IOSR Journal of Mathematics (IOSR-JM)
e-ISSN: 2278-5728, p-ISSN: 2319-765X. Volume 16, Issue 4 Ser. II (Jul.–Aug. 2020), PP 35-40
www.iosrjournals.org

Homotopy Analysis Method for Solving Some Partial Time Fractional Differential Equation

Abdallah Habila Ali[1, 2], Abu Bakr Ahmed[2], Hiba Mohamed[2, 3]
[1]Sudan University of Science & Technology, College of Science, Department of Mathematics (Sudan),
[2] AlMughtaribeen University, College of Engineering, Department of General Sciences (Sudan),
[3]Ibn Sina University, Department of Statistics (Sudan).

Abstract: *In this paper, the homotopy analysis method (HAM) is applied to solve a time-fractional nonlinear partial differential equation. The fractional derivatives are described by Caputo's sense, and the (HAM) gives a series of solutions which converge rapidly within a few terms with the help of the nonzero convergence control parameter ℏ. After applying this method we reach the conclusion that the HAM is very efficient and accurate. Graphical representations of the solution obtained..*

Keywords: *Fractional calculus, Homotopy Analysis Method, Time-fractional nonlinear partial differential equation.*

Date of Submission: 14-07-2020 Date of Acceptance: 29-07-2020

I. Introduction

Several methods have been used to solve fractional differential equations, such as Laplace transform method, Fourier transform method [1], Adomian's decomposition method [2], homotopy analysis method [3] and so on. A substantial amount of research work has been directed to the study of the nonlinear fractional heat conduction, Kaup–Kupershmidt, Fisher and Huxley equations respectively. Dehghan et al. [3] have applied homotopy analysis method for solving nonlinear fractional partial differential equations. Wazwaz [4] has investigated exact solitary solutions for the nonlinear equation of heat conduction in two dimensions. Babolian et. al [5] have obtained analytic approximate solutions to a class of nonlinear PDEs such as Burgers, Fisher, Huxley equations and two combined forms of these equations using the homotopy analysis method. Analytic study on Burgers, Fisher, Huxley equations and combined forms of these equations with the tanh-coth method is used to determine these sets of travelling wave solutions by Wazwaz [6]. Ozi¸s et. al [7] has applied Exp-function method for solving the Fisher equation. The homotopy analysis method is a combination of the classical perturbation technique [8–12] and the homotopy, a concept in topology, and does not rely on the small/large parameter. The advantage of the homotopy analysis method over the existing techniques is the excellent freedom of choosing the initial guess and the existence of the so-called nonzero convergence-control parameter. Based on the basic idea of the homotopy analysis method, many numerical and analytical techniques have been proposed. Marinca and Herisa suggested the optimal homotopy analysis method [13] in2008. In 2009, Niu and Wang introduced a one-step optimal homotopy analysis method [14], and the spectral homotopy analysis method based on the Chebyshev pseudo spectral method [15] was proposed by Motsa et al. in 2010. The predictor homotopy analysis method [16] was also suggested in 2010, and recently in 2018 Singh et al. successfully applied the homotopy analysis method and the Sumudu transform method to fractional Drinfeld–Sokolov–Wilson equation [17]. Besides, many authors have discovered that the Adomian decomposition method (ADM), the homotopy perturbation method (HPM), and the variational iteration method (VIM) are all special cases of the homotopy analysis method (HAM) when the nonzero convergence-control parameter $\hbar = -1$ (see [18–21]). In our paper, we have used HAM successfully to find the approximate analytical solutions of linear / nonlinear PDEs with time-fractional derivatives.

The current paper is organized as follows: In section 2, some necessary definitions and mathematical preliminaries of the fractional calculus theory. In section 3, Basic ideas of the homotopy analysis method. In section 4, the proposed method is applied to several examples. Also a conclusion is given in the section 5.

II. Fractional Calculus

In this section**,** we gives some basic definitions and properties of the Fractional calculus.

Definition 1 A real function $f(x), x > 0$, is said to be in the space $C_{\square}, \square \in \mathbb{R}$, if there exists a real number $p > \square$, such that $f(x) = x^p f(x)$, where $f(x) \in C[a, b]$, and it is said to be in the space $C_{\square}^{n}$, if and only if $f^{(n)} \in C_{\square}, n \in \mathbb{N}$.

