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Comparison between Some Methods for Solving Fractional Differential Equations

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Abstract:
In this paper, the Homotopy Analysis Method (HAM) is applied to obtain the solution of fractional differential equations. The fractional derivatives are described in the Caputo sense. The solution obtained by this method has been compared with those obtained by Homotopy Perturbation Method (HPM) and the Variational Iteration Method (VIM). Results show that (HPM) and (VIM) are all special cases of the homotopy analysis method (HAM) when the nonzero convergence-control parameter $\hbar = -1$.

Keywords: *Ordinary Fractional differential equation, Homotopy Analysis Method, Homotopy Perturbation Method, Variational Iteration Method.*

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I. Introduction

Fractional differential equations have been successfully modelled for many physical and engineering phenomena such as seismic analysis, rheology, fluid flow, viscous damping, viscoelastic materials, and polymer physics. Consequently, considerable attention has been given to the solution of fractional ordinary differential equations, integral equations and fractional partial differential equations of physical interest. Since most fractional differential equations do not have exact analytic solutions, approximation and numerical techniques, therefore, are used extensively.

Recently, the variational iteration method, which proposed by the Chinese mathematician He [1–6], was successfully applied to autonomous ordinary and partial differential equations [7–11] and other fields. He [2] was the first to apply the variational iteration method to fractional differential equations. The Perturbation techniques are based on the existence of small/large parameters, the so-called perturbation quantity. Unfortunately, many nonlinear problems in science and engineering do not contain such kind of perturbation quantities at all. Some non-perturbative techniques, such as the artificial small parameter method, the δ-expansion method and the Adomian's decomposition method, have been developed. Different from perturbation techniques, these non-perturbative methods are independent upon small parameters. However, both of the perturbation techniques and the non-perturbative methods cannot provide us with a simple way to adjust and control the convergence region and rate of given approximate series.

Dr. Shijun Liao [Liao92] proposed a powerful analytic method for nonlinear problems, namely the homotopy analysis method. Different from all reported perturbation and non-perturbative techniques mentioned above, the homotopy analysis method itself provides us with a convenient way to control and adjust the convergence region and rate of approximation series, when necessary. The HAM has been developing greatly in theory and applied successfully to numerous types of nonlinear equations in lots of different fields by scientists, researchers and engineers. All of these verify the originality, novelty, validity and generality of the HAM.
Our aim in this paper is to compare the homotopy perturbation method, variational iteration method with the homotopy analysis method for solving the linear fractional differential equation.

II. Formation of the Problem

In this paper, the Homotopy Analysis Method (HAM) is applied to obtain the solution of fractional differential equation given by

$$y''(t) - D^{\alpha} y(t) = t\,, \tag{1}$$

Subject to the initial condition

$$y(0) = 0, \quad y'(0) = 0 \tag{2}$$

Where $D^{\alpha}(.)$ is the fractional differential operator (Caputo derivative) of order $0 < \alpha < 1$. Two other methods, viz., HPM and VIM, have also been used to solve the above equation (1) for comparison of the solution using HAM.

In particular case, if we set $\alpha = 1$, then the equation (1) becomes linear differential equation of second order which has the following exact solution

$$y(t) = e^t - \frac{t^2}{2} - t - 1\,.$$

III. Basic definitions

In this section**,** we gives some basic definitions and properties of the Fractional calculus.

Definition 1 A real function $f(t), t > 0$, is said to be in the space C_{μ}, $\mu \in \mathbb{R}$, if there exists a real number $p > \mu$, such that $f(t) = t^p f_1(t)$, where $f_1(t) \in C[a, b]$, and it is said to be in the space C_{μ}^{n}, if and only if $f^{\,(n)} \in C_{\mu}\,, n \in \mathbb{N}$.

Definition 2 A function $f(t), t\ >\ 0$, is said to be in the space $C_{\mu}^{m} \in \mathbb{N} \cup \{0\}$, if $f^{\,m} \in C\mu$.

