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The Legendre-WENO Method

Mrumun C. Soomiyol[1], Terhemen Aboiyar[2] and MsuurM. Shior[3]
[1,3]*Department of Mathematics and Computer Science, Benue State University Makurdi, Nigeria.*
[2]*Department of Mathematics/Statistics/Computer Science University of Agriculture Makurdi, Benue State, Nigeria.*

Abstract: *In this work, the Legendre Polynomial is used as a basis function in the reconstruction step of the Weighted Essentially Non Oscillatory (WENO) method for the numerical solution of two dimensional scalar conservation laws. The WENO method is a high order high accurate finite volume method that has been designed for problems that have piecewise smooth solutions but still contain some discontinuities. The most common basis used in the reconstruction step of the finite volume methods are polynomial basis. The minimum error property of the Legendre polynomial makes it a good choice for the basis function to be used. In this work, we used the two-dimensional Legendre polynomial as a basis function and the resulting method is called the Legendre-WENO (L-WENO) method. The reconstruction procedure for the L-WENO method is clearly highlighted. Ten cells were used for a cubic reconstruction on triangular meshes. Two numerical tests confirm the efficiency and accuracy of the L-WENO method.*

***Key Word**: Finite Volume Method; Weighted Essentially Non Oscillatory (WENO) Scheme; Legendre Polynomials*

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I. Introduction

The Essentially Non-Oscillatory (ENO) method which was first proposed by Harten*et al.*[1] was the first successful attempt to obtain a uniformly high order accurate extension of the high resolution van Leer approach to higher order of accuracy. The key idea of the ENO method is to consider a suitable number of possible stencils covering a given control volume and to select the smoothest one, using any appropriate criterion. A reconstruction polynomial is then built using this stencil. Weighted Essentially Non-Oscillatory (WENO) schemes were first developed by Liu *et al.*[2] and more comprehensively by Jiang and Shu[3] using a convex combination of all involved stencils instead of just one as in the original ENO scheme. The construction of the WENO schemes was based on the successes of the ENO schemes with additional advantage. The distinguishing idea of the WENO scheme is the reconstruction procedure.Friedrich[4] constructed WENO schemes for unstructured triangular grids in two space dimensions.Polynomials are probably the most common basis used in the reconstruction step of finite volume methods. But in practical applications, when the order of the polynomial becomes large, the polynomial basis tends to exhibit some numerical instability. The use of orthogonal polynomial basis to replace the basis function is a common way of avoiding such problem. The Legendre and Chebyshev polynomials are well known orthogonal polynomials due to their minimum error property.In this work we intend to implement the WENO method for which the Legendre polynomial is used in the reconstruction step as the polynomial basis.

II. Methods

WENO Approximation: The WENO approximation is based on the ENO approximation. We shall be assuming a uniform grid, that is $\Delta x_i = \Delta x$, in one space dimension.
Suppose we have k candidate stencils

$$S_r(i) = \{x_{i-r}, \dots, x_{i-r+k-1}\}, r = 0, \dots, k-1 \qquad (1)$$

which produce k different reconstructions to the value

$$u_{i+\frac{1}{2}} = \sum_{j=0}^{k-1} C_{r,j}\bar{u}_{k-r+j} \qquad (2)$$

as

$$u_{i+\frac{1}{2}}^{(r)} = \sum_{j=0}^{k-1} C_{r,j}\bar{u}_{k-r+j}\ , r = 0, \dots, k-1 \qquad (3)$$

The WENO reconstruction takes the convex combination of all $u_{i+\frac{1}{2}}^{(r)}$ defined in (3) as a new approximation to the cell boundary value $u\left(x_{i+\frac{1}{2}}\right)$ given as

$$u_{i+\frac{1}{2}} = \sum_{r=0}^{k-1} \omega_r u^{(r)}_{i+\frac{1}{2}} \quad (4)$$

Evidently, the success of WENO scheme lies in the choice of the weights ω_r. For stability and consistency, we require that

$$\omega_r \geq 0, \sum_{r=0}^{k-1} \omega_r = 1$$

If the function $u(x)$ is smooth for all stencils in (1) then

$$u_{i+\frac{1}{2}} = \sum_{r=0}^{k-1} d_r u^{(r)}_{i+\frac{1}{2}} = u\left(x_{i+\frac{1}{2}}\right) + O(\Delta x^{2k-1}) \quad (5)$$

whered_r is a positive constant and $\sum_{r=0}^{k-1} d_r = 1$, due to consistency we would in this smooth case like to have $\omega_r = d_r + O(\Delta x^{2k-1})$, $r = 0, \dots, k-1$, implying $(2k-1)^{th}$ order of accuracy, such that

$$u_{i+\frac{1}{2}} = \sum_{r=0}^{k-1} \omega_r u^{(r)}_{i+\frac{1}{2}} = u\left(x_{i+\frac{1}{2}}\right) + O(\Delta x^{2k-1})$$

Also, where the function $u(x)$ has a discontinuity in one or more of the stencils (1) we would expect that the corresponding weight(s) ω_r is zero. We would also consider that the weights should be smooth functions of the involved cell averages and should be computationally efficient. All these analysis lead to the following form of weights:

$$\omega_r = \frac{\alpha_r}{\sum_{s=0}^{k-1} \alpha_s}, r = 0, \dots, k-1 \quad (6)$$

with

$$\alpha_r = \frac{d_r}{(\epsilon + \beta_r)^2} \quad (7)$$

where$\epsilon > 0$ is introduced to avoid the denominator to be zero and β_r are the smoothness indicators of the stencil $S_r(i)$.