Definition 2 The Riemann-Liouville fractional integral operator (I^{α}) of order $\alpha \geq 0$, of a function $f(x) \in C_{?}, ? \in \mathbb{N}$.

$$\begin{Bmatrix} I^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_{0}^{t} (x-\tau)^{\alpha-1} f(\tau)\, d\tau \\ I^{0} f(x) = f(x) \end{Bmatrix} \tag{1}$$

Properties of the operator I^{α} can be found in [5-8], we mention only the following:

$$I^{\alpha} I^{\beta} f(x) = I^{\alpha+\beta} f(x) = I^{\beta} I^{\alpha} f(x) \tag{2}$$

$$I^{\alpha} x^{v} = \frac{\Gamma(v+1)}{\Gamma(v+\alpha+1)} x^{v+\alpha} \tag{3}$$

For $f(x) \in C_{?}, ? \in \mathbb{N}, \alpha, \beta \geq 0$ and $v \geq -1$.

Definition 3 Suppose that $\alpha > 0, x > 0, \alpha, x \in \mathbb{R}, n \in \mathbb{N}, n-1 < \alpha \leq n$. The Caputo fractional differential operator of order α define as:

$$D^{\alpha} f(x) = \begin{cases} \frac{1}{\Gamma(n-\alpha)} \int_{0}^{t} (x-\tau)^{n-\alpha-1} f^{(n)}(\tau)\, d\tau \\ \frac{d^{n}}{dx^{n}} f(x) & \alpha = n \in \mathbb{N} \end{cases} \tag{4}$$

According to the Caputo's derivative, we can obtain:

$$D^{\alpha} C = 0, \qquad \text{C is constant,}$$

$$D^{\alpha} x^{v} = \frac{\Gamma(v+1)}{\Gamma(v-\alpha+1)} x^{v-\alpha} \tag{5}$$

Definition 4 The generalized Mittag-Leffler function is defined by the power series:

$$E_{\alpha}(z) = \sum_{k=0}^{\infty} \frac{z^{k}}{\Gamma(\alpha k+1)}, \quad E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^{k}}{\Gamma(\alpha k+\beta)}, \qquad \alpha > 0, \beta \in \mathbb{R} \tag{6}$$

III. Homotopy Analysis Method (HAM)

To describe the basic ideas of the HAM, we consider the following differential equation:

$$N[u(x,t)] = 0, \ t > 0 \tag{7}$$

where N is a nonlinear differential operator, and $u(x,t)$ is unknown function of the independent variables $x \ and \ t$.

Based on the zero-order deformation equation constructed by Liao [5-9], we give the following zero-order deformation equation in the similar way:

$$(1-q)\ell[\varphi(x,t;q) - u_0(x,t)] = q\hbar H(x,t)[N\varphi(x,t;q)], \tag{8}$$

Where $q \in [0,1]$ is an embedding parameter, $\hbar$ are non-zero auxiliary parameter for $H(t)$ are non-zero auxiliary function, N is nonlinear differential operator, $\varphi(x,t;q)$ is an unknown function, and $u_0(x,t)$ is an initial guess of $u(x,t)$, ℓ is an auxiliary linear integral order operator and it possesses the property $\ell(\mathrm{c}) = 0$.

Obviously when $q = 0$ and $q = 1$, we have

$$\varphi(x,t;0) = u_0(x,t) \quad , \qquad \varphi(x,t;1) = u(x,t) \tag{9}$$

Expanding φ in Taylor series with respect to q, one has

$$\varphi(x,t;q) = u_0(x,t) + \sum_{m=1}^{\infty} u_m(x,t) \tag{10}$$

Where

$$u_m(x,t) = \frac{1}{m!} \frac{\partial^m \varphi(x,t;q)}{\partial q^m}\bigg|_{q=0}, \ \ i = 1,2,3,\dots,n \tag{11}$$

