IOSR Journal of Mathematics (IOSR-JM)
e-ISSN: 2278-5728, p-ISSN: 2319-765X. Volume 16, Issue 4 Ser. II (Jul.–Aug. 2020), PP 63-67
www.iosrjournals.org

Weak separation axioms in terms of R-I-open sets

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Abstract: *Throughout this paper, we study some weak separation axioms in ideal topological spaces using R-I-open sets and certain properties of the same.*
2010 Mathematics Subject Classification: 54A05
Key Word: *R-I-R_0 space, R-I-R_1 space.*

Date of Submission: 14-07-2020 Date of Acceptance: 29-07-2020

I. Introduction

The concept of ideal in topological space was first introduced by Kuratowski and Vaidyanathswamy [8]. They have also defined local function in ideal topological space. Further Hamlett and Jankovic in [2] studied the properties of ideal topological spaces. The notion of R_0 topological spaces was introduced by Shanin [7] in 1943. In 1961, Davis [1] studied some properties of the same and also introduced the notion of R_1 topological space. Further investigations of the properties of R_0 topological spaces were carried out by many topologists as in [3, 5, 9]. In this paper we define weak separation axioms using the notion of R-I-open sets and certain of its properties.

II. Preliminaries

By a space (X,τ), we mean a topological space with a topology τ defined on X on which no separation axioms are assumed unless otherwise explicitly stated. For a given point x in a space (X,τ), the system of open neighborhoods of x is denoted by $N(x) = \{U \in \tau : x \in U\}$. For a given subset A of a space (X,τ), $Cl(A)$ and $Int(A)$ are used to denote the closure of A and interior of A, respectively, with respect to the topology.

A nonempty collection of subsets of a set X is said to be an ideal I on X, if it satisfies the following two conditions: (i) If $A \in I$ and $B \subset A$, then $B \in I$; (ii) If $A \in I$ and $B \in I$, then $A \cup B \in I$. An ideal topological space (or ideal space) (X,τ,I) means a topological space (X,τ) with an ideal I defined on X. Let (X,τ) be a topological space with an ideal I defined on X. Then for any subset A of X, $A^*(I,\tau) = \{x \in X / A \cap U \notin I \; for\; every\; U \in N(x)\}$ is called the local function of A with respect to I and τ . If there is no ambiguity, we will write $A^*(I)$ or simply A^* for $A^*(I,\tau)$. Also, $Cl^*(A) = A \cup A^*$.It defines a Kuratowski closure operator for the topology $\tau^*(I)$ (or simply τ^*) which is finer than τ. [2, 4, 6]

A subset A of an ideal topological space (X,τ,I) is said to be R-I-open (resp. regular open) if $Int(Cl^*(A)) = A \; (resp. Int(Cl(A)) = A)$. We call a subset A of (X,τ,I) is R-I-closed if its complement is R-I-open. The intersection of all R-I-closed sets containing A is called the R-I-closure of A and is denoted by $R-I-Cl(A)$. The R-I-interior of A is defined by the union of all R-I-open sets contained in A and is denoted by $R-I-Int(A)$. The family of all R-I-open (resp. R-I-closed) sets of (X,τ,I) containing a point $x \in X$ is denoted by $RIO(X,x)$ (resp. $RIC(X,x)$). A subset N of X is called an R-I-nbd of a point $x \in X$ if there exists an R-I-open set U of (X,τ,I) such that $x \in U \subset N$.

Definition 2.1.[1] A topological space (X,τ,I) is said to be:
(i) R_0 if every open set contains the closure of each of its singletons.
(ii) R_1 if for $x,y \in X$ with $Cl(\{x\}) \neq Cl(\{y\})$ there exists disjoint open sets U,V such that $Cl(\{x\}) \subset U \; and \; Cl(\{y\}) \subset V$.

III. R-I-R0 spaces

Definition 3.1. A topological space (X,τ,I) is said to be:
(i) R-I-T_0 if for each pair of distinct points x and y in X, there exists R-I-open set containing x but not y.
(ii) R-I-T_1 if for each pair of distinct points x and y in X, there exist R-I-open sets U and V of X, such that $x \in U,\; y \notin U$ and $y \in V,\; x \notin V$.
(iii) R-I-T_2 if for each pair of distinct points x and y in X, there exist disjoint R-I-open sets U and V in X such that $x \in U$ and $y \in V$.