Definition 3 The Riemann-Liouville fractional integral operator (I^{α}) of order $\alpha \geq 0$, of a function $f(t) \in C_{\mu}, n \in \mathbb{N}$.

$$I^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha\, -\, 1} f(\tau)\, d\tau$$

$$I^0 f(t) = f(t)$$

Some properties of the operator (I^{α}), are as follows [5]:

$$I^{\alpha} I^{\beta} f(t) = I^{\beta} I^{\alpha} f(t) = I^{\alpha+\beta} f(t),$$

$$I^{\alpha} t^n = \frac{\Gamma(n+1)}{\Gamma(n+\alpha+1)} t^{n+\alpha}.$$

Definition 4 let $\alpha > 0, x > 0, \alpha, t \in \mathbb{R}, n \in \mathbb{N}, n - 1 < \alpha < n.$ Then the Caputo fractional derivative of $\alpha > 0$, define as [12]

$$D^{\alpha} f(t) = \frac{d^{\alpha} f}{dt^{\alpha}} = \begin{cases} \dfrac{1}{\Gamma(n-\alpha)} \displaystyle\int_0^t (t-\tau)^{n-\alpha\,-\,1} f^{(n)}(\tau)\, d\tau \\ \dfrac{d^n}{dx^n} f(x)\ ; \qquad\qquad \alpha = n \in \mathbb{N} \end{cases}$$

Is called the Caputo fractional derivatives or Caputo fractional differential operator of order α. According to this definition, when $f(x) = C$, where C is the constant function, since the n^th derivative $C^{(n)}$ of constant equal 0, it follows $D_c^{\alpha} C = 0$, and

$$D_c^{\alpha} x^{v} = \frac{\Gamma(v+1)}{\Gamma(v-\alpha+1)} x^{v-\alpha}$$

IV. The Homotopy Analysis Method (HAM)

To describe the basic ideas of the HAM, we consider the following differential equation:

$$N[y(t)] = 0, \qquad t > 0 \tag{3}$$

Where N is a nonlinear differential operator, and $y(t)$ is unknown function of the independent variable t. Based on the zero-order deformation equation constructed by Liao [21, 22], we give the following zero-order deformation equation in the similar way:

$$(1 - q)\ell[\varphi(t; q) - y_0(t)] = q\hbar H(t)[N\varphi(t; q)]\,, \tag{4}$$

Where $q \in [0,1]$ is an embedding parameter, $\hbar$ is the nonzero convergence-control parameter, $H(t)$ are non-zero auxiliary function, N is nonlinear differential operator, $\varphi(t; q)$ is an unknown function, and $y_0(t)$ is an initial guess of $y(t)$, ℓ is an auxiliary linear integral order operator and it possesses the property $\ell(\mathrm{c}) = 0$.

Obviously when $q = 0$ and $q = 1$, we have

$$\varphi(t; 0) = y_0(t) \quad , \qquad \varphi(t; 1) = y(t) \tag{5}$$

Expanding φ in Taylor series with respect to q, one has

$$\varphi(t; q) = y_0(t) + \sum_{m=1}^{\infty} y_m(t) \tag{6}$$

Where

$$y_m(t) = \frac{1}{m!} \frac{\partial^m \varphi(t; q)}{\partial q^m}\bigg|_{q=0} \,, \quad i = 1,2,3, \dots, n \tag{7}$$

Define the vector

$$\vec{y}_{m-1} = \{y_{0,} y_1, y_2, \dots, y_{m-1}\}.$$

Differentiating equation (4) m-times with respect to embedding parameter , then setting $q = 0$, and dividing them by $m!$, we have, using (7), the so-called m^th-order deformation equation

$$y_m(t) = \chi_m y_{m-1}(t) + \hbar I^{\alpha}\left[H(t) R_m\left(\overrightarrow{y_{m-1}}(t)\right)\right] \quad (8)$$

$$R_m\left(\overrightarrow{y_{m-1}}(t)\right) = \frac{1}{(m-1)!} \frac{\partial^{m-1}[N\varphi(t;q)]}{\partial q^{m-1}}\bigg|_{q=0} \quad (9)$$

