IOSR Journal of Mathematics (IOSR-JM)
e-ISSN: 2278-5728, p-ISSN: 2319-765X. Volume 16, Issue 3 Ser. V (May – June 2020), PP 09-13
www.iosrjournals.org

The Fermat Classes And The Proof Of Beal Conjecture

Mohamed Sghiar
UNIVERSITÉ DE BOURGOGNE DIJON
Faculté des sciences Mirande Département de mathématiques et informatiques
9 Av Alain Savary 21078 DIJON CEDEX , France

Abstract : *If after 374 years the famous theorem of Fermat-Wiles was demonstrated in 150 pages by A. Wiles [4], The purpose of this article is to give a proofs both for the Fermat last theorem and the Beal conjecture by using the Fermat class concept.*
Résumé : *Si après 374 ans le célèbre théorème de Fermat-Wiles a été démontré en 150 pages par A. Wiles [4] , le but de cet article est de donner des démonstrations à la fois du dernier théorème de Fermat et de la conjecture de Beal en utilisant la notion des classes de Fermat.*
Keywords : *Fermat, Fermat-Wiles theorem, Fermat's great theorem, Beal conjecture, Diophantine equation.*
The Subject Classification Codes : *11D41 - 11G05 - 11G07 - 26B15 - 26B20 - 28A10 - 28A75 - 26A09 - 26A42*

Date of Submission: 09-06-2020 Date of Acceptance: 26-06-2020

I. Introduction, notations and definitions

Set out by Pierre de Fermat [3], it was not until more than three centuries ago that Fermat's great theorem was published, validated and established by the British mathematician Andrew Wiles [4] in 1995.

In mathematics, and more precisely in number theory, the last theorem of Fermat [3], or Fermat's great theorem, or since his Fermat-Wiles theorem demonstration [4], is as follows: There are no non-zero integers a, b, and c such that : $a^n+b^n=c^n$, as soon as n is an integer strictly greater than 2 ".

The Beal conjecture [2] is the following conjecture in number theory : If $a^x+b^y=c^z$ where a, b, c, x, y and z are positive integers with x, y, z > 2, then a, b, and c have a common prime factor. Equivalently, There are no solutions to the above equation in positive integers a, b, c, x, y, z with a, b and c being pairwise coprime and all of x, y, z being greater than 2.

The purpose of this article is to give a proofs both for the Fermat last theorem and the Beal conjecture by using the **Fermat class concept**.

Let be two equations $x^a+y^b-z^c=0$ with $(x,y,z)\in E^3$ and $(a,b,c)\in F^3$, and $X^A+Y^B-Z^c$ with $(X,Y,Z)\in E'^3$ and $(A,B,C)\in F'^3$. In the following $F=F'=\mathbb{N}$ and E and E' are subsets of $\mathbb{R}$.

The two equations $x^a+y^b-z^c=0$ with $(x,y,z)\in E^3$ and $(a,b,c)\in F^3$; and $X^A+Y^B-Z^c$ with $(X,Y,Z)\in E'^3$ and $(A,B,C)\in F'^3$, are said to be equivalent if the resolution of one is reduced to the resolution of the other.

In the following, an equation $x^a+y^b-z^c=0$ with $(x,y,z)\in E^3$ and $(a,b,c)\in F^3$ is considered at **close equivalence**, and we say $x^a+y^b-z^c=0$ is a **Fermat class**.

Example: The equation $x^{15}+y^{15}-z^{15}=0$ with $(x,y,z)\in\mathbb{Q}^3$ is equivalent to the equation $X^3+Y^3-Z^3=0$ with $(X,Y,Z)\in\mathbb{Q}_5^3$ and where $\mathbb{Q}_5=\{q^5,q\in\mathbb{Q}\}$.

II. The proof of Fermat's last theorem

Theorem 1 : There are no non-zero a, b, and c three elements of E with $E\subset\mathbb{Q}$ such that: $a^n+b^n=c^n$, with n an integer strictly greater than 2.