Definition 3.2. Let (X,τ,I) be an ideal topological space and $A \subset X$. The R-I-kernel of A is denoted by $I_R Ker(A)$ and is defined to be the set $I_R Ker(A) = \cap \{G \in RIO(X) : A \subset G\}$.

Lemma 3.3. For subsets A, B of an ideal topological space (X, τ, I), the following properties hold:

1. $A \subset I_R Ker(A)$.
2. If $A \subset B$, then $I_R Ker(A) \subset I_R Ker(B)$.
3. If A is R-I-open, then $I_R Ker(A) = A$.
4. $x \in I_R Ker(A)$if and only if $A \cap D \neq \varphi$ for any R-I-closed set D of X such that $x \in D$.

Theorem 3.4. Let (X,τ,I) be an ideal topological space and $x,y \in X$. Then $y \in I_R Ker(\{x\})$ if and only if $x \in R - I - Cl(\{y\})$.

Proof. Let $x,\ y \in X.$ Suppose $y \notin I_R Ker(\{x\})$. Then there exists $U \in RIO(X,x)$ such that $y \notin U$. So $X - U$ is a R-I-closed set containing y but not x. Therefore $x \notin R - I - Cl(\{y\})$.

Conversely suppose $x \notin R - I - Cl(\{y\})$. Then there exists $V \in RIC(X,y)$ such that $x \notin V$. So $X - V$ is a R-I-open set containing x but not y. Hence $y \notin I_R Ker(\{x\})$.

Theorem 3.5. Let (X,τ,I) be an ideal topological space and S a subset of X. Then$I_R Ker(S) = \{x \in X / \ R - I - Cl(\{x\}) \cap S \neq \varphi\}$.

Proof. Let $S \subset X$ and let $x \in I_R Ker(S)$. Suppose $S \cap R - I - Cl(\{x\}) = \varphi$. Hence $X - (R - I - Cl(\{x\}))$ is an R-I-open set not containing x. But $S \subset X - (R - I - Cl(\{x\}))$. This implies$x \notin I_R Ker(S)$, which is a contradiction. Hence $S \cap R - I - Cl(\{x\}) \neq \varphi$. Now suppose $x \in X$ and $S \cap R - I - Cl(\{x\}) \neq \varphi$ and suppose that $x \notin I_R Ker(S)$. Then there exists an R-I-open set U such that $S \subset U$ and $x \notin U$. Let $y \in S \cap R - I - Cl(\{x\})$. Thus $y \in S \subset U\ or\ y \in U$ and so U is a R-I-nbd of y and $x \notin U$. But this will make a contradiction that $y \in R - I - Cl(\{x\}) \subset X - U$. Hence the proof.

Definition 3.6. An ideal topological space (X,τ,I) is called an R-I-R_0 space if every R-I-open set contains the R-I-closure of each of its singletons.

Theorem 3.7. Let (X,τ,I) be an ideal topological space. Then X is R-I-T_1 if and only if it is R-I-T_0 and R-I-R_0.

Proof. Let X be a R-I-T_1 space. Then clearly X is a R-I-T_0 space and also X is a R-I-R_0 space. Conversely, let X be both R-I-T_0 and R-I-R_0. Let $x,\ y$ be any two distinct points of X. Since X is R-I-T_0, there exists a R-I-open set U such that $x \in U$ and $y \notin U$ or there exists a R-I-open set V such that $y \in V$ and $x \notin V$. Since X is R-I-R_0, then $R - I - Cl(\{x\}) \subset U$ for $x \in U$. Since $y \notin U,\ y \notin R - I - Cl(\{x\})$. So$y \in X - (R - I - Cl(\{x\})) = W$, say. Thus U and W are R-I-open sets containing x and y respectively. Also $x \notin W$ and $y \notin U$. Hence X is R-I-T_1.

Remark 3.8. Every R-I-T_1 space is R-I-R_0 space, since in R-I-T_1space every singletonis R-I-closed. The converse is not true in general.

