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Scalar Field Influenced Robertson –Walker Model

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Abstract: *Earlier work was done by [1] considering when a dust(i.e., $p = 0$) universe has always $q = \frac{1}{2}$ for all values of spatial curvature where they have shown that if a scalar field similar to the one mediating inflation is effective even now, then the Robertson – Walker model may have some interesting properties under certain conditions. In this case even a spatially closed universe may expand forever. The critical density is also less than in the standard Robertson – Walker model. In this work we have considered not only for $q = \frac{1}{2}$ but the other two conditions of $q < \frac{1}{2}$ and $q > \frac{1}{2}$ for all values of spatial curvature. We again found that all the properties of the Robertson – Walker model satisfy considering the scalar field to the Robertson – Walker model. So we claim that this work is the general solution obtained by the earlier authors.*

Key Word: *Relativity, Scalar Field, Cosmology*.

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I. Introduction

Brans- Dicke [2, 3] have proposed a modification of the general relativistic theory of gravitation in an attempt to be more consistent with Mach's principle and less reliant on absolute properties of space. The modification involves a violation of the strong principle of the equivalence [4] on which Einsteins theory is based. This was brought about by the introduction of a scalar function, ϕ, into the variational principle and field equations in a manner analogous to G^{-1} in Einstein theory. This was done in such a way however, as to keep the Lagrangian and action for matter itself unchanged. This ensures continued satisfaction of the *"weak principle of equivalence"* that is, the statement that the paths of test particles in a gravitational field are independent of their masses. For convenience the field equations will be restated here [4]

$$\phi\left(R_{ij} - \frac{1}{2} g_{ij} R\right) = 8\pi T_{ij} + \frac{\omega}{\phi}\left(\phi_{,i}\phi_{,j} - \frac{1}{2} g_{ij}\phi_{,k}\phi^{,k}\right) + \phi_{;i;j} - g_{ij}\phi^{;k}{}_{;k} \quad \ldots (1)$$

$$\phi^{;k}{}_{;k} = \frac{8\pi T}{2\omega + 3} \quad \ldots (2)$$

Here ω is a dimensionless constant number and T_{ij} is the stress-energy tensor for matter itself. In this theory, matter contributes to the locally measured gravitational constant in a manner consistent with the conjecture:

$$G^{-1} \approx A \sum \frac{m}{r} \quad \ldots (3)$$

Where *A* is a dimensionless number.

A similar analysis has been carried out for Einstein theory and has shown that no such dependence as (3) occurs in standard general relativity, although there had been same expectations to the contrary. Hence the consensus of opinion among modern cosmologists is that spontaneous symmetry breaking in the early universe led to the inflation resulting in the de-Sitter expansion [5]. Solutions are presented for a scalar field coupled conformally to Einstein field equations, in the case that the space time metric is spatially homogeneous and isotropic.

That the ϕ - field theory might lead to a relationship of the form (3) is suggested by (2). In fact, if ϕ does correspond to the reciprocal of the locally measured gravitational *"constant"* and if curvature effects can be neglected.

II. The Field Equations

The Robertson-Walker metric are

$$ds^2 = dt^2 - R^2(t)\left[\frac{dr^2}{1-kr^2} + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)\right] \quad \text{… (4)}$$

And the Einstien field equation are given by

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = -8\pi G T_{\mu\nu} \quad \text{… (5)}$$

Where $T_{\mu\nu}$ is the energy momentum tensor of the source producing for radiation, gravitational waves, matter, dusts, clouds, clusters, superclustersetc.

Now the energy momentum tensor of a perfect fluid together with the scalar field is [6].

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu + \phi_{,\mu}\phi_{,\nu} - g_{\mu\nu}\left[p + \frac{1}{2}g^{\alpha\beta}\phi_{,\alpha}\phi_{,\beta} - V(\phi)\right] \quad \text{… (6)}$$

ϕ is here taken as a function of time alone.

The energy momentum tensor has density ρ , pressure p ,four velocity vector $u^\mu = (1,0,0,0)$ and the metric components.