Differentiating equation (8) m-times with respect to embedding parameter , then setting $q = 0$, and dividing them by $m!$, we have, using (11), the so-called m$^{\text{th}}$-order deformation equation

$$u_m(x,t) = \chi_m u_{m-1}(x,t) + \hbar I^{\alpha}\left[H(x,t) R_m\left(\overrightarrow{u_{m-1}}(x,t)\right)\right] \tag{12}$$

$$R_m\left(\overrightarrow{u_{m-1}}(x,t)\right) = \frac{1}{(m-1)!} \frac{\partial^{m-1}[N\varphi(x,t;q)]}{\partial q^{m-1}}\bigg|_{q=0} \tag{13}$$

And

$$\chi_m = \begin{cases} 0, \ m \leq 1 \\ 1, \ m > 1 \end{cases} \tag{14}$$

Here $\vec{u}_{m-1} = \{u_{0,}u_1, u_2, \dots, u_{m-1}\}$.

These equations can be easily solved using software such as Maple, Matlab and so on.

IV. An application

Example 1. We consider the one-dimensional linear inhomogeneous fractional Burger equation

$$\frac{\partial^\alpha u}{\partial t^\alpha}+\frac{\partial u}{\partial x}-\frac{\partial^2 u}{\partial x^2}=2x-2+2\frac{t^{2-\alpha}}{\Gamma(3-\alpha)}\,,\quad t>0, x\in\mathbb{R},\qquad 0<\alpha\le 1 \tag{15}$$

subject to initial condition

$$u(x,0)=x^2$$

We can start with an initial approximation $u_0(x,t)=x^2$, and we choose the auxiliary linear operator:

$$\ell = I^\alpha\,,$$

with the property $\ell(c)=0$, where c is an integral constant. We also choose the auxiliary function to be:

$$H(x,t)=1\,,$$

Hence, the mth-order deformation can be given by:

$$u_m(x,t)=\chi_m u_m(x,t)+\hbar I^\alpha[R_m(\overrightarrow{u_{m-1}})],$$

Where

$$R_m(\overrightarrow{u_{m-1}})=\frac{\partial^\alpha u}{\partial t^\alpha}+\frac{\partial u}{\partial x}-\frac{\partial^2 u}{\partial x^2}-\left(2x-2+2\frac{t^{2-\alpha}}{\Gamma(3-\alpha)}\right)(1-\chi_m)\,,$$

Consequently, the first few terms of the HAM series solution are as follows:

$$u_1(x,t)=-\hbar t^2\,,$$

$$u_2(x,t)=-(\hbar+\hbar^2)t^2,$$

$$u_3(x,t)=-(\hbar+2\hbar^2+\hbar^3)t^2$$

And so on, Then n-term approximate solution for (35) is

$$u_{HAM}=u_0(x,t)+\sum_{m=1}^{n-1}u_m(x,t),$$

$$u_{HAM}=x^2-\hbar t^2-(\hbar+\hbar^2)t^2-(\hbar+2\hbar^2+\hbar^3)t^2+\cdots,$$

Hence, the HAM series solution (for $\hbar=-1$) is

$$u_{HAM}=x^2+t^2\,.$$

which is the Analytical solution of the one-dimensional linear inhomogeneous fractional Burger equation.

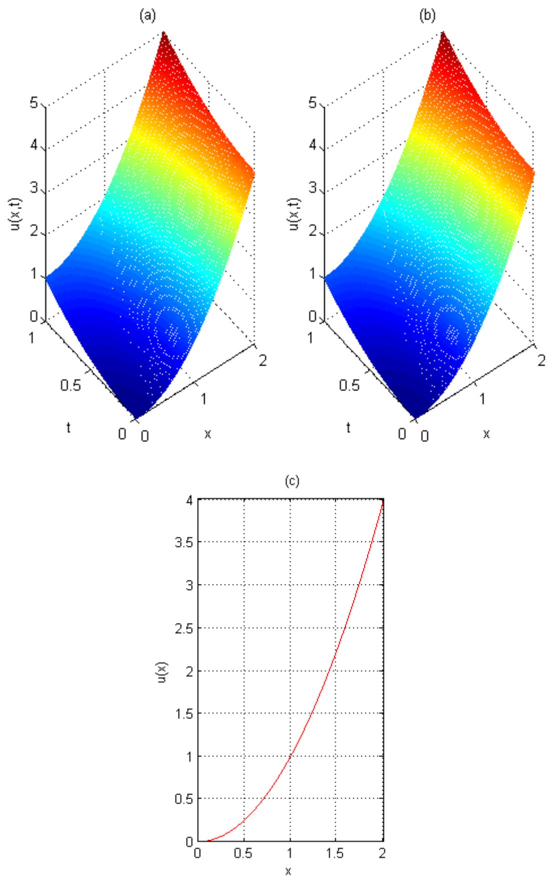

Figure (1): The surface shows the solution $u(x,t)$. (a) Exact Solution (b) HAM (C) Exact solution, in case $t=0$.