Example 3.9. Let $X = \{a,b,c\}$, $\tau = \{\varphi, X, \{a,b\}, \{c\}\}$, $I = \{\varphi, \{a,b\}, \{a\}, \{b\}\}$. (X,τ,I) is R-I-R_0 but not R-I-T_1.

Example 3.10. Let $X = \{a,b,c\}$, $\tau = \{\varphi, X, \{a,b\}\}$, $I = \{\varphi, \{a,b\}\}$. (X,τ,I) is not R-I-R_0 and not R-I-T_0.

Remark 3.11. R-I-T_0 and R-I-R_0 are independent, which is clear from the above two examples.

Theorem 3.12. Let (X,τ,I) be an ideal topological space. Then for $x,\ y\ in\ X$, $I_R Ker(\{x\}) \neq I_R Ker(\{y\}) \Leftarrow \Rightarrow R - I - Cl(\{x\}) \neq R - I - Cl(\{y\})$.

Proof. Let $I_R Ker(\{x\}) \neq I_R Ker(\{y\})$. Then there exists $z \in X$ such that $z \in I_R Ker(\{x\})$ and$z \notin I_R Ker(\{y\})$. Also, by theorem 3.4, $y \notin R - I - Cl(\{z\})$ and $x \in R - I - Cl(\{z\})$. So$R - I - Cl(\{x\}) \subset R - I - Cl(\{z\})$ and so $y \notin R - I - Cl(\{x\})$. Hence $R - I - Cl(\{x\}) \neq R - I - Cl(\{y\})$.

Now, let $R - I - Cl(\{x\}) \neq R - I - Cl(\{y\})$. Then there exists $z \in X$ such that$z \in R - I - Cl(\{x\})$and $z \notin R - I - Cl(\{y\})$. This implies there exists a R-I-open set containing z and x but not y. So $y \notin I_R Ker(\{x\})$. Hence$I_R Ker(\{x\}) \neq I_R Ker(\{y\})$.

Theorem 3.13. Let (X,τ,I) be an ideal topological space. Then the following are equivalent.

(i) (X,τ,I) is a R-I-R_0 space.

(ii) For any $P \in RIC(X),\ x \notin P$ implies $P \subset U$ and $x \notin U$ for some $U \in RIO(X)$.

(iii) For any $P \in RIC(X),\ x \notin P$ implies $P \cap R - I - Cl(\{x\}) = \varphi$.

(iv) For any distinct points x and y of X, $R - I - Cl(\{x\}) = R - I - Cl(\{y\})$ or $R - I - Cl(\{x\}) \cap R - I - Cl(\{y\}) = \varphi$.

Proof. (i) ⇒ (ii)

Let P be a R-I-closed set of X and $x \notin P$. Since X is R-I-R_0, $R - I - Cl(\{x\}) \subset X - P$. Denote$U = X - (R - I - Cl(\{x\}))$. Then U is a R-I-open set and $P \subset U$ and $x \notin U$.

(ii) ⇒ (iii)

Let $P \in RIC(X)$ and $x \in P$. Then there exists $U \in RIO(X)$ such that $P \subset U$ and $x \notin U$. Since$U \in RIO(X),\ U \cap R - I - Cl(\{x\}) = \varphi$ and so $P \cap R - I - Cl(\{x\}) = \varphi$.

(iii) ⇒ (iv)

Suppose $R - I - Cl(\{x\}) \neq R - I - Cl(\{y\})$ for $x \neq y \in X$. Then there exists $z \in X$ with$z \in R - I - Cl(\{x\})$ and $z \notin R - I - Cl(\{y\})$. So there exists a R-I-open set V in X such that $y \notin V$ and $z \in V$ and so $x \in V$. Also, we get, $x \notin R - I - Cl(\{y\})$. Hence $R - I - Cl(\{x\}) \cap R - I - Cl(\{y\}) = \varphi$. The other statement can be proved in a similar way.

(iv) ⇒ (i)

Let V be a R-I-open set in X. For each $y \notin V$,$x \neq y$ and $x \notin R - I - Cl(\{y\})$. So, by (iv),$R - I - Cl(\{x\}) \cap R - I - Cl(\{y\}) = \varphi$for each $y \notin V$. Hence $(R - I - Cl(\{x\})) \cap (\bigcup_{y \in X-V} R - I - Cl(\{y\})) = \varphi$. Since V is R-I-open and$y \in X - V$, $R - I - Cl(\{y\}) \subset X - V$ and so $X - V = y \in X - V R - I - Cl(\{y\})$. Therefore $(X - V) \cap R - I - Cl(\{x\}) = \varphi$ or $R - I - Cl(\{x\}) \subset V$. Hence (X, τ, I) is a R-I-R0 space.