Lemma 1 : If $n\in\mathbb{N}$, a, b and c are a non-zero three elements of $\mathbb{R}$ with $a^n+b^n=c^n$ then:

$$\int_0^b x^{n-1} - \left(\frac{c-a}{b}x+a\right)^{n-1} \frac{c-a}{b} \, dx = 0$$

Proof :

$$a^n + b^n = c^n \Leftrightarrow \int_0^a n x^{n-1} dx + \int_0^b n x^{n-1} dx = \int_0^c n x^{n-1} dx$$

But as :

$$\int_0^c n x^{n-1} dx = \int_0^a n x^{n-1} dx + \int_a^c n x^{n-1} dx$$

So :

$$\int_0^b n x^{n-1} dx = \int_a^c n x^{n-1} dx$$

And as by changing variables we have :

$$\int_a^c n x^{n-1} dx = \int_0^b n \left(\frac{c-a}{b} y + a\right)^{n-1} \frac{c-a}{b} \, dy$$

Then :

$$\int_0^b x^{n-1} dx = \int_0^b \left(\frac{c-a}{b} y + a\right)^{n-1} \frac{c-a}{b} \, dy$$

It results:

$$\int_0^b x^{n-1} - \left(\frac{c-a}{b}x+a\right)^{n-1} \frac{c-a}{b} \, dx = 0$$

Corollary 1. If $N, \ n \in \mathbb{N}^*$, a, b and c are a non-zero three elements of $\mathbb{R}$ and $a^n + b^n = c^n$ then :

$$\int_0^{\frac{b}{N}} x^{n-1} - \left(\frac{c-a}{b}x + \frac{a}{N}\right)^{n-1} \frac{c-a}{b} \, dx = 0$$

Proof : It results from **lemma 1** by replacing a, b and c respectively by $\frac{a}{N}$, $\frac{b}{N}$ and $\frac{c}{N}$.

Lemma 2 : If $a^n + b^n = c^n$ is a **Fermat class**, where $n \in \mathbb{N}$, a, b and c are a non-zero three elements of $E \subset \mathbb{R}^+$ with $n > 2$ and $0 < a \leqslant b \leqslant c$. Then we can choose a not zero integer N, a, b , c and n in the class, such that : $f(x) = x^{n-1} - \left(\frac{c-a}{b}x + \frac{a}{N}\right)^{n-1} \frac{c-a}{b} \leqslant 0 \quad \forall \ x \in \left[0, \frac{b}{N}\right]$.

Proof :

$$\frac{df}{dx} = (n-1)x^{n-2} - (n-1)\left(\frac{c-a}{b}x + \frac{a}{N}\right)^{n-2} \left(\frac{c-a}{b}\right)^2$$

The function f decreases in the right of 0 in $[0, \epsilon[$.

But :

$$f(x)=0 \Leftrightarrow x=\frac{\frac{a}{N}\left(\frac{c-a}{b}\right)^{\frac{1}{n-1}}}{1-\left(\frac{c-a}{b}\right)^{1+\frac{1}{n-1}}} \quad .$$

So :

$$f(x)\leqslant 0 \quad \forall\ x \quad \text{such that} \quad 0\leqslant x\leqslant\frac{\frac{a}{N}\left(\frac{c-a}{b}\right)^{\frac{1}{n-1}}}{1-\left(\frac{c-a}{b}\right)^{1+\frac{1}{n-1}}}$$

And :

$$f(x)\geqslant 0 \quad \forall\ x \quad \text{such that} \quad x\geqslant\frac{\frac{a}{N}\left(\frac{c-a}{b}\right)^{\frac{1}{n-1}}}{1-\left(\frac{c-a}{b}\right)^{1+\frac{1}{n-1}}}$$

Otherwise if $\mu\in]0,1]$ we have :

$$\frac{b(1-\mu)}{N} < \frac{\frac{a}{N}\left(\frac{c-a}{b}\right)^{\frac{1}{n-1}}}{1-\left(\frac{c-a}{b}\right)^{1+\frac{1}{n-1}}} \Leftrightarrow 1-\mu\left(1-\left(\frac{c-a}{b}\right)^{1+\frac{1}{n-1}}\right)<\left(\frac{c-a}{b}\right)^{1+\frac{1}{n-1}}+\frac{a}{b}\left(\frac{c-a}{b}\right)^{\frac{1}{n-1}}$$

By replacing a, b and c respectively with $a'=a^{\frac{1}{k}}$, $b'=b^{\frac{1}{k}}$, and $c'=c^{\frac{1}{k}}$ with $k\in\mathbb{N}^{*}$, we get another Fermat class : $a'^{kn}+b'^{kn}=c'^{kn}$.
we will show that for this class and for k large enough,

$$1-\mu\left(1-\left(\frac{c'-a'}{b'}\right)^{1+\frac{1}{kn-1}}\right)\leqslant\left(\frac{c'-a'}{b'}\right)^{1+\frac{1}{kn-1}}+\frac{a'}{b'}\left(\frac{c'-a'}{b'}\right)^{\frac{1}{kn-1}} \quad :$$

First we have : $1-\mu\left(1-\left(\frac{c'-a'}{b'}\right)^{1+\frac{1}{kn-1}}\right)\leqslant 1-\mu\left(1-\left(\frac{c'-a'}{b'}\right)^{2}\right)<1$.