And

$$\chi_m = \begin{cases} 0, & m \leq 1 \\ 1, & m > 1 \end{cases} \quad (10)$$

4.1 Application of HAM

We consider the non-homogenous fractional differential equation (1), we define, according to the equation (1), the nonlinear operators,

$$N_{\alpha}[y(t)] = y''(t) - D^{\alpha} y(t) - t\,,$$

According to (14) and (16), we have

$$R_m\left(\overrightarrow{y_{m-1}}(t)\right) = y''_{m-1}(t) - D^{\alpha} y_{m-1}(t) - t\,(1 - \chi_m),$$

If we take the auxiliary function $H(t) = 1$, and the parameter $\alpha = 2$, then the auxiliary linear operator becomes

$$\ell = \frac{d^2}{dx^2}\,,$$

and

$$y_m(t) = \chi_m y_{m-1}(t) + \hbar I^{2}\left[R_m\left(\overrightarrow{y_{m-1}}(t)\right)\right],$$

Subject to the initial conditions

$$y(0) = 0\,, \qquad y'(0) = 0\,,$$

If we choose the initial guess approximation

$$y_0(t) = 0\,,$$

then we have

$$y_1(t) = -\hbar \frac{t^3}{6}\,,$$

$$y_2(t) = -(\hbar + \hbar^2)\frac{t^3}{6} + \hbar^2 \frac{t^{5-\alpha}}{\Gamma(6-\alpha)}\,,$$

$$y_3(t) = -(\hbar + 2\hbar^2 + \hbar^3)\frac{t^3}{6} + 2(\hbar^2 + \hbar^3)\frac{t^{5-\alpha}}{\Gamma(6-\alpha)} - \hbar^3 \frac{t^{7-2\alpha}}{\Gamma(8-2\alpha)}\,.$$

And so on. Thus, The 3rd order approximation of $y(t)$ is given by

$$y_{HAM}(t) = y_0(t) + y_1(t) + y_2(t) = -\hbar \frac{t^3}{6} = -(\hbar + \hbar^2)\frac{t^3}{6} + \hbar^2 \frac{t^{5-\alpha}}{\Gamma(6-\alpha)}\,.$$

The 4th order approximation of $y(t)$ is given by

$$\begin{aligned} y_{HAM}(t) &= y_0(t) + y_1(t) + y_2(t) + y_3(t) \\ &= -\hbar \frac{t^3}{6} - (\hbar + \hbar^2)\frac{t^3}{6} + \hbar^2 \frac{t^{5-\alpha}}{\Gamma(6-\alpha)} - (\hbar + 2\hbar^2 + \hbar^3)\frac{t^3}{6} + 2(\hbar^2 + \hbar^3)\frac{t^{5-\alpha}}{\Gamma(6-\alpha)} \\ &\quad - \hbar^3 \frac{t^{7-2\alpha}}{\Gamma(8-2\alpha)} \end{aligned}$$

The mth order approximation of $y(t)$ is given by

$$y_{HAM}(t) = y_0(t) + \sum_{m=1}^{\infty} y_m(t)$$

$$y_{HAM}(t) = -\hbar \frac{t^3}{6} - (\hbar + \hbar^2)\frac{t^3}{6} + \hbar^2 \frac{t^{5-\alpha}}{\Gamma(6-\alpha)} - (\hbar + 2\hbar^2 + \hbar^3)\frac{t^3}{6} + 2(\hbar^2 + \hbar^3)\frac{t^{5-\alpha}}{\Gamma(6-\alpha)} - \hbar^3 \frac{t^{5-\alpha}}{\Gamma(6-\alpha)} + \cdots$$