The metric components and christoffel symbols which are non zero are given as follows (recall that $\left(x^0, x^1, x^2, x^3\right) = (ct, r, \theta, \phi)$)

$$\left.\begin{aligned} &g_{00} = 1, g_{11} = -\frac{R^2}{1-kr^2}, g_{22} = -r^2R^2, g_{33} = -r^2R^2\sin^2\theta \\ &g^{00} = 1, g^{11} = -\frac{1-kr^2}{R^2}, g^{22} = -\frac{1}{r^2R^2}, g^{33} = -\frac{1}{r^2R^2\sin^2\theta} \end{aligned}\right\} \quad \text{... (7)}$$

The Ricci tensor is defined by the contraction

$$R_{jk} = \Gamma^i_{rk}\Gamma^r_{ji} - \Gamma^i_{ri}\Gamma^r_{jk} + \frac{\partial\Gamma^i_{ji}}{\partial x^k} - \frac{\partial\Gamma^i_{jk}}{\partial x^i} \quad \text{… (8)}$$

Putting different value of j,k in equation (8), we have the Ricci tensor are

$$R_{00} = \frac{3\ddot{R}}{R} \quad \text{… (9)}$$

$$R_{11} = -\frac{1}{1-kr^2}\left(2\dot{R}^2 + 2k + R\ddot{R}\right) \quad \text{… (10)}$$

$$R_{22} = -r^2\left(2\dot{R}^2 + 2k + R\ddot{R}\right) \quad \text{… (11)}$$

$$R_{33} = -r^2\sin^2\theta\left(2\dot{R}^2 + 2k + R\ddot{R}\right) \quad \text{… (12)}$$

Hence the Ricci scalar is

$$R = \frac{6\left(\dot{R}^2 + k + R\ddot{R}\right)}{R^2} \quad \text{… (13)}$$

The Einstein modified field equation is

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = -8\pi G T_{\mu\nu} \quad \text{… (14)}$$

Putting the value of T_{00} in equation (14) for time-time component ,we have

$$\frac{3\dot{R}^2}{R^2} + \frac{3k}{R^2} = 8\pi G\left(\rho + \frac{1}{2}\dot{\phi}^2 + V(\phi)\right) \quad \text{… (15)}$$

Putting the value of T_{11} in equation (14) for space-space component, we have

$$2\frac{\ddot{R}}{R}+\frac{\dot{R}^2}{R^2}+\frac{k}{R^2}=-8\pi G\left(p+\frac{1}{2}\dot{\phi}^2-V(\phi)\right) \quad \ldots (16)$$

Therefore the identity $T^{\mu\nu}_{;\nu}=0$ gives the condition for conservation law as follows

$$\left\{\left(\rho+\frac{1}{2}\dot{\phi}^2+V(\phi)\right)R^3\right\}_{,t}+3\left(p+\frac{1}{2}\dot{\phi}^2-V(\phi)\right)R^2\dot{R}=0 \quad \ldots (17)$$

This implies that $\dot{\rho}+\dot{\phi}.\ddot{\phi}+\dot{\phi}.V'(\phi)+3\frac{\dot{R}}{R}\left(\rho+p+\dot{\phi}^2\right)=0$ … (18)

When $\rho = p = 0$ the equation (18) reduces to $\ddot{\phi}+V'(\phi)+3\frac{\dot{R}}{R}.\dot{\phi}=0$ … (19)

When $\phi = 0$ the equation (18) reduces to $\dot{\rho}+3\frac{\dot{R}}{R}(\rho+p)=0$ … (20)

However the equation (19) is not independent of (15) and (16), since it can be deduced from them .Here we have two equations involving unknowns R, p, ρ and ϕ . So we may add two more equations to make them determinate, one of which may be the equation of state. We take the other equation in the from

$$4\pi G\left(\dot{\phi}^2-2V\right)=-\frac{k}{R^2} \quad \ldots (21)$$

The exact form of *V as* a function of ϕ is leftundetermined. The field equation (15) becomes

$$\frac{\dot{R}^2}{R^2}+\frac{4k}{3R^2}=\frac{8\pi G(\rho+2V)}{3} \quad \ldots (22)$$

And the field equation (16) becomes

$$2\frac{\ddot{R}}{R}+\frac{\dot{R}^2}{R^2}=-8\pi Gp \quad \ldots (23)$$

Equation (23) corresponds to the field equation of Robertson-Walker model with k=0 in the absence of the scalar field, (i.e., $\phi = 0$) although here the spatial curvature in (4) may be positive or negative.

III. Properties of the R-W model with new results

Subtracting (22) from (23), we have

$$6\frac{\ddot{R}}{R}=\frac{4k}{R^2}-8\pi G(3p+\rho+2V) \quad \ldots (24)$$

Equation (23) shows that if *p* is always positive, then there is no minimum volume in any case $\left(\ddot{R}/R<0 \quad for \quad \dot{R}/R=0\right)$.But a minimum volume will occur if the effective pressure is negative at a turning point whatever the value of k.