In consideration for a smooth flux and for a higher order variation we are led to the following measurement for smoothness; let $P_r(x)$ be the reconstruction polynomial on the stencil $S_r(i)$. Then we define

$$\beta_r = \sum_{l=1}^{k-1} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \Delta x^{2l-1} \left(\frac{\partial^l P_r(x)}{\partial^l x}\right) dx \quad (8)$$

which is based on L^2 norms. We note that (2.8) is smooth and renders an accurate WENO scheme for the case $k = 2$ and 3 giving a third order and fifth order WENO scheme respectively[5].

Reconstruction procedure in two dimensions

Given the cell averages of a function $u(x, y)$ as

$$\bar{u}_{ij} = \frac{1}{\Delta x_i \Delta y_j} \int_{y_{i-\frac{1}{2}}}^{y_{i+\frac{1}{2}}} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} u(x, y) dx dy \quad (9)$$

we need to find a polynomial $p_{ij}(x, y)$ of degree $k-1$, for each cell I_{ij}, such that it is a $k-th$ order accurate approximation to the function $u(x, y)$ inside I_{ij}:

$$p_{ij}(x, y) = u(x, y) + O(\Delta^k) \quad (10)$$

for$(x, y) \in I_{ij}, i = 1, 2, \dots, N_x, j = 1, 2, \dots, N_y$.

This is the polynomial we will use to reconstruct the values at cell interface. When evaluated at cell boundaries, the polynomial gives the following approximations

$$u^-_{i+\frac{1}{2},j} = p_{ij}\left(x_{i+\frac{1}{2}}, y\right), u^+_{i-\frac{1}{2},j} = p_{ij}\left(x_{i-\frac{1}{2}}, y\right)$$

$$i = 1, \dots, N_x, y_{j-\frac{1}{2}} \leq y \leq y_{j+\frac{1}{2}}$$

$$u^-_{i,j+\frac{1}{2}} = p_{ij}\left(x, y_{j+\frac{1}{2}}\right), u^+_{i,j-\frac{1}{2}} = p_{ij}\left(x, y_{j-\frac{1}{2}}\right)$$

$$j = 1, \dots, N_y, x_{i-\frac{1}{2}} \leq x \leq x_{i+\frac{1}{2}}$$

which are $k-th$ order accurate.

On a two dimensional stencil

$$S_{rs}(i,j) = \left\{\left(x_{l+\frac{1}{2}}, y_{m+\frac{1}{2}}\right) : i-r-1 \leq l \leq i+k-1-r, j-s-1 \leq m \leq j+k-1-s\right\}$$

there exist a unique polynomial $P(x, y)$ which interpolates U at every point in $S_{rs}(i,j)$.

For the grid cell denoted by $C_{ij} = \left(x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}\right) \times \left(y_{j-\frac{1}{2}}, y_{j+\frac{1}{2}}\right)$ the cell average $\bar{u}^n_{ij}$ over the ij^{th} interval is given by

$$\bar{u}_{ij}^{n}(t)=\frac{1}{\Delta x_i \Delta y_j}\int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} u(x,y,t)dxdy$$

where the length of the cell i$\Delta x = x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}}$ and $\Delta y = y_{i+\frac{1}{2}} - y_{i-\frac{1}{2}}$

implies a numerical method of the form

$$\bar{u}_{ij}^{n+1} = \bar{u}_{ij}^{n} - \frac{\Delta t}{\Delta x}\left(F_{i+\frac{1}{2},j}^{n} - F_{i-\frac{1}{2},j}^{n}\right) - \frac{\Delta t}{\Delta y}\left(G_{i,j+\frac{1}{2}}^{n} - G_{i,j-\frac{1}{2}}^{n}\right)$$

whereF^n and G^n are approximations to the average flux along $x_{i\pm\frac{1}{2}}$ and $y_{i\pm\frac{1}{2}}$ respectively.

A useful approach in developing methods with order of accuracy greater than two is the semi-discrete method. The semi-discrete finite volume formulation on Cartesian grids is given by

$$\frac{d}{dt}\bar{u}_{ij}(t) = -\frac{1}{\Delta x_i}\left(\hat{f}_{i+\frac{1}{2},j} - \hat{f}_{i-\frac{1}{2},j}\right) - \frac{1}{\Delta y_j}\left(\hat{g}_{i,j+\frac{1}{2}} - \hat{g}_{i,j-\frac{1}{2}}\right) \quad (11)$$

with the numerical flux defined by:

$$\hat{f}_{i+\frac{1}{2},j} = \sum_\alpha \omega_\alpha h(u^{-}_{i+\frac{1}{2},y_j+\beta_\alpha \Delta y_j}, u^{+}_{i+\frac{1}{2},y_j+\beta_\alpha \Delta y_j}) \quad (12)$$

$$\hat{g}_{i,j+\frac{1}{2}} = \sum_\alpha \omega_\alpha h(u^{-}_{x_i+\beta_\alpha \Delta x_i, j+\frac{1}{2}}, u^{+}_{x_i+\beta_\alpha \Delta x_i, j+\frac{1}{2}}) \quad (13)$$

whereω_α and β_α are nodes and weights of the Gaussian quadrature for approximating the integrals

$$\frac{1}{\Delta y_j}\int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} f\left(u\left(x_{i+\frac{1}{2}}, y, t\right)\right)dy \text{ and } \frac{1}{\Delta x_i}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} g\left(u\left(x, y_{j+\frac{1}{2}}, t\right)\right)dx$$

which are the interface fluxes.