Taking $\hbar = -1$, we have

$$y_{HAM}(t) = \frac{t^3}{6} + \frac{t^{5-\alpha}}{\Gamma(6-\alpha)} + \frac{t^{7-2\alpha}}{\Gamma(8-2\alpha)} + \frac{t^{9-3\alpha}}{\Gamma(10-3\alpha)} + \cdots,$$

In particular if $\alpha = 1$, we have

$$y_{HAM}(t) = \frac{t^3}{6} + \frac{t^4}{\Gamma(5)} + \frac{t^5}{\Gamma(6)} + \frac{t^6}{\Gamma(7)} + \cdots,$$

Thus, the above equation can be written in the form

$$y_{HAM}(t) = -1 - t - \frac{t^2}{2!} + 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \frac{t^5}{5!} + \frac{t^6}{6!} + \cdots$$

$$y_{HAM}(t) = -1 - t - \frac{t^2}{2} + e^t\,.$$

Which is the exact solution of equation (1).

V. The Homotopy Perturbation Method (HPM)

The HPM was first proposed by Chinese mathematician He [13, 14]. It is a powerful mathematic tool to solve nonlinear problems, especially engineering problems.

Now, we consider the non-homogenous fractional differential equation (1), In view of HPM, we construct the following homotopy:

$$y''(t) - p[D^\alpha y(t) + t] = 0\,, \tag{11}$$

Where $p \in [0,1]$ is an embedding parameter. If $p = 0$, equation (11) become

$$y''(t) = 0\,,$$

when $p = 1$, equation (11) turn out to be the original non-homogenous FDE (1).

Using the parameter p, the solution $y(t)$ can expanded in the following form

$$y(t) = y_0(t) + py_1(t) + p^2 y_2(t) + p^3 y_3(t) + \cdots \tag{12}$$

Setting $p = 1$, gives the approximate solution

$$y(t) = y_0(t) + y(t) + y_2(t) + y_3(t) + \cdots \tag{13}$$

Now, substituting (12) into (11) and equating the terms with the identical powers of p, we obtain

p^0: $y_0''(t) = 0\,, \quad y_0(t) = 0\,,$ (14)

p^1: $y_1''(t) = D^\alpha y_0 + t\,,$ (15)

p^2: $y_2''(t) = D^\alpha y_1\,,$ (16)

p^3: $y_3''(t) = D^\alpha y_2,$ (17)

p^4: $y_4''(t) = D^\alpha y_3,$ (18)

Applying the operator I^2, the inverse operator of $\frac{d^2}{dt^2}$, on both sides of the linear equations (14) – (18) and using the initial condition (2), we obtain

$y_0(t) = y(0) = 0\,,$

$y_1(t) = I^2[D^\alpha y_0 + t] = \dfrac{t^3}{6},$

$y_2(t) = I^2[D^\alpha y_1] = \dfrac{t^{5-\alpha}}{\Gamma(6-\alpha)},$

$y_3(t) = I^2[D^\alpha y_2] = \dfrac{t^{7-2\alpha}}{\Gamma(8-2\alpha)},$

⋮

And so on. Hence the 3rd order approximate solution of $y(t)$ is given by

$$y_{HPM}(t) = y_0(t) + y_1(t) + y_2(t) = \frac{t^3}{6} + \frac{t^{5-\alpha}}{\Gamma(6-\alpha)}.$$

The 4th order approximate solution of $y(t)$ is given by

$$y_{HPM}(t) = y_0(t) + y_1(t) + y_2(t) + y_3(t) = \frac{t^3}{6} + \frac{t^{5-\alpha}}{\Gamma(6-\alpha)} + \frac{t^{7-2\alpha}}{\Gamma(8-2\alpha)}.$$

VI. The Variational Iteration Method (VIM)

The principles of the variation iteration method and its applicability for various kinds of differential equations are given in [15-20].