Expressed in term of Hubble's constant H and the deceleration parameter *q* the field equations (22) become

$$\frac{8\pi G(\rho+2V)}{3}=H^2+\frac{4k}{3R^2} \quad \ldots (25)$$

And the field equation (23) become $8\pi Gp = H^2(2q-1)$ … (26)

When $\rho > 0$ $q>\frac{1}{2}$ .However in the standard Robertson-Walker model *q* may have any positive value in this case.

From equation (25) and (26) we have $$8\pi G(\rho + 3p + 2V) = \frac{4k}{R^2} + 6qH^2 \qquad \ldots (27)$$

From equation (27), if $p < 0$, $q < \frac{1}{2}$, we obtain the condition for negative *q* as

$$8\pi G(\rho + 3p + 2V) < \frac{4k}{R^2} \qquad \ldots(28)$$

When *k=0*, from equation (21) we have, $V = \frac{\dot{\phi}^2}{2}$ i.e. $V > 0$

When *k=+1*, from equation (21) we have, $V = \frac{\dot{\phi}^2}{2} + \frac{1}{8\pi G R^2}$ i.e, $V > 0$

when *k=0* the condition (28)cannot be satisfied if $(\rho + 3p) > 0$ .when *k=+1* , however *q* may be negative even when $(\rho + 3p) > 0$.

When *k=-1* , *V* has the value $V = \frac{\dot{\phi}^2}{2} - \frac{1}{8\pi G R^2}$

This may be negative, but since $16\pi.V = 8\pi\dot{\phi}^2 - \frac{2}{R^2}$

The above condition cannot be satisfied when $(\rho + 3p) > 0$. To sum up may be negative *for k= 0, -1*, only if $(\rho + 3p) < 0$ .But for *k= +1* then it is possible for $(\rho + 3p) > 0$. However, in the standard Robertson-Walker model *q* cannot be negative if $(\rho + 3p) > 0$.

We know that at the present epoch we may write *p = 0* and the equation (23) would give $R = t^{\frac{2}{3}}$ …(29)

As in the Einstein-de Sitter universe where there is no turning round. this shows that the scalar field prevents a standard Robertson – Walker model with *k =+1* from turning round although the universe is closed . The Hubble constant at the present time t_0 is given by

$$H_0 = \frac{2}{3t_0} \qquad \ldots(30)$$

And it does not depend on the scalar curvature, unlike the standard case .

Now from equation (29), we have $\dot{R} = \frac{2}{3} t^{-\left(\frac{1}{3}\right)}$

So the equation (22) becomes

$$\rho + 2V = \left(6\pi G t^2\right)^{-1} + \frac{k}{2\pi G} t^{-\left(\frac{4}{3}\right)} \qquad \ldots(31)$$

In the absence of the scalar field the Einstein – de Sitter universe gives

$$\rho = \left(6\pi G t^2\right)^{-1} \quad \text{(Since V = 0 = k)} \qquad \ldots(32)$$

In this case $q = \frac{1}{2}$ always, whatever the value of k. But in the standard case q is greater than ,equal to , or less than $\frac{1}{2}$,according as k is *+1 , 0 , or, -1* respectively .

When k = 0, we have from (25) and (31) $$\rho_c + 2V_0 = \frac{3H^2}{8\pi G} \qquad \ldots(33)$$

And from equation (31), we have $$\rho_c + 2V_0 = \left(6\pi G t^2\right)^{-1} \qquad \ldots(34)$$

Now from equation (33) and (34), we have $$\rho_c + 2V_0 = \frac{3H^2}{8\pi G} = \left(6\pi G t^2\right)^{-1} \qquad \ldots(35)$$

Putting *k=0* in equation (21), we have $V_0 = \frac{\dot{\phi}^2}{2}$

Putting $V_0 = \frac{\dot{\phi}^2}{2}$ in equation (35), we have

$$\rho_c = \frac{3H^2}{8\pi G} - \dot{\phi}^2 = \left(6\pi G t^2\right)^{-1} - \dot{\phi}^2 \qquad \text{…(36)}$$

Comparing this equation with (31) we find that it is less than in the standard case.
When $k = 1$, then from equation (25) and (31), we have

$$\rho_1 + 2V_1 = \frac{3H^2}{8\pi G} + \frac{1}{2\pi G R^2} \qquad \text{…(37)}$$

And
$$\rho_1 + 2V_1 = \left(6\pi G t^2\right)^{-1} + \frac{1}{2\pi G} t^{-\left(\frac{4}{3}\right)} \qquad \text{...(38)}$$

