairs satisfy the following relations.

$$5965 = 77^2 + 06^2 \qquad 7706 = 59^2 + 65^2$$
$$43354 = 127^2 + 165^2 \qquad 127165 = 43^2 + 354^2$$
$$137461 = 231^2 + 290^2 \qquad 231290 = 137^2 + 461^2$$
$$1261485 = 222^2 + 1101^2 \qquad 2221101 = 126^2 + 1485^2$$
$$1528804 = 298^2 + 1200^2 \qquad 2981200 = 1528^2 + 804^2$$
$$7414650 = 2217^2 + 1581^2 \qquad 22171581 = 741^2 + 4650^2$$

(2) In the second type of amicable pairs, each number of the pair is divided into two parts. Then, each number of the pair is expressed as the sum of the squares of two parts of the other number with a negative coefficient before the square of one part. Few examples of this type of amicable pairs are (16, 35), (28, 68), (240, 1604),(369, 1215), (1155, 3146), (2205, 42021), (2880, 5616), (21384, 147015), (42471, 220077), (60912, 37100277) etc. These amicable pairs satisfy the following relations.

$$16 = -3^2 + 5^2 \qquad 35 = -1^2 + 6^2$$
$$28 = -6^2 + 8^2 \qquad 68 = 2^2 + 8^2$$
$$240 = 16^2 - 04^2 \qquad 1604 = 2^2 + 40^2$$
$$369 = 12^2 + 15^2 \qquad 1215 = 36^2 - 9^2$$
$$1155 = -31^2 + 46^2 \qquad 3146 = 11^2 + 55^2$$
$$2205 = 42^2 + 021^2 \qquad 42021 = -2^2 + 205^2$$
$$2880 = 56^2 - 16^2 \qquad 5616 = -28^2 + 80^2$$
$$21384 = 147^2 - 015^2 \qquad 147015 = -21^2 + 384^2$$
$$42471 = 220^2 - 077^2 \qquad 220077 = -42^2 + 471^2$$
$$60912 = 371^2 - 00277^2 \qquad 37100277 = 6091^2 - 2^2$$

(3) In the third type of amicable pairs, each number of the pair is formed by two digits 0 and 1 only. Here, each number of the pair is divided into two parts. Then, each number of the pair is expressed as the sum of the powers of two parts of the other number. In this case, same number forms few amicable pairs with other numbers. For example, the number 10001 forms three amicable pairs with three other numbers as mentioned below. Few examples of this type of amicable pairs are (1001, 1000001), (10001, 100000001), (10001, 1000001), (10001, 1000000000001), (100001, 10000000001) etc. These amicable pairs satisfy the following relations.

$$1001 = 10^3 + 00001^3 \qquad 1000001 = 100^3 + 1^3$$
$$10001 = 10^4 + 0000001^4 \qquad 100000001 = 100^4 + 01^4$$
$$10001 = 100^2 + 0001^2 \qquad 1000001 = 1000^2 + 1^2$$
$$10001 = 10^4 + 00000000001^4 \qquad 1000000000001 = 1000^4 + 1^4$$
$$100001 = 10^5 + 000000001^5 \qquad 10000000001 = 100^5 + 001^5$$

X. Conclusion

Some interesting facts about Harshad numbers and amicable pairs were discussed in this article with examples. Harshad amicable pairs, happy amicable pairs, quasi-amicable pairs [4] and three other types of amicable pairs [6] with examples were also mentioned in this article.

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