To illustrate the basic idea of this method, we consider the following general nonlinear system

$$D^\alpha y(x) + Ly(x) + Ny(x) = g(x)\,, \tag{19}$$

Where L is a linear operator, N is a nonlinear operator, and $g(x)$ is the source term, and D^α is the Caputo fractional derivative of order α with $m - 1 < a < m$. According to the variation iteration method, we can construct the correction functional as follows:

$$y_{n+1}(x) = y_n(x) + \int_0^x \lambda(t)\,[D^\alpha y_n(t) + Ly_n(t) + N\widetilde{y_n}(t) - g(t)]dt \tag{20}$$

Where λ is a general Lagrange multiplier [14], which can be identified optimally via the variation theory [14], the subscript n denotes the nth-order approximation, $\tilde{u}_n$ is considered as a restricted variation, i.e. $\delta\tilde{y}_n = 0$ .Successive approximations $y_{n+1}(x), n \geq 0$ of solution $y(x)$.

The zero-th approximations y_0 can be any selection function .However, guesses the initial values $y(0), y'(0), and\ y''(0)$ are preferably for the selective zeroth approximation y_0 as will be seen later. Consequently, the solution is given by:

$$y(x) = \lim_{n\to\infty} y_n(x).$$

Now, In view of equation (20), the correction functional for equation (1) can be given by

$$y_{n+1} = y_n - I^2[y_n''(t) - D^\alpha y_n(t) - t\,],$$

Consequently, begin with $y_0 = 0$, we find the following approximations

$$y_1(t) = -I^2[y_0''(t) - D^\alpha y_0(t) - t] = \frac{t^3}{6},$$

$$y_2(t) = y_1 - I^2[y_1''(t) - D^\alpha y_1(t) - t\,] = \frac{t^3}{6} + \frac{t^{5-\alpha}}{\Gamma(6-\alpha)},$$

$$y_3(t) = y_2 - I^2[y_2''(t) - D^\alpha y_2(t) - t\,] = \frac{t^3}{6} + \frac{t^{5-\alpha}}{\Gamma(6-\alpha)} + \frac{t^{7-2\alpha}}{\Gamma(8-2\alpha)},$$

⋮

And so on.

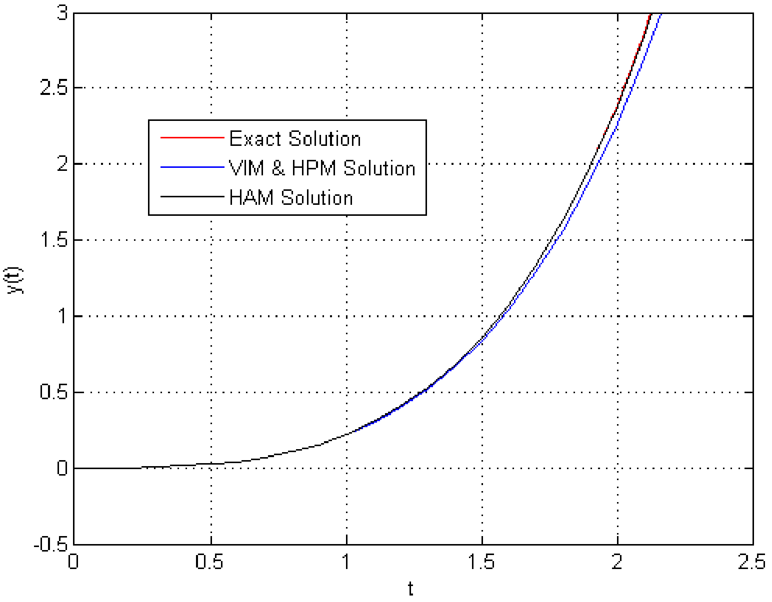

Figure 1: Graph of HAM solution (4 term approximation) **at** $\hbar = -1.2$, HPM (4 term approximation), VIM (4 term approximation) and exact solution, in case $\alpha = 1$.