The algorithm for the two dimensional WENO reconstruction is given as

1. Perform the WENO reconstruction of the values at the Gaussian points $u^{\pm}_{i+\frac{1}{2},y_j+\beta_\alpha \Delta y_j}$ and $u^{\pm}_{x_i+\beta_\alpha \Delta x_i, j+\frac{1}{2}}$.
2. Compute the fluxes $\hat{f}_{i+\frac{1}{2},j}$ and $\hat{g}_{i,j+\frac{1}{2}}$ as in (12) and (13).
3. Form the scheme (11).

Legendre Polynomials

The Legendre polynomial is a class of special functions that is widely used in applied mathematics. They are applied in quadrature, approximation theory, solution of partial differential equations and several other areas[6]. They are solutions to the Legendre equation

$$(1-x^2)y'' - 2xy' + \lambda y = 0 \quad (14)$$

which can also be written in self-adjoint form as

$$[(1-x^2)y']' + \lambda y = 0 \quad (15)$$

The singular points for this equation are $x = 1$ and $x = -1$ while $x = 0$ is an ordinary point.

The two variable Legendre polynomials are of higher importance in this thesis because the reconstruction to be carried out is for the two-dimensional conservation laws.

We will use the definition of the two variable Legendre polynomial $P_n(x,y)$ as defined by Khan and Al-Gonah[7] in which they gave the polynomial to be

$$R_n(x,y) = (n!)^2 \sum_{k=0}^{n} \frac{(-1)^{n-k} y^k x^{n-k}}{(k!)^2[(n-k)!]^2} \quad (16)$$

specified by the following generating function

$$C_0(-yt)C_0(-yt) = \sum_{n=0}^{\infty} R_n(x,y)\frac{t^n}{(n!)^2}$$

where$C_0(x)$ denotes the 0^{th} order Tricomi function. The n^{th} order Tricomi functions $C_n(x)$ are defined by Srivastava and Manocha[8] as

$$C_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r x^r}{r!(n+r)!}.$$

The two variable Legendre polynomial $R_n(x,y)$ is linked to the one variable Legendre polynomial $P_n(x)$ by the following relation

$$R_n\left(\frac{1-x}{2}, \frac{1+x}{2}\right) = P_n(x)$$

The first few two-variable Legendre polynomial (2.16) are given as:
$R_0(x,y) = 1$,
$R_1(x,y) = -x + y$,
$R_2(x,y) = x^2 - 4xy + y^2$,
$R_3(x,y) = -x^3 + 9x^2y - 9xy^2 + y^3$.

The Legendre-WENO Reconstruction Procedure
The reconstruction on one dimensional intervals or multi-dimensional Cartesian grids is much easier than the reconstruction of high order polynomials on triangulations. The set $\mathcal{P}_n$ is a vector space of dimension $N(n) = \frac{1}{2}(n+1)(n+2)$. We assume that the computational domain $\Omega \subset \mathbb{R}^2$ is discretized by a conforming triangulation $\mathcal{T}$, formed by the set $\mathcal{T} = \{T_\ell\}_\ell$ of triangles $T_\ell \in \Omega$, $\ell = 1, \dots, \#\mathcal{T}$. In the finite volume structure, each triangle (control volume) has a cell average value

$$\bar{u}_\ell = \frac{1}{|T_\ell|}\int_{T_\ell} u(x)\,dx \qquad (17)$$

where$|T_\ell|$ is the area of the triangle T_ℓ.
In the reconstruction, we desire to solve the following problem:
Given the space of polynomials$\mathcal{P}_n$, and cell average values $\bar{u}_{\ell_k}$, $k = 1, \dots, N$ (where N =dim $\mathcal{P}_n$) of the function u on each control volume T_{ℓ_k} , find a polynomial $p \in \mathcal{P}_n$, that satisfies

$$p_{\ell_1} = \bar{u}_{\ell_1}$$
$$p_{\ell_2} = \bar{u}_{\ell_2}$$

. .
. .
..

$$p_{\ell_N} = \bar{u}_{\ell_N}$$

where the system has a unique solution iff the associated Vandermonde matrix is non-singular (Liu and Zhang[9]).
In our computation, we will use three cells for linear reconstruction, six cells for quadratic reconstruction and ten cells for cubic reconstruction.
Now we consider a basis function of the form

$$P^n(x,y) = \sum_{i=0}^{n} a_i R_i(x,y) = \sum_{\ell+m=0}^{n} c_{\ell,m} x^\ell y^m \qquad (18)$$

where$R_i(x,y), i = 0, 1, 2, \dots,$ n is the i^{th} degree two-variable Legendre polynomial and P^n is a polynomial of degree n.
Then we have,
for$n = 1$

$$P^1(x,y) = a_0 R_0(x,y) + a_1 R_1(x,y)$$
$$= a_0 1 - a_1 x + a_1 y$$
$$= c_{1,00} 1 + c_{1,10} x + c_{1,01} y$$

and in the same way we get
$P^2(x,y) = c_{2,00} 1 + c_{2,10} x + c_{2,01} y + c_{2,20} x^2 + c_{2,11} xy + c_{2,02} y^2$,

$$P^3(x,y) = c_{3,00} 1 + c_{3,10} x + c_{3,01} y + c_{3,20} x^2 + c_{3,11} xy + c_{3,02} y^2 + c_{3,30} x^3 + c_{3,21} x^2 y + c_{3,12} xy^2 + c_{3,03} y^3 \text{ .} \qquad (19)$$