And as :

$$\left(\frac{c'-a'}{b'}\right)^{1+\frac{1}{kn-1}}+\frac{a'}{b'}\left(\frac{c'-a'}{b'}\right)^{\frac{1}{kn-1}} = \frac{c'}{b'}\left(\frac{c'-a'}{b'}\right)^{\frac{1}{kn-1}}\geqslant\left(\frac{c^{\frac{1}{k}}-a^{\frac{1}{k}}}{b^{\frac{1}{k}}}\right)^{\frac{1}{kn-1}}\geqslant\left(1-\left(\frac{a}{b}\right)^{\frac{1}{k}}\right)^{\frac{1}{kn-1}} \quad ,$$

By using the **logarithm**, we have $\lim_{k\to+\infty}\left(1-\left(\frac{a}{b}\right)^{\frac{1}{k}}\right)^{\frac{1}{kn-1}}=\lim_{k\to+\infty}\left(1-\left(\frac{a}{b}\right)^{\frac{1}{k}}\right)^{\frac{1}{kn}}=1$ because :

$\left(1-\left(\frac{a}{b}\right)^{\frac{1}{k}}\right)^{\frac{1}{k}}=e^{\frac{1}{k}\ln\left(1-\left(\frac{a}{b}\right)^{\frac{1}{k}}\right)}$, by posing : $1-\left(\frac{a}{b}\right)^{\frac{1}{k}}=e^{-l}$, we will have : $\frac{1}{k}=\frac{\ln\left(1-e^{-l}\right)}{\ln\left(\frac{a}{b}\right)}$ and

$\lim_{k\to+\infty}\left(1-\left(\frac{a}{b}\right)^{\frac{1}{k}}\right)^{\frac{1}{k}}=\lim_{l\to+\infty} e^{\left(\frac{\ln\left(1-e^{-l}\right)}{\ln\left(\frac{a}{b}\right)}\right)(-l)}=1$ which shows the result.

So, for k large enough, we deduce that there exists a class $a'^{kn}+b'^{kn}=c'^{kn}$ such that :
$f(x)=x^{kn-1}-\left(\frac{c'-a'}{b'}x+\frac{a'}{N}\right)^{kn-1}\frac{c'-a'}{b'} < 0 \ \forall\ x\in\left[0,\frac{b'(1-\mu)}{N}\right]$ **independently of N**.

Let's fix an N and put $S=\sup\left\{f(x), x\in\left[0,\frac{b'(1-\mu)}{N}\right]\right\}$ By replacing a', b' and c' respectively with $a'(1-\mu)=a''$, $b'(1-\mu)=b''$, and $c'(1-\mu)=c''$, we get another **Fermat class** : $a''^{kn}+b''^{kn}=c''^{kn}$ And we will have for M large enough :

$f(x)=x^{kn-1}-\left(\frac{c''-a''}{b''}x+\frac{a''}{M}\right)^{kn-1}\frac{c''-a''}{b''} < 0 \ \forall\ x\in\left[0,\frac{b''}{M}\right]$ Because
$f(x)\leqslant S+\operatorname{Sup}\left\{\frac{P(x,\mu)}{M},\ x\in\left[0,\frac{b'(1-\mu)}{N}\right]\right\}$ where P is a polynomial, and as for M large enough
$\left|\operatorname{Sup}\left\{\frac{P(x,\mu)}{M}, x\in\left[0,\frac{b'(1-\mu)}{N}\right]\right\}\right|<|S|$ and $S<0$, the result is deduced.

Proof of Theorem:

If $a^n+b^n=c^n$ is a **Fermat class**, where $n\in\mathbb{N}$, a, b and c are a non-zero three elements of $E\subset\mathbb{R}^+$ with $n>2$ and $0<a\leqslant b\leqslant c$. Then, by the **lemma 2**, for **well chosen** N, and a, b, c, and n in the class , we will have :

$$f(x)=x^{n-1}-\left(\frac{c-a}{b}x+\frac{a}{N}\right)^{n-1}\frac{c-a}{b}\leqslant 0 \ \forall\ x\in\left[0,\frac{b}{N}\right]$$

And by using the **corollary 1**, we have : $\int_0^{\frac{b}{N}} x^{n-1}-\left(\frac{c-a}{b}x+\frac{a}{N}\right)^{n-1}\frac{c-a}{b}\,dx=0$

So : $x^{n-1}-\left(\frac{c-a}{b}x+\frac{a}{N}\right)^{n-1}\frac{c-a}{b}=0 \ \forall\ x\in\left[0,\frac{b}{N}\right]$

And therefore $\frac{c-a}{b}=1$ because f(x) is a null polynomial as it have more than n zeros. So $c=a+b$ and $a^n+b^n\neq c^n$ which is absurd .