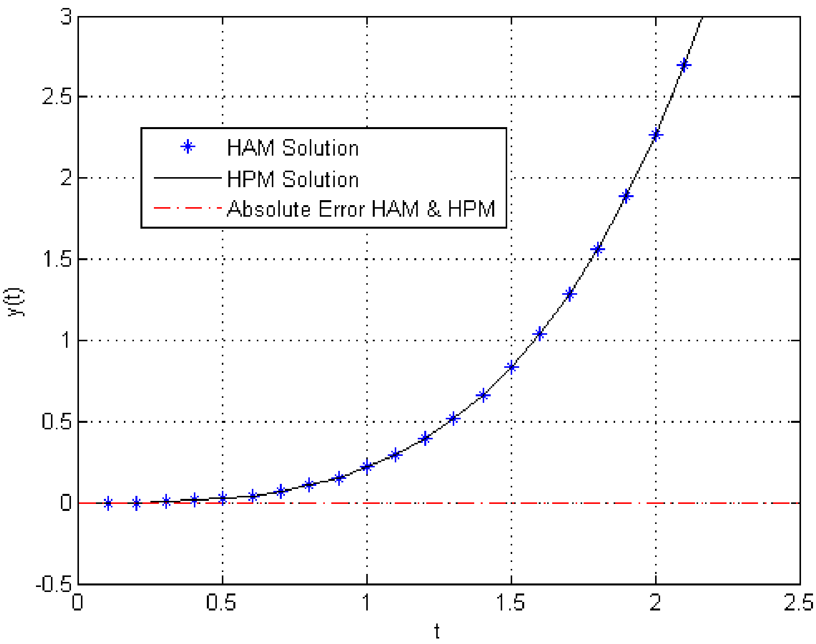

Figure 2: Graph of HAM solution (4 term approximation) at $\hbar = -1$, VIM (4 term approximation), and absolute error, in case $\alpha = 1$.

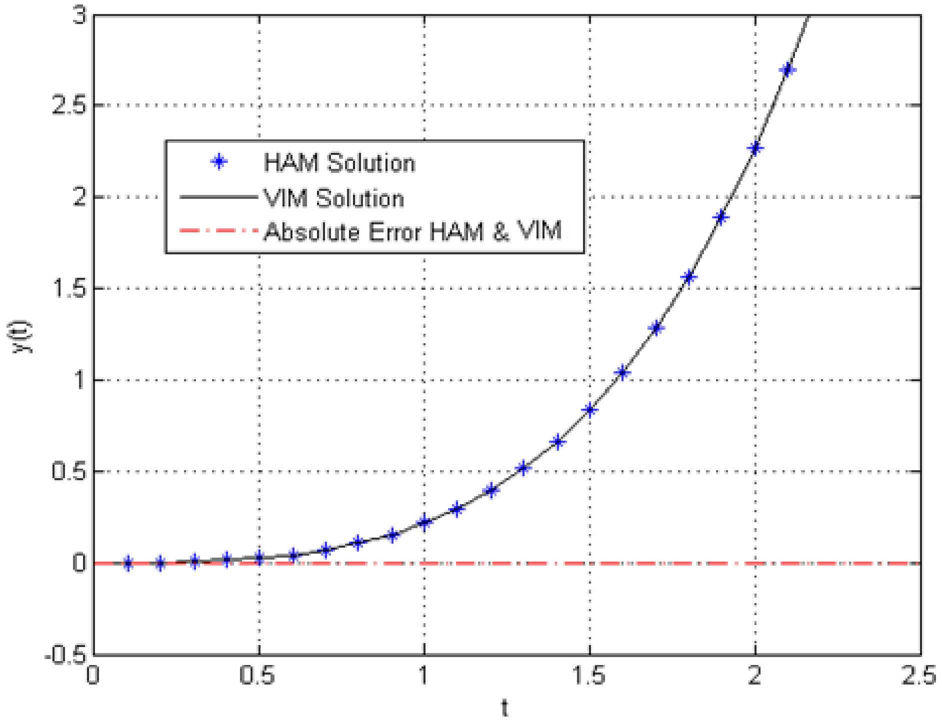

Figure 3: Graph of HAM solution (4 term approximation) at $\hbar = -1$, VIM (4 term approximation), and absolute error, in case $\alpha = 1$.