To ensure that the scheme is conservative, it needs to satisfy

$$\bar{p}_\ell^k = \bar{u}_\ell \ , \ \ell = 1, \dots, N \qquad (20)$$

where$N$ is the stencil size.
On a cell T, we compute the polynomial for each stencil S_i

$$\bar{p}_{\ell,i}^k = \frac{1}{|T_\ell|}\int_{T_\ell} P^k(x,y)dx = \bar{u}_{T_\ell} k = 1, 2, 3 l = 1, 2, \dots, \#N_i \qquad (21)$$

where$\#N_i$ is the number of triangles in the stencilS_i. Using (21) enables us to obtain the coefficients for p_i^k on stencil S_i.
The WENO reconstruction is the weighted sum

$$\sum_i \ \omega_i P_i^k(x,y) \qquad (22)$$

whereω_i is the weight which is defined by

$$\omega_i = \frac{(\varepsilon + \mathrm{I}_i)^{-\rho}}{\sum_{i=1}^{k}(\varepsilon + \mathrm{I}_i)^{-\rho}}$$

whereε is a small positive number to avoid division by zero, I_i is the oscillation indicator for the polynomial in each stencil and ρ is a measure of the sensitivity of the weights with respect to the oscillation indicator.
In this work, the following values were used:
$\varepsilon = 10^{-6}$and$\rho = 4$

III. Result

The efficiency and accuracy of the Legendre-WENO method is tested on two examples. The two dimensional linear advection equation and the Doswell's Frontogenesis problem. The results are obtained with the use of the MATLAB 7.5.0 program on windows 7 operating system

3.1 Example one: lonear advection equation

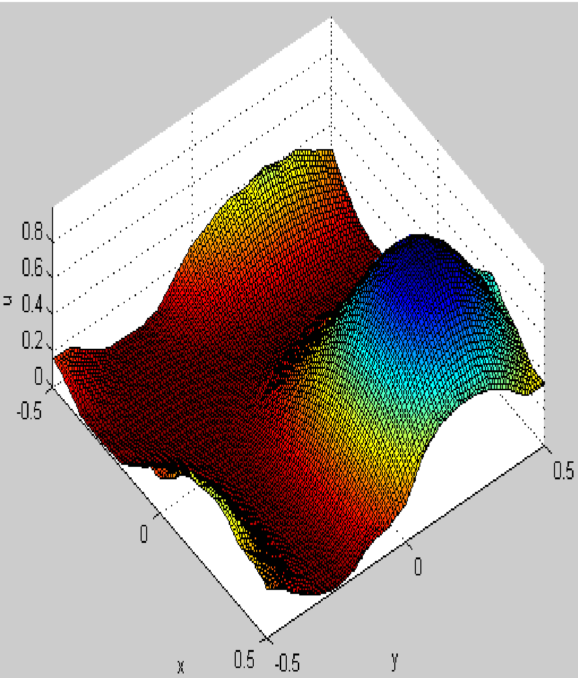
The first illustration considers the two-dimensional Linear Advection Equation

$$u_t + u_x + u_y = 0,\ u \equiv u(t, \mathbf{x}),\ \mathbf{x} = (x, y) \in \mathbb{R}^2 \qquad (23)$$

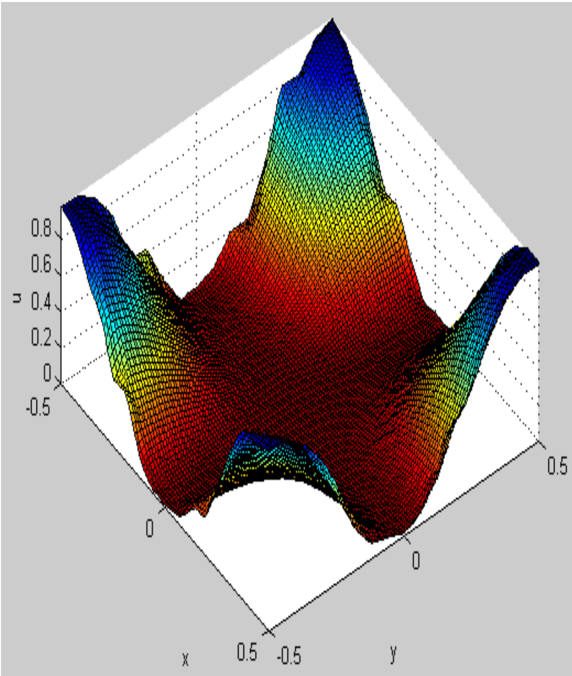
with the initial condition

$$u_0(x, y) = \sin^2\left(\pi\left(x + \frac{1}{2}\right)\right)\sin^2\left(\pi\left(y + \frac{1}{2}\right)\right) \qquad (24)$$

The results are evaluated on the computational domain $\Omega = [-0.5.0.5] \times [-0.5,0.5] \subset \mathbb{R}^2$. The numerical experiment is performed on a sequence of triangular meshes of sizes $h = \frac{1}{16}$ and $\frac{1}{64}$.