Theorem 3.14. An ideal topological space (X, τ, I) is R-I-R_0 if and only if for any two points $x,\ y \in X$,$R - I - Cl(\{x\}) \neq R - I - Cl(\{y\})$implies $R - I - Cl(\{x\}) \cap R - I - Cl(\{y\}) = \varphi$.

Proof. Let (X, τ, I) be R-I-R_0. Then by theorem 3.13, the statement holds. Conversely let U be a R-I-open set of X containing x. We claim $R - I - Cl(\{x\}) \subset U$. For that let $y \in X - U$. So, $x \notin R - I - Cl(\{y\})$. This implies $R - I - Cl(\{x\}) \neq R - I - Cl(\{y\})$. By assumption,$R - I - Cl(\{x\}) \cap R - I - Cl(\{y\}) = \varphi$. Thus $y \notin R - I - Cl(\{x\})$ and hence the claim.

Theorem 3.15. An ideal topological space (X, τ, I) is R-I-R_0 if and only if for any two points $x, y \in X$, $I_R Ker(\{x\}) \neq I_R Ker(\{y\})$implies $I_R Ker(\{x\}) \cap I_R Ker(\{y\}) = \varphi$.

Proof. Let (X, τ, I) be R-I-R_0. By theorem 3.12, for any two points $x, y \in X$, if $I_R Ker(\{x\}) \neq I_R Ker(\{y\})$, then $R - I - Cl(\{x\}) \neq R - I - Cl(\{y\})$. Assume the contrary and suppose$z \in I_R Ker(\{x\}) \cap I_R Ker(\{y\})$. Since $z \in I_R Ker(\{x\})$, $x \in R - I - Cl(\{z\})$. Also, $x \in R - I - Cl(\{x\})$. Then by theorem 3.13(iv), $R - I - Cl(\{x\}) = R - I - Cl(\{z\})$. Thus, in a similar way, we get $R - I - Cl(\{x\}) = R - I - Cl(\{z\}) = R - I - Cl(\{y\})$. From this contradiction, we have$I_R Ker(\{x\}) \cap I_R Ker(\{y\}) = \varphi$.

Now assume the converse. By theorem 3.12, if $R - I - Cl(\{x\}) \neq R - I - Cl(\{y\})$, then $I_R Ker(\{x\}) \neq I_R Ker(\{y\})$. So by assumption, $I_R Ker(\{x\}) \cap I_R Ker(\{y\}) = \varphi$. Now let $z \in R - I - Cl(\{x\})$ and this implies $x \in I_R Ker(\{z\})$. Therefore $I_R Ker(\{x\}) \cap I_R Ker(\{y\}) \neq \varphi$. Then by hypothesis, $I_R Ker(\{x\}) = I_R Ker(\{z\})$. So $z \in R - I - Cl(\{x\}) \cap R - I - Cl(\{y\})$ will imply that $I_R Ker(\{x\}) = I_R Ker(\{z\}) = I_R Ker(\{y\})$. From this contradiction, we get $R - I - Cl(\{x\}) \cap R - I - Cl(\{y\}) = \varphi$. Thus, by theorem 3.14, (X, τ, I) is a R-I-R_0 space.

Theorem 3.16. Let (X, τ, I) be a R-I-R_0 space. Then $x \in R - I - Cl(\{y\})$ if and only if$y \in R - I - Cl(\{x\})$ for any $x, y \in X$. The converse is also true.

Proof. Let (X, τ, I) be R-I-R_0. Let $x \in R - I - Cl(\{y\})$ and let U be a R-I-open set of X containing y. Then $R - I - Cl(\{y\}) \subset U$. So, by hypothesis, $x \in R - I - Cl(\{y\})$ implies $x \in U$. That means, every R-I-open set containing y contains x. Hence $y \in R - I - Cl(\{x\})$.