Example 4.1 Consider the nonlinear time-fractional gas dynamics [25]

$$D_t^\alpha u + \frac{1}{2}(u^2)_x - u(1-u) = 0, \qquad t > 0, \qquad 0 < \alpha \le 1, \tag{16}$$

with the initial condition:

$$u(x,0) = e^{-x}$$

If we set $\alpha = 1$ then the equation (16) reduces to the classical gas dynamics equation of order one which has

$$u(x,t) = e^{t-x}$$

Since, $N_\alpha[u(x,t)] = D_t^\alpha u + \frac{1}{2}(u^2)_x - u(1-u)$, according to (11) and (13), we have

$$R_m\left(\overrightarrow{u_{m-1}}(x)\right) = D_t^\alpha u_{m-1} + \frac{1}{2}\Big(\sum_{i=0}^{m-1} u_i\, u_{m-1-i}\Big)_x - u_{m-1} + \sum_{i=0}^{m-1} u_i\, u_{m-1-i}$$

If we take the auxiliary function $H(x,t) = 1$, we obtain the mth order deformation equation

$$u_m(x,t) = \chi_m u_{m-1}(x,t) + \hbar I^\alpha\left[R_m\left(\overrightarrow{u_{m-1}}(x,t)\right)\right]$$

subject to the initial conditions

$$u(x,0) = e^{-x},$$

If we choose the initial guess approximation

$$u_0(x,t) = e^{-x}\ ,$$

then we have

$$u_1(x,t) = -\frac{\hbar e^{-x}}{\Gamma(\alpha+1)} t^\alpha\ ,$$

$$u_2(x,t) = -\frac{(\hbar+\hbar^2)e^{-x}}{\Gamma(\alpha+1)} t^\alpha + \frac{\hbar^2 e^{-x}}{\Gamma(2\alpha+1)} t^{2\alpha}\ ,$$

$$u_3(x,t) = -\frac{(\hbar+2\hbar^2+\hbar^3)e^{-x}}{\Gamma(\alpha+1)} t^\alpha + 2\frac{(\hbar+\hbar^2)e^{-x}}{\Gamma(2\alpha+1)} t^{2\alpha} - \frac{\hbar^3 e^{-x}}{\Gamma(3\alpha+1)} t^{3\alpha}\ .$$

and so on, the 4th order approximation $u(x,t)$ is given by

$$u_{HAM}(x,t) = u_0(x,t) + \sum_{m=1}^{3} u_{\mathrm{m}}(x,t)$$

$$u_{HAM}(x,t) = e^{-x} - \frac{\hbar e^{-x}}{\Gamma(\alpha+1)} t^\alpha - \frac{(\hbar+\hbar^2)e^{-x}}{\Gamma(\alpha+1)} t^\alpha + \frac{\hbar^2 e^{-x}}{\Gamma(2\alpha+1)} t^{2\alpha} - \frac{(\hbar+2\hbar^2+\hbar^3)e^{-x}}{\Gamma(\alpha+1)} t^\alpha + 2\frac{(\hbar+\hbar^2)e^{-x}}{\Gamma(2\alpha+1)} t^{2\alpha} - \frac{\hbar^3 e^{-x}}{\Gamma(3\alpha+1)} t^{3\alpha}.$$

Finally, if we take $\hbar = -1$, then the 4th approximation solution of this problem can be obtained as:

$$u_{HAM}(x,t) = e^{-x}\left[1 + \frac{1}{\Gamma(\alpha+1)} t^\alpha + \frac{1}{\Gamma(2\alpha+1)} t^{2\alpha} + \frac{1}{\Gamma(3\alpha+1)} t^{3\alpha} + \cdots\right]$$