(a)



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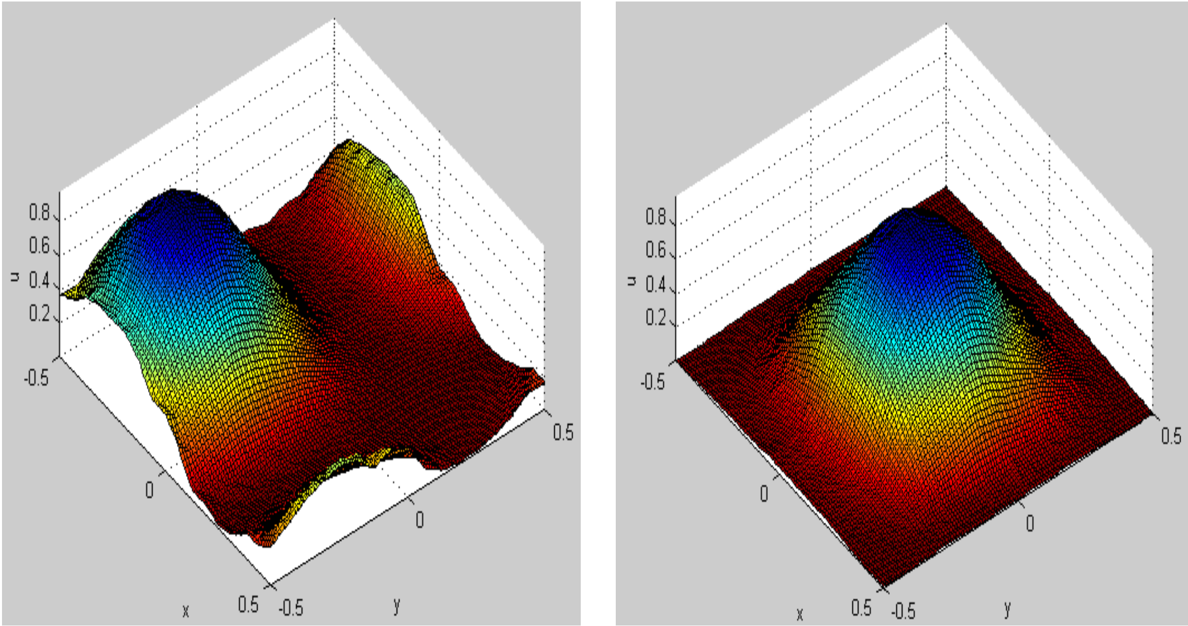
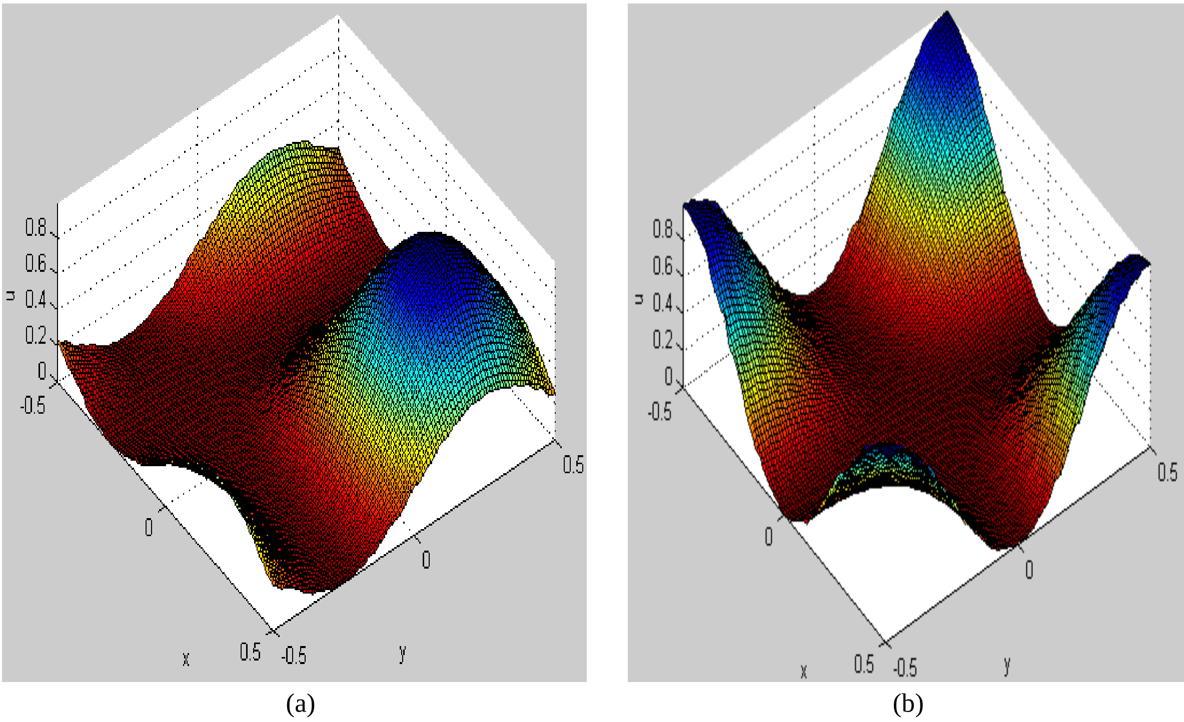
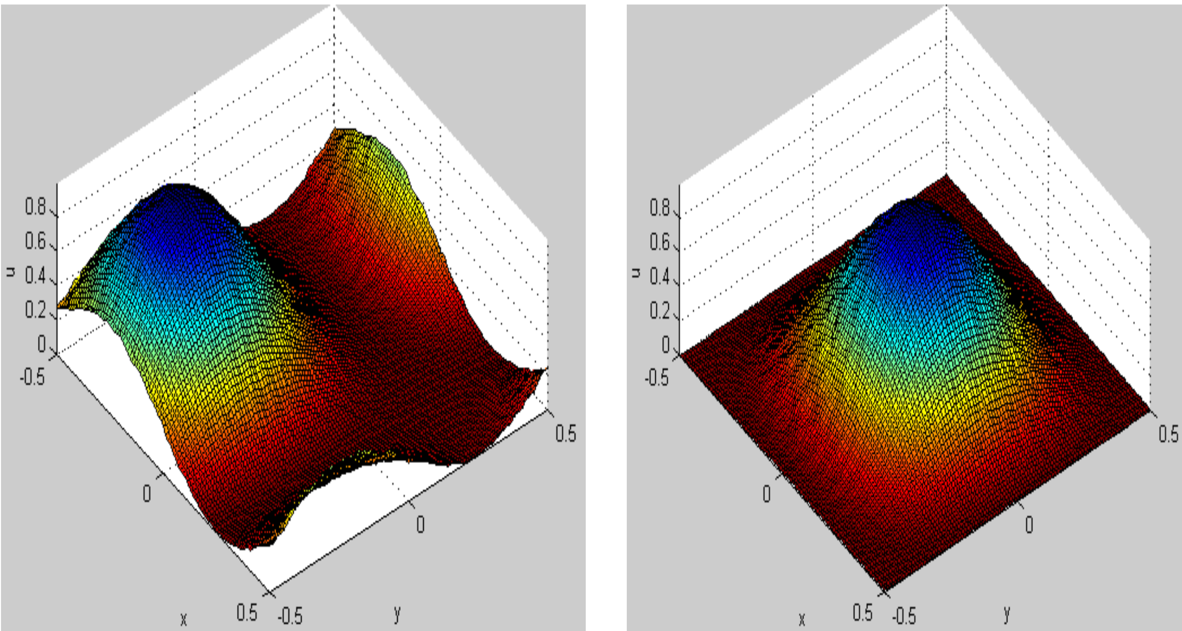