Therefore
$$\rho_1 + 2V_1 = \frac{3H^2}{8\pi G} + \frac{1}{2\pi G R^2} = \left(6\pi G t^2\right)^{-1} + \frac{1}{2\pi G} t^{-\left(\frac{4}{3}\right)} \qquad \text{…(39)}$$

Putting $k = 1$ in equation (21), we have $V_1 = \frac{\dot{\phi}^2}{2} + \frac{1}{8\pi G R^2}$

Putting the value of V_1 in equation (39),we have

$$\rho_1 = \frac{3H^2}{8\pi G} + \frac{1}{4\pi G R^2} - \dot{\phi}^2 = \left(6\pi G t^2\right)^{-1} + \frac{1}{4\pi G} t^{-\left(\frac{4}{3}\right)} - \dot{\phi}^2 \qquad \text{…(40)}$$

Or, $$\rho_1 = \rho_c + \frac{1}{4\pi G R^2} = \rho_c + \frac{1}{4\pi G} t^{-\left(\frac{4}{3}\right)}$$

So we have $\rho_1 > \rho_c$ … (41)

When $k = -1$, then from equation (25) and (31), we have

$$\rho_2 + 2V_2 = \frac{3H^2}{8\pi G} - \frac{1}{2\pi G R^2} \qquad \text{…(42)}$$

And
$$\rho_2 + 2V_2 = \left(6\pi G t^2\right)^{-1} - \frac{1}{2\pi G} t^{-\left(\frac{4}{3}\right)} \qquad \text{...(43)}$$

Therefore
$$\rho_2 + 2V_2 = \frac{3H^2}{8\pi G} - \frac{1}{2\pi G R^2} = \left(6\pi G t^2\right)^{-1} - \frac{1}{2\pi G} t^{-\left(\frac{4}{3}\right)} \qquad \text{…(44)}$$

Putting $k = -1$ in equation (34), we have $V_2 = \frac{\dot{\phi}^2}{2} - \frac{1}{8\pi G R^2}$

Putting the value of V_2 in equation (44),we have

$$\rho_2 = \frac{3H^2}{8\pi G} - \frac{1}{4\pi G R^2} - \dot{\phi}^2 = \left(6\pi G t^2\right)^{-1} - \frac{1}{4\pi G} t^{-\left(\frac{4}{3}\right)} - \dot{\phi}^2 \qquad \text{…(45)}$$

Or, $$\rho_2 = \rho_c - \frac{1}{4\pi G R^2} = \rho_c - \frac{1}{4\pi G} t^{-\left(\frac{4}{3}\right)}$$

So we have $\rho_2 < \rho_c$ ... (46)

In order to avoid having to reformulate the whole of atomic and nuclear physics, Dirac chose G as the 'constant' that varies with time, and he proposed that $G \propto \frac{\dot{R}}{R}$ But $\frac{\dot{R}}{R} = \frac{2}{3} t^{-1}$

Therefore, $G \propto \frac{2}{3}t^{-1}$ Or, $G \propto t^{-1}$ ...(47)

The gravitational 'constant' measured by the observation of slowly moving particles or in the time dilatation experiments[4] is $G = \frac{2\omega+4}{2\omega+3}\phi^{-1}$ …(48)

Where ω is the dimensionless coupling parameter.

Therefore $G \propto \phi^{-1}$ (Since $\frac{2\omega+4}{2\omega+3}$ =constant) …(49)

Now from (47) and (49), we have $t^{-1} \propto \phi^{-1}$ i.e, $t \propto \phi$

So we can write $t = \phi$ …(50)

Now $\frac{d\phi}{dt} = 1 \therefore \dot{\phi} = 1$ ...(51)

Case I

When *k=0* then from equation (36), we have $\rho_c = \frac{1}{6\pi G\phi^2} - 1$ ...(52)

Again from (31), we have $\rho = \frac{1}{6\pi G t^2} - \dot{\phi}^2 + \frac{k}{4\pi G} t^{-\left(\frac{4}{3}\right)}$

Putting $t = \phi, \dot{\phi} = 1, and \quad k = 0$, we have $\rho = \frac{1}{6\pi G\phi^2} - 1$ …(53)

From equation (52) and (53), we have $\rho = \rho_c$ …(54)

Now the equation (16) becomes

$$q = \frac{4\pi G R^2}{\dot{R}^2}\left(p + \frac{1}{2}\dot{\phi}^2 - V(\phi)\right) + \frac{k}{2\dot{R}^2} + \frac{1}{2} \quad \text{….(55)}$$