Thus, we have

$$u_{HAM}(x,t) = e^{-x} E_\alpha(t^\alpha). \tag{17}$$

Where $E_\alpha(z)$, is Mittag-Leffler function in one parameter. Equation (6), in particular if $\alpha = 1$ then the equation (17) becomes,

$$u_{HAM}(x,t) = e^{-x} E_1(t) = e^{-x} e^t = e^{t-x}$$

Which is the same as given by HPM, VISTM and HPSTM (see [23 - 25]). The solution by graphical of equation (16) obtained by using HAM for difference values of given by

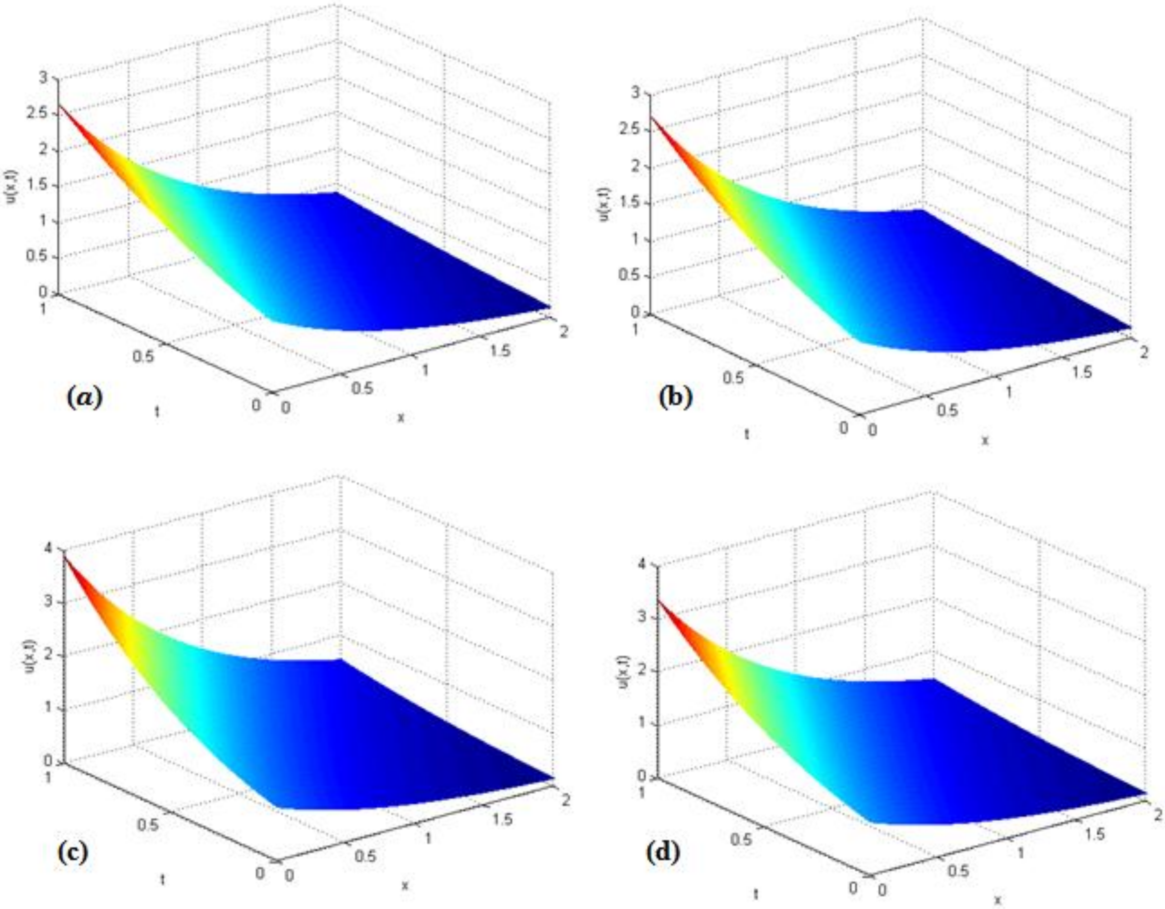

Figure (2): (a) the surface shows the Solution $u(x,t)$ by HAM (4 term approximation) of equation (15), for $\alpha = 1$. (b) Exact Solution for $\alpha = 1$. (c) The surface shows the Solution $u(x,t)$ of equation (16), for $\alpha = 0.5$ (d) The surface shows the Solution $u(x,t)$ of equation (1), for $\alpha = 0.7$