(c) (d)

Figure 1: Solution of (3.1) subject to the initial condition (3.2) at times (a) t=0.25, (b) t= 0.5, (c) t=0.75 and (d) t=1 using the L-WENO scheme on mesh size h=1/16.



(a) (b)



(c) (d)

Figure 2: Solution of (3.1) subject to the initial condition (3.2) at times (a) t=0.25, (b) t= 0.5, (c) t=0.75 and (d) t=1 using the L-WENO scheme on mesh size h=1/64.

Example two: Doswell's frontogenesis

Another test worth considering is the kinematic frontogenesis problem. It is a standard atmospheric modelling test which helps to rate the performance of a scheme in the treatment of sharp fronts. It is a challenging case for advection schemes but is able to numerically test the ability of a scheme in treating discontinuities that move with regard to each other.

The linear equation to be solved is

$$u_t + \sigma_1(\mathrm{x,y})u_x + \sigma_2(\mathrm{x,y})u_y = 0 \qquad (25)$$

where the velocity field is a steady circular vortex with tangential velocity

$v_t(r) = \frac{1}{v_{max}} \cdot \frac{\tanh(r)}{\cosh^2(r)}$.

This means that

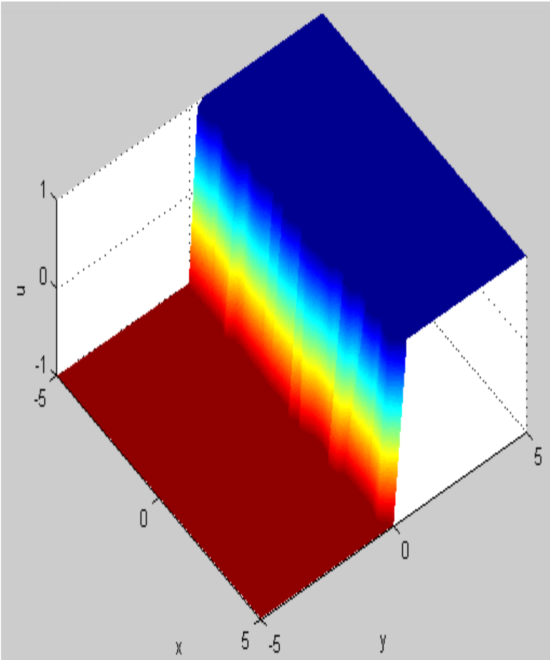
$\sigma_1(\mathrm{x,y}) = \frac{y-c_1}{r} v_t(r)$ and $\sigma_2(\mathrm{x,y}) = \frac{x-c_2}{r} v_t(r)$