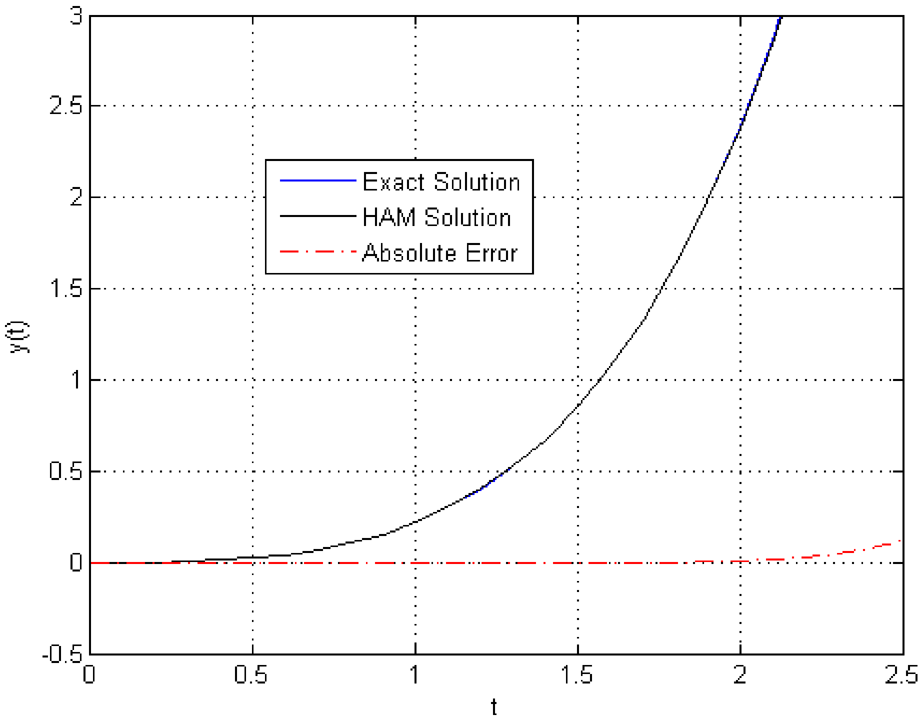

Figure 4: Graph of HAM solution (4 term approximation) at $\hbar = -1.2$, exact solution, and absolute error in case $\alpha = 1$.