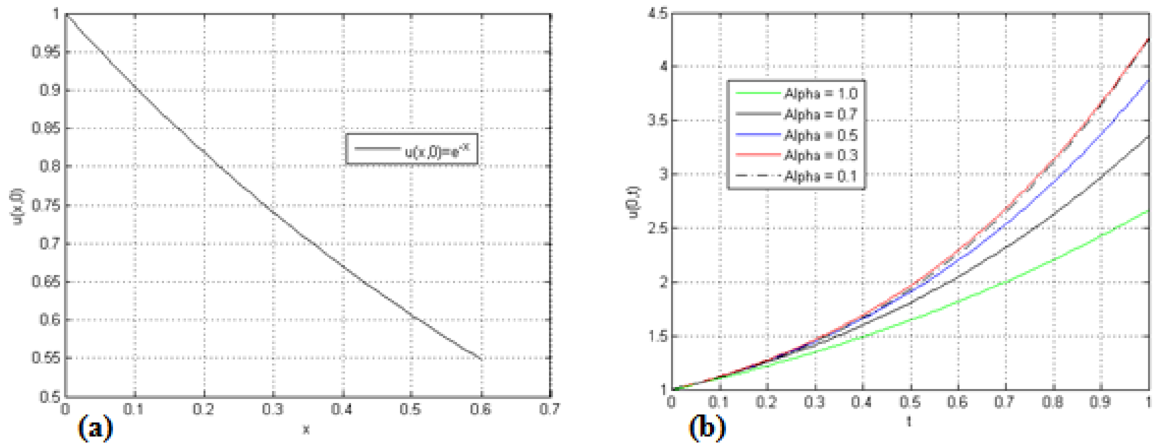

Figure (3): (a) Plots of the solution $u(x,t)$, in case $t = 0$. (b) Plots of the solution $u(x,t)$ for different values of α, in case $x = 0$.

V. Conclusion

In this paper, the Homotopy Analysis Method (HAM) has been successfully applied to obtain the exact solutions for solving fractional nonlinear gas dynamics equation and time fractional Fornberg-Whitham equation. It is easy to see that Homotopy Analysis Method (HAM) is very powerful, and professional techniques for solving different kinds of linear and nonlinear fractional differential equations arising in different fields of science and engineering. Furthermore, this method does not require any transformation technique, linearization, or discretization of the variables and it does not make closure approximation or smallness assumption. The fractional derivatives are described by Caputo’s sense.

References

[1]. S. Kemple, H. Beyer, Global and causal solutions of fractional differential equations, Transform methods and special functions: Varna 96, Proceedings of 2nd international workshop (SCTP), Singapore, 96 (1997) 210-216.

[2]. S. Momani, NT. Shawagfeh, Decomposition method for solving fractional Riccati differential equations,Appl. Math. Comput, 182 (2006) 1083-1092.

[3]. M. Dehghan, J. Manafian, A. Saadatmandi, Solving nonlinear fractional partial differential equations using the homotopy analysis method, Num. Meth. Par. Diff. Equ. J, 26 (2010) 448-479.

[4]. A. M. Wazwaz, The tanh method for generalized forms of nonlinear heat conduction and Burgers- Fisher equations, Appl. Math. Comput, 169 (2005) 321 -338.

[5]. E. Babolian, J. Saeidian, Analytic approximate solutions to Burgers, Fisher, Huxley equations and two combined forms of these equations, Commu. Nonlinear Sci. Num. Simu, 14 (2009) 1984-1992.

[6]. A. M. Wazwaz, Analytic study on Burgers, Fisher, Huxley equations and combined forms of these equations, Appl. Math. Comput, 195 (2008) 754-761.

[7]. T. ¨ Ozi¸s, C. K¨oroˇglu, A novel approach for solving the Fisher equation using Exp-function method, Phys. Let. A, 372 (2008) 3836-3840.