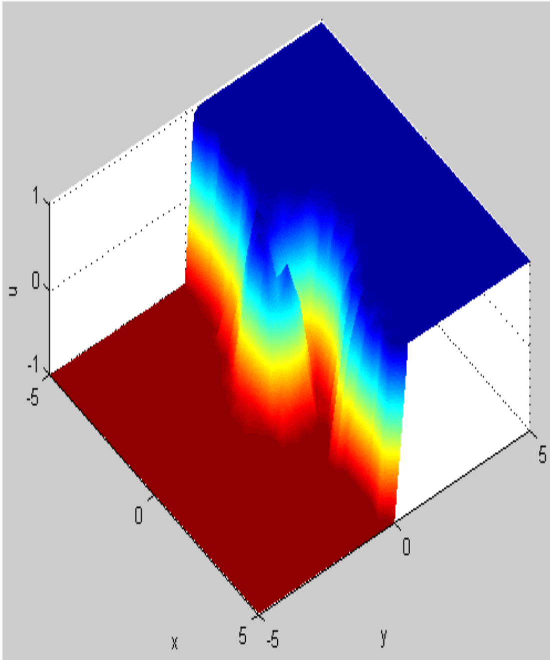
where (c_1, c_2) is the center of the rotation and $r = \sqrt{x-c_1)^2 + (y-c_2)^2}$ is the distance of any point in the domain from the center of rotation. The variable angular velocity is given as $\omega = {}^{v_t}/_{r}$. The initial condition in this test case is defined as

$$u(\mathrm{x,y},0) = u_0(y) = \tanh\left(\frac{y-c_2}{\delta}\right) \qquad (26)$$

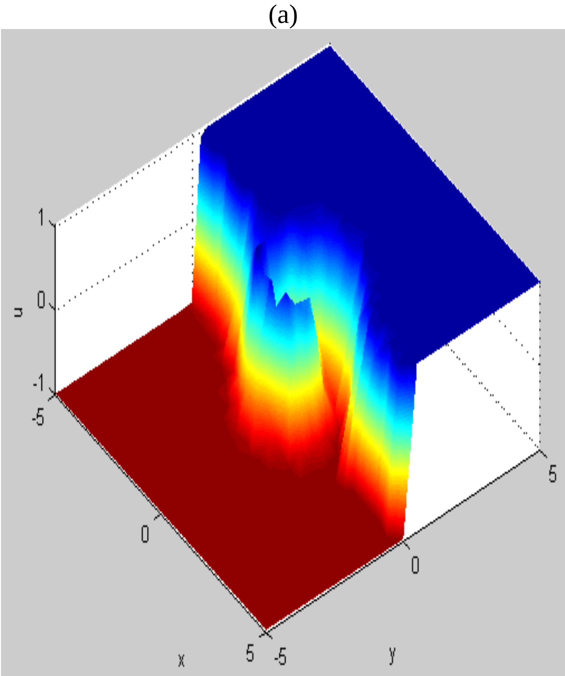
on the computational domain $\Omega = [-5,5] \times [-5,5] \subset \mathbb{R}^2$ and the final time of $T = 4$.

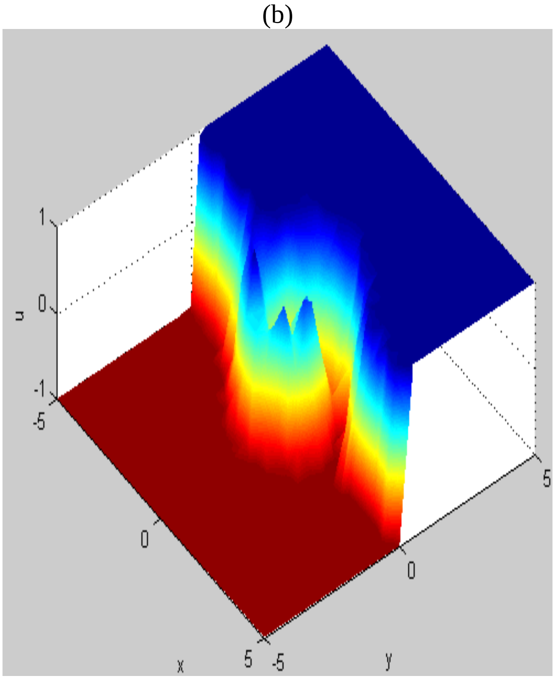
The numerical experiment is performed on a sequence of triangular meshes of sizes $\mathrm{h} = \frac{1}{20}$ and $\frac{1}{80}$.





(a) (b)





(c) (d)

Figure 3: Solution of (3.3) subject to the initial condition (3.4) at times (a) t=0, (b) t=2, (c) t=3 and (d) t=4 using the L-WENO scheme on mesh size h=1/20.