III. The proof of Beal conjecture

Corollaire 2 [Beal conjecture] : If $a^x+b^y=c^z$ where a, b, c, x, y and z are positive integers with x, y, z > 2, then a, b, and c have a common prime factor.
Equivalently, there are no solutions to the above equation in positive integers a, b, c, x, y, z with a, b and c being pairwise coprime and all of x, y, z being greater than 2.

Proof:

Let $a^x+b^y=c^z$. If a, b and c are not pairwise coprime, then by posing $a=ka'$, $b=kb'$, and $c=kc'$. Let $a'=u'^{yz},\ b'=v'^{xz},\ c'=w'^{xy}$ and $k=u^{yz},\ k=v^{xz},\ k=w^{xy}$.
As $a^x+b^y=c^z$, we deduce that $(uu')^{xyz}+(vv')^{xyz}=(ww')^{xyz}$.
So :

$$k^x u'^{xyz}+k^y v'^{xyz}=k^z w'^{xyz}$$

This equation does not look like the one studied in the first theorem.
But if a, b and c are pairwise coprime, we have $k=1$ and $u=v=w=1$ and we will have to solve the equation :

$$u'^{xyz}+v'^{xyz}=w'^{xyz}$$

The equation $u'^{xyz}+v'^{xyz}=w'^{xyz}$ have a solution if and only if at least one of the equations : $(u'^{xy})^z+(v'^{xy})^z=(w'^{xy})^z$, $(u'^{xz})^y+(v'^{xz})^y=(w'^{xz})^y$, $(u'^{yz})^x+(v'^{yz})^x=(w'^{yz})^x$ have a solution .

So by the proof given in the proof of the first Theorem we must have : $z\leqslant 2$ or $y\leqslant 2$, or $x\leqslant 2$.
We therefore conclude that if $a^x+b^y=c^z$ where a, b, c, x, y, and z are positive integers with x, y, z > 2, then a, b, and c have a common prime factor.

IV. Important notes

1- If a, b, and c are not pairwise coprime, someone, by applying the proof given in the corollary like this : $a=u^{yz}, b=v^{xz}, c=w^{xy}$ we will have $u^{xyz}+v^{xyz}=w^{xyz}$, and could say that all the x, y and z are always smaller than 2. What is false: $7^3+7^4=14^3$
The reason is sipmle: it is the common factor k which could increase the power, for example if $k=c'^r$ in the proof, then $c^z=(kc')^z=c'^{(r+1)z}$. You can take the example : $2^r+2^r=2^{r+1}$ where $k=2^r$.
2- These techniques do not say that the equation $a^n+b^n=c^n$ where $a,b,c\in]0,+\infty[$ has no solution since in the proof the Fermat class $X^2+Y^2=Z^2$ can have a sloution(We take $a=X^{\frac{2}{n}}$ $b=Y^{\frac{2}{n}}$ and $C=Z^{\frac{2}{n}}$).
3- In [1] I proved the abc conjecture which implies only that the equation $a^x+b^y=c^z$ has only a finite number of solutions with a, b, c, x, y, z a positive integers and a, b and c being pairwise coprime and all of x, y, z being greater than 2.

V. Conclusion

The Fermat class used in this article have allowed to prove both the Fermat' last theorem and the Beal' conjecture and have shown that the Beal conjecture is only a corollary of the Fermat' last theorem.

Acknowledgments.
I want to thank everyone who contributed to the success of the result of this article.

References

[1]. M. Sghiar, La preuve de la conjecture abc, IOSR Journal of Mathematics, 14.4:22-26, 2018.
[2]. https://en.wikipedia.org/wiki/Beal conjecture
[3]. https://en.wikipedia.org/wiki/Fermat last theorem
[4]. Andrew Wiles, Modular elliptic curves and fermat's last theorem, Annal of mathematics, 10:443--551, september-december 1995.

Mohamed Sghiar. "The Fermat Classes And The Proof Of Beal Conjecture." *IOSR Journal of Mathematics (IOSR-JM)*, 16(3), (2020): pp. 09-13.