[8]. Liao, S.J.: Beyond Perturbation: Introduction to the Homotopy Analysis Method. CRC Press, Boca Raton (2003).

[9]. Liao, S.J.: Comparison between the homotopy analysis method and homotopy perturbation method. Appl. Math. Comput. 169(2), 1186–1194 (2005).

[10]. Liao, S.J.: An optimal homotopy-analysis approach for strongly nonlinear differential equations. Commun. Nonlinear Sci. Numer. Simul. 362, 2003–2016 (2010).

[11]. Liao, S.J.: Homotopy Analysis Method in Nonlinear Differential Equations. Springer, Heidelberg (2012).

[12]. Marinca, V., Herisanu, N.: Application of optimal homotopy asymptotic method for solving nonlinear equations arising in heat transfer. Int. Commun. Heat Mass Transf. 35, 710–715 (2008).

[13]. Niu, Z., Wang, C.: A one-step optimal homotopy analysis method for nonlinear differential equations. Commun. Nonlinear Sci. Numer. Simul. 15, 2026–2036 (2010).

[14]. Motsa, S.S., Sibanda, P., Shateyi, S.: A new spectral homotopy analysis method for solving a nonlinear second order BVP. Commun. Nonlinear Sci. Numer. Simul. 15, 2293–2302 (2010).

[15]. Abbabandy, S., Magyari, E., Shivanian, E.: The homotopy analysis method for multiple solutions of nonlinear boundary value problems. Commun. Nonlinear Sci. Numer. Simul. 14, 3530–3536 (2009).

[16]. Singh, J., Kumar, D., Baleanu, D., Rathore, S.: An efficient numerical algorithm for the fractional Drinfeld–Sokolov–Wilson equation. Appl. Math. Comput. 335, 12–24 (2018).

[17]. Songxin, L., Jeffrey, D.J.: Comparison of homotopy analysis method and homotopy perturbation method through an evolution equation. Commun. Nonlinear Sci. Numer. Simul. 14(12), 4057–4064 (2009).

[18]. Bataineh, A.S., Noorani, M.S.M., Hashim, I.: Solutions of time-dependent Emden–Fowler type equations by homotopy analysis method. Phys. Lett. A 371, 72–82 (2007).

[19]. Van Gorder, R.A., Vajravelu, K.: Analytic and numerical solutions to the Lane Emden equation. Phys. Lett. A 372, 6060–6065 (2008).

[20]. Hayat, T., Sajid, M.: On analytic solution for thin film flow of a fourth grade fluid down a vertical cylinder. Phys. Lett. A 361, 316–322 (2007).

[21]. Sajid, M., Hayat, T.: Comparison of HAM and HPM methods in nonlinear heat conduction and convection equations. Nonlinear Anal. (B) 9, 2290–2295 (2008).

[22]. S. Das, and R. Kumar, “Approximate analytical solutions of fractional gas dynamics equations,” Applied Mathematics and Computation, vol. 217, pp. 9905-9915, 2011.

[23]. H. Jafari , M. Zabihi and M. Saidy,"Application of Homotopy-perturbation method for Solving gas dynamics equation," Applied Mathematical Sciences, Vol. 2, no. 48, 2393 – 2396,2008.

[24]. J Singh, D Kumar, and A. Kılıçman," Homotopy perturbation method for fractional gas dynamics equation using sumudu transform," Hindawi Publishing Corporation, Vol.8, 2013.

[25]. A. S. Abedl-Rady, S. Z. Rida, A. A. M. Arafa, H. R. Abedl-Rahim, Variational Iteration Sumudu Transform Method for Solving Fractional Nonlinear Gas Dynamics Equation, Volume 1, Issue 9, December 2014, PP 82-90

[26]. R.Aruldoss and G.Jasmine, Numerical Solutions of Time Fractional Nonlinear Partial Differential Equations Using Yang Transform Combined with Variational Iteration Method, ISSN 0973-1768 Volume 16, Number 2 (2020), pp. 249-260

Abdallah Habila Ali, et. al. "Homotopy Analysis Method for Solving Some Partial Time Fractional Differential Equation." *IOSR Journal of Mathematics (IOSR-JM)*, 16(4), (2020): pp. 35-40.