Assume the converse. Let U be a R-I-open set in X containing x. If $y \notin U$, then $x \notin R - I - Cl(\{y\})$ and hence $y \notin R - I - Cl(\{x\})$. This means $R - I - Cl(\{x\}) \subset U$. Then (X, τ, I) is R-I-R_0.

Theorem 3.17. Let (X, τ, I) be an ideal topological space. Then the following are equivalent:

(i) (X, τ, I) is a R-I-R_0 space.

(ii) For any $\varphi \neq P \in X$ and $U \in RIO(X)$ with $P \cap U \neq \varphi$, there exists $V \in RIC(X)$ such that $P \cap V \neq \varphi$ and $V \subset U$.

(iii) For any $U \in RIO(X)$, $U = \cup\{V \in RIC(X) : V \subset U\}$.

(iv) For any $V \in RIC(X)$, $V = \cap\{U \in RIO(X) : V \subset U\}$.

(v) For any $x \in X$, $R - I - Cl(\{x\}) \subset I_R Ker(\{x\})$.

Proof. (i) ⇒ (ii)

Let $\varphi \neq P \in X$ and U be an R-I-open set with $P \cap U \neq \varphi$. Let $x \in P \cap U$. Since $x \in U$ and X is R-I-R_0,$R - I - Cl(\{x\}) \subset U$. Let $V = R - I - Cl(\{x\})$. Then $V \in RIC(X)$ and $V \subset U$ and $P \cap V \neq \varphi$.

(ii) ⇒ (iii)

Let $U \in RIO(X)$. Then clearly $\cup\{V \in RIC(X) : V \subset U\} \subset U$. Now let $x \in U$. Then there exists $V \in RIC(X)$ such that $x \in V \subset U$ by (ii). Thus $x \in V \subset \cup\{V \in RIC(X) : V \subset U\}$. Hence$U = \cup\{V \in RIC(X) : V \subset U\}$.

(iii) ⇒ (iv)

Let $V \in RIC(X)$. Consider all R-I-open sets containing V. Then $\cap\{U \in RIO(X) : V \subset U\} \subset V$. Now let $x \in V$. Then $x \in U$ for all $U \in RIO(X)$ with $V \subset U$. So $x \in \cap\{U \in RIO(X) : V \subset U\}$. Thus $V = \cap\{U \in RIO(X) : V \subset U\}$.

(iv) ⇒ (v)

Let $x \in X$ and let $y \in I_R Ker(\{x\})$. Then there exists $G \in RIO(X)$ such that $x \in G$ and $y \notin G$. So $R - I - Cl(\{y\}) \cap G = \varphi$. Then by (iv), $(\cap\{U \in RIO(X) : R - I - Cl(\{y\}) \subset U\}) \cap G = \varphi$. So there exists

an R-I-open set U such that $x \notin U$ and $R - I - Cl(\{y\}) \subset U$. Hence$R - I - Cl(\{x\}) \cap U = \varphi$ and $y \notin R - I - Cl(\{x\})$. Thus $R - I - Cl(\{x\}) \subset I_R Ker(\{x\})$.
(v) ⇒ (i)
Let U be an R-I-open set in X and $x \in U$. Let $y \in I_R Ker(\{x\})$. Then $x \in R - I - Cl(\{y\})$. Also $y \in U$. Then $I_R Ker(\{x\}) \subset U$. Thus $x \in R - I - Cl(\{x\}) \subset I_R Ker(\{x\}) \subset U$. Hence X is a R-I-R_0 space.
Theorem 3.18. Let (X, τ, I) be an ideal topological space. Then the following are equivalent:
(i) (X, τ, I) is a R-I-R_0 space.
(ii) If V is a R-I-closed subset of X, then $V = I_R Ker(V)$.
(iii) If V is a R-I-closed subset of X and $x \in V$, *then* $I_R Ker(\{x\}) \subset V$.
(iv) If $x \in X$, then $I_R Ker(\{x\}) \subset R - I - Cl(\{x\})$.
Proof. (i) ⇒ (ii)
Let V be a R-I-closed subset of X and let $x \in X - V$. Since X is a R-I-R_0 space and $X - V \in RIO(X, x)$, $R - I - Cl(\{x\}) \subset X - V$. By theorem 3.5, $I_R Ker(V) \subset X - (R - I - Cl(\{x\}))$. Also $x \notin I_R Ker(V)$. Thus$I_R Ker(V) = V$.
(ii) ⇒ (iii)
Since $U \subset V$ implies $I_R Ker(U) \subset I_R Ker(V)$, it follows that $I_R Ker(\{x\}) \subset I_R Ker(V)$ for $x \in V$. Therefore $I_R Ker(\{x\}) \subset V$ from (ii).
(iii) ⇒ (iv)
Let $x \in X$ and clearly $x \in R - I - Cl(\{x\})$. From (iii) $I_R Ker(\{x\}) \subset R - I - Cl(\{x\})$.
(iv) ⇒ (i)
Let $x \in R - I - Cl(\{y\})$. Then by theorem 3.4, $y \in I_R Ker(\{x\})$. Thus we get$y \in I_R Ker(\{x\}) \subset R - I - Cl(\{x\})$ by (iv). Hence $x \in R - I - Cl(\{y\})$ implies $y \in R - I - Cl(\{x\})$. Clearly the reverse implication holds. Thus, by theorem 3.16, X is a R-I-R_0 space.
Corollary 3.19. Let (X, τ, I) be an ideal topological space. If (X, τ, I) is R-I-R_0, then $I_R Ker(\{x\}) = R - I - Cl(\{x\})$ for all $x \in X$. The converse is also true.
Proof. Suppose (X, τ, I) is a R-I-R_0 space. By theorem 3.17 and theorem 3.18 the statement is obvious. The converse is trivial by theorem 3.18.