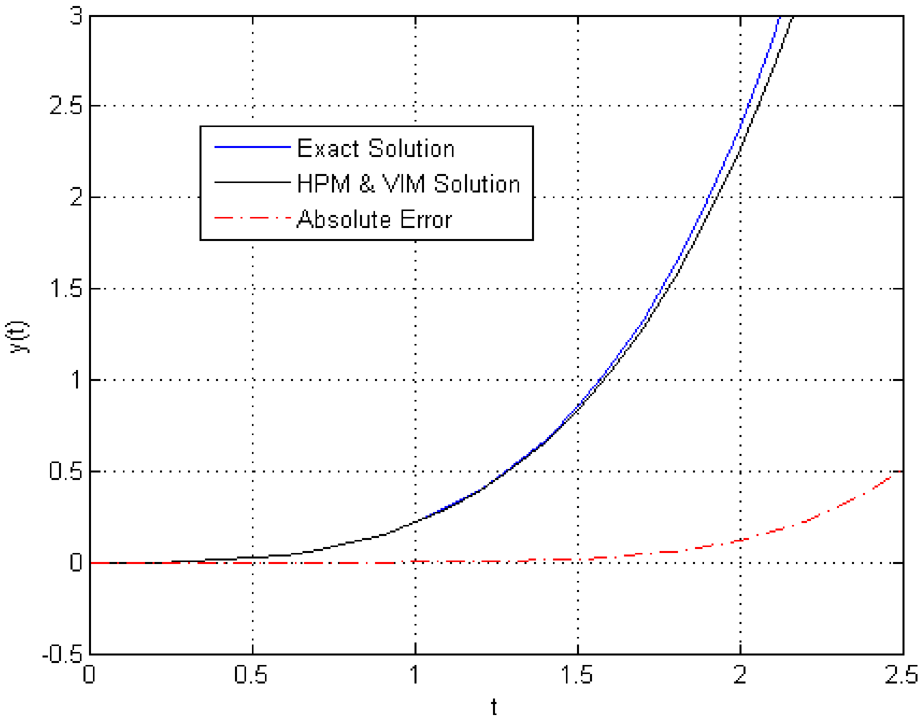

Figure 5: Graph of HPM solution (4 term approximation), VIM solution (4 term approximation) and exact solution, and absolute error in case $\alpha = 1$.

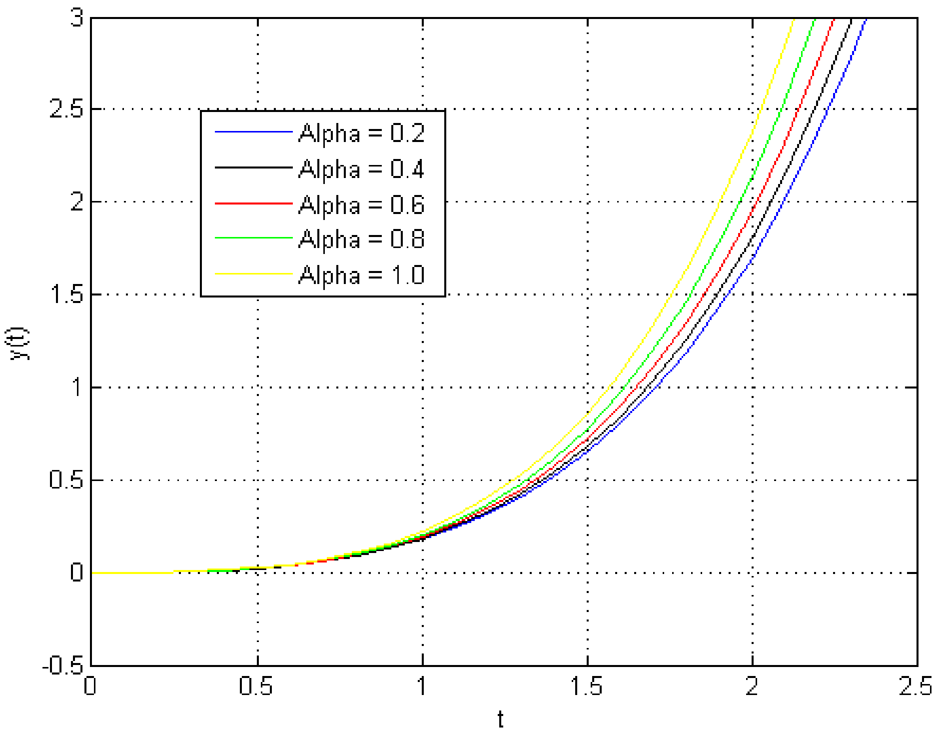

Figure 6: Graph of HAM solution (4 term approximation) at $\hbar = -1.2$, for different values of α .

VII. Conclusion

In this paper, we have successfully applied HAM to obtain the solution of fractional differential equations. The graphs are obtained by using matlab. From the above graphs we observe that HPM solution is in good agreement with the VIM solution and HAM solution. All of these methods gives very good approximations in a few terms. And also through the results we can say, (HPM) and (VIM) are all special cases of the homotopy analysis method (HAM) when the nonzero convergence-control parameter $\hbar = -1$.

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