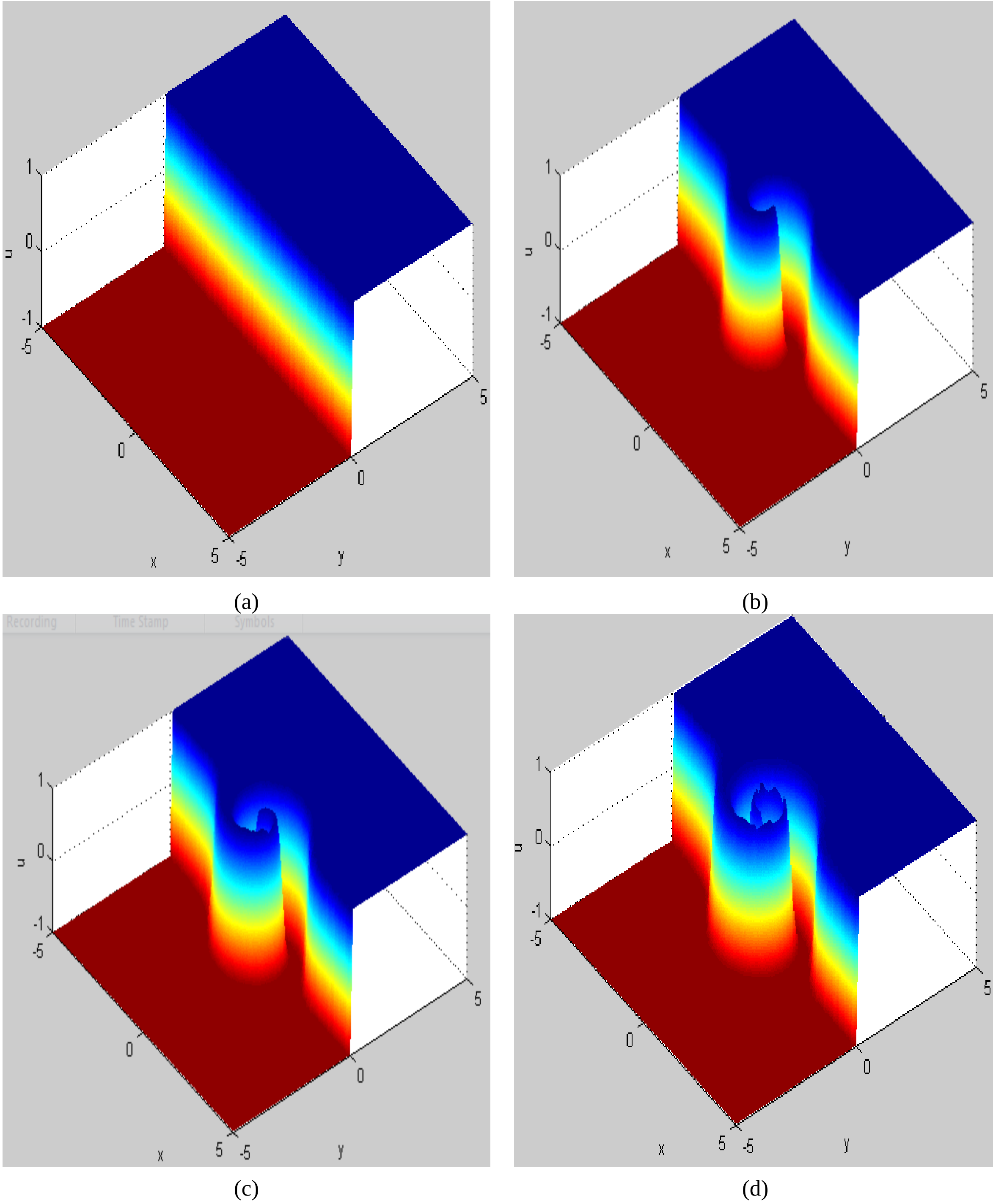


Figure 4: Solution of (3.3) subject to the initial condition (3.4) at times (a) t=0, (b) t=2, (c) t=3 and (d) t=4 using the L-WENO scheme on mesh size h=1/80.

IV. Discussion

It can be seen from Figure 1 and 2 that as the mesh is further refined, the solutions becomes smoother. The oscillations that can be viewed when the mesh is $\frac{1}{16}$ is difficult to see when the mesh is $\frac{1}{64}$. A further refinement of the mesh will produce better results. The simulation of the Doswell's frontogenesis problem agrees with the analytic solution. The mesh size $\frac{1}{20}$ (Figure 3) yields the poorest. The solution on mesh $\frac{1}{80}$ (Figure 4) produces a better result with less diffusion. Here also, it is observed that the mesh that is increasingly refined produces superior results. The method is seen to be numerically stable.

V. Conclusion

The implementation of the Legendre-WENO method yielded a good degree of convergence for a standard test problem; the linear advection equation and the Doswell's frontogenesis problem which exhibits a multi scale behavior. The scheme performed very well.

References

[1]. Harten, A., Engquist, B., Osher, S. and Chakravarthy, S. R. (1987). Uniformly High Order Accurate Essentially Non-oscillatory Schemes, III. *Journal of Computational Physics*, 71(2): 231-303.

[2]. Liu, X. D., Osher, S. and Chan, T. (1994). Weighted Essentially Non-oscillatory Schemes. *Journal of Computational Physics*, 115(1): 200-212.

[3]. Jiang, G. S. and Shu, C. W. (1996). Efficient Implementation of Weighted ENO Schemes. *Journal of Computational Physics*, 126(1), 202-228.

[4]. Friedrich, O. (1998). Weighted Essentially Non-oscillatory Schemes for the Interpolation of Mean Values on Unstructured Grids. *Journal of Computational Physics*, 144(1): 194-212.

[5]. Xing, Y. and Shu, C. W. (2014). A Survey of High Order Schemes for the Shallow Water Equations. *Journal of Mathematical Study*, 47(3): 221-249.

[6]. Li, G. and Wen, C. (2010). Legendre Polynomials in Signal Reconstruction and Compression. In *Industrial Electronics and Applications, the 5th IEEE Conference* Pp1636-1640.

[7]. Khan, S. and Al-Gonah, A. A. (2012). Summation Formulae for the Legendre Polynomials. *ActaMathematicaUniversitaticsComenianae, 81*(1): 127-139.

[8]. Srivastava, H. and Manocha, H. L. (1984). *Treatise on generating functions*. 605 third avenue New York,10158, USA: John Wiley and Sons, Incoporation.500pp.

[9]. Liu, Y. and Zhang, Y. T. (2013). A Robust Reconstruction for Unstructured WENO Schemes. *Journal of Scientific Computing*, 54(2-3): 603-621.

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