IV. R-I-R1 spaces

Definition 4.1. An ideal topological space (X, τ, I) is called an R-I-R_1 space if for $x, y \in X$ with$R - I - Cl(\{x\}) \neq R - I - Cl(\{y\})$ there exist disjoint R-I-open sets U, V such that $R - I - Cl(\{x\}) \subset U$ *and* $R - I - Cl(\{y\}) \subset V$.
Theorem 4.2. Every R-I-R_1 space is R-I-R_0.
Proof. Let (X, τ, I) be a R-I-R_1 space and $x, y \in X$. Let U be a R-I-open set containing x but not y. So,$x \notin R - I - Cl(\{y\})$. Then we have $R - I - Cl(\{x\}) \neq R - I - Cl(\{y\})$. By hypothesis, then there exists R-I-open set V such that $R - I - Cl(\{y\}) \subset V$. Therefore $x \notin V$ and this implies $y \notin R - I - Cl(\{x\})$. Thus $R - I - Cl(\{x\}) \subset U$. Hence (X, τ, I) is R-I-R_0.
Remark 4.3. The converse of the above theorem is not true in general.
Example 4.4. Let $X = \{a, b, c\}, \tau = \{\varphi, X, \{a\}, \{b\}, \{a, b\}, \{b, c\}\}, I = \{\varphi, \{a\}\}$. The R-I-open sets are $\{a\}, \{b, c\}, X$. Then (X, τ, I) is R-I-R_0 but not R-I-R_1.
Remark 4.5. R_0 implies R-I-R_0 but the converse is not true.
Example 4.6. Consider the same example written above (Example 4.4). (X, τ, I) is R-I-R_0 but not R_0.
Remark 4.7. R_1 implies R-I-R_1 but the converse is not true.
Example 4.8. Let $X = \{a, b, c\}, \tau = \{\varphi, X, \{a\}, \{c\}, \{a, c\}, \{b, c\}\}, I = \{\varphi, \{a\}\}$. The R-I-open sets are $\{a\}, \{b, c\}, X$. Then (X, τ, I) is R-I-R_1 but not R_1.
Theorem 4.9. An ideal topological space (X, τ, I) is R-I-R_1 if and only if for any two points $x, y \in X$, $I_R Ker(\{x\}) \neq I_R Ker(\{x\})$ implies there exists disjoint R-I-open sets U, V such that$R - I - Cl(\{x\}) \subset U$ *and* $R - I - Cl(\{y\}) \subset V$.
Proof. By theorem 3.12, theorem directly follows.
Theorem 4.10. Let (X, τ, I) be a R-I-R1 space. Then for x, y ∈ X with $R - I - Cl(\{x\}) \neq R - I - Cl(\{y\})$, there exists R-I-closed sets K_1 and K_2 such that $x \in K_1$, $y \in K_2$, $y \notin K_1$, $x \notin K_2$ and $K_1 \cup K_2 = X$.
Proof. Let (X, τ, I) be R-I-R_1. Suppose x, $y \in X$ with $R - I - Cl(\{x\}) \neq R - I - Cl(\{y\})$. Then there exist disjoint R-I-open sets U, V such that $R - I - Cl(\{x\}) \subset U$and $R - I - Cl(\{y\}) \subset V$. Let$K_1 = X - V$ and$K_2 = X - U$. Then K_1 and K_2 are R-I-closed sets such that $x \in K_1$, $y \in K_2$, $y \notin K_1$, $x \notin K_2$and $K_1 \cup K_2 = X$.
Assume the converse. To show X is R-I-R_1, we first prove X is R-I-R_0. For that suppose U be a R-I-open set containing x and suppose $R - I - Cl(\{x\})$ is not a subset of U. So, $R - I - Cl(\{x\}) \cap U^c \neq \varphi$. Let