When *k=0* , then $V = \frac{1}{2}\dot{\phi}^2$ and for dust $p = 0$

$$\therefore q = \frac{8\pi G R^2}{2\dot{R}^2}\left(0 + \frac{1}{2}\dot{\phi}^2 - \frac{1}{2}\dot{\phi}^2\right) + \frac{0}{2\dot{R}^2} + \frac{1}{2}$$

$$\therefore q = \frac{1}{2}$$

Hence when $k = 0$ then $\rho = \rho_c \Rightarrow \frac{\rho}{\rho_c} = 1 \Rightarrow \Omega = 1,\ q = \frac{1}{2}$ ….(56)

Where $\Omega =$ density parameter $= \frac{average\ density}{critical\ density} = \frac{\rho}{\rho_c}$

Case I I

When *k=1, the* equation (40) becomes

$$\rho_1 = \frac{1}{6\pi G\phi^2} - 1 + \frac{1}{4\pi G}\phi^{-\left(\frac{4}{3}\right)} \quad \text{Or, } \rho_1 = \rho_c + \frac{1}{4\pi G}\phi^{-\frac{4}{3}}$$

$$\therefore \rho_1 > \rho_c \Rightarrow \frac{\rho_1}{\rho_c} > 1 \Rightarrow \Omega > 1 \quad \text{...(57)}$$

When *k=1* , then $V = \frac{1}{2}\dot{\phi}^2 + \frac{1}{8\pi G R^2}$ putting this in equation (55) ,we have $q = 9\pi G p\phi^2 + \frac{1}{2}$

Therefore $q > \frac{1}{2}$ when $p > 0$

Hence when $k = 1$ then $\rho_1 > \rho_c \Rightarrow \frac{\rho_1}{\rho_c} > 1 \Rightarrow \Omega > 1,\ q > \frac{1}{2}$ ….(58)

Case III

When *k=-1* ,the equation (45) becomes $\rho_2 = \frac{1}{6\pi G\phi^2} - 1 - \frac{1}{4\pi G}\phi^{-\left(\frac{4}{3}\right)}$

Or, $\rho_2 = \rho_c - \frac{1}{4\pi G}\phi^{-\left(\frac{4}{3}\right)}$

$$\therefore\ \rho_2 < \rho_c \Rightarrow \frac{\rho_2}{\rho_c} < 1 \Rightarrow \Omega < 1 \quad …..(59)$$

When *k=-1* , then $V = \frac{1}{2}\dot{\phi}^2 - \frac{1}{8\pi GR^2}$ putting this in equation (55) ,we have

$$q = 9\pi Gp\phi^2 + \frac{1}{2}$$

Therefore when $p < 0$ then $q < \frac{1}{2}$ ….(60)

Hence when $k = -1$ then $\rho_2 < \rho_c \Rightarrow \frac{\rho_2}{\rho_c} < 1 \Rightarrow \Omega < 1,\ q < \frac{1}{2}$ ….(61)

Therefore

when $k = 0$ then $\rho = \rho_c \Rightarrow \frac{\rho}{\rho_c} = 1 \Rightarrow \Omega = 1,\ q = \frac{1}{2}$ ….(62)

when $k = 1$ then $\rho_1 > \rho_c \Rightarrow \frac{\rho_1}{\rho_c} > 1 \Rightarrow \Omega > 1,\ q > \frac{1}{2}$ ….(63)

when $k = -1$ then $\rho_2 < \rho_c \Rightarrow \frac{\rho_2}{\rho_c} < 1 \Rightarrow \Omega < 1,\ q < \frac{1}{2}$ ….(64)

All properties of R-W model are satisfied by using the scalar field too.

IV. Conclusion

We have seen that the scalar field prevents a closed RW model from turning round. It makes q always greater than $\frac{1}{2}$ for positive pressure, equal to $\frac{1}{2}$ for $p = 0$ for dust and less than $\frac{1}{2}$ for negative pressure, independent of spatial curvature. The dust model in this case expands at the same rate as the standard Einstien de-Sitter model and therefore has the same age at present whatever the spatial curvature. The density for closing the universe is in this case less than in the standard model. The earlier result was shown by the two authors *L.P.Sarker* and *S.Banarjee*, the department of physics, the university of Burdwan, Burdwan 713-104,India. But in this work we have carried out all parameters in our calculations to satisfy the different properties of the Friedmann Robertson-Walker model of the universe and found that the scalar field influenced effectively not only for $q = \frac{1}{2}$ but for other two cases $q > \frac{1}{2}$, and $q < \frac{1}{2}$ too. So this is the general solution satisfying all properties of the R-W model considering the scalar field ϕ.

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