$y \in R - I - Cl(\{x\}) \cap U^c$. Then $R - I - Cl(\{x\}) \neq R - I - Cl(\{y\})$. Then by hypothesis there exists R-I-closed sets K_1 and K_2 such that $x \in K_1,\ y \in K_2,\ y \notin K_1,\ x \notin K_2$ and $K_1 \cup K_2 = X$. Thus, there exists a R-I-closed set containing x but not y, which is a contradiction. Hence (X, τ, I) is R-I-R_0. Now assume that $u, v \in X$ with $R - I - Cl(\{u\}) \neq R - I - Cl(\{v\})$. Then as earlier there exist R-I-closed sets L_1 and L_2 such that $x \in L_1, y \in L_2, y \notin L_1, x \notin L_2$ and $L_1 \cup L_2 = X$. Thus $u \in L_1 - L_2$ and $v \in L_2 - L_1$. But $L_1 - L_2 = X - L_2$ and $L_2 - L_1 = X - L_1$ and both are R-I-open. Since X is R-I-R_0, $R - I - Cl(\{u\}) \subset L_1 - L_2$ and $R - I - Cl(\{v\}) \subset L_2 - L_1$. Therefore (X, τ, I) is R-I-R_1.

Theorem 4.11. The following statements are equivalent:

(i) (X, τ, I) is a R-I-R1 space.

(ii) For each $x, y \in X$ either (a) or (b) holds. (a) if U is R-I-open, then $x \in U$ if and only if $y \in U$. (b) there exist disjoint R-I-open sets U and V such that $x \in U$ and $y \in V$.

(iii) If $x, y \in X$ with $R - I - Cl(\{x\}) \neq R - I - Cl(\{y\})$, there exists R-I-closed sets K_1 and K_2 such that $x \in K_1,\ y \in K_2,\ y \notin K_1,\ x \notin K_2$ and $K_1 \cup K_2 = X$.

Proof. (i) ⇒ (ii)

Let $x, y \in X$. Case 1: $R - I - Cl(\{x\}) = R - I - Cl(\{y\})$. Let U be an R-I-open set. Then $x \in U$ implies $y \in R - I - Cl(\{x\}) \subset U$ and $y \in U$ implies $x \in R - I - Cl(\{y\}) \subset U$. Thus $x \in U$ if and only if $y \in U$. Case 2: $R - I - Cl(\{x\}) \neq R - I - Cl(\{y\})$. Then there exist disjoint R-I-open sets U and V such that $x \in R - I - Cl(\{x\}) \subset U$ and $y \in R - I - Cl(\{y\}) \subset V$.

(ii) ⇒ (iii)

Let $x, y \in X$ such that $R - I - Cl(\{x\}) \neq R - I - Cl(\{y\})$. Then either $x \notin R - I - Cl(\{y\})$ or $y \notin R - I - Cl(\{x\})$. Suppose $x \notin R - I - Cl(\{y\})$. Then there exists a R-I-open set S such that $x \in S$ and $y \notin S$. So, by (ii) there exists disjoint R-I-open sets U and V such that $x \in U$ and $y \in V$. Let $K_1 = V^c$ and $K_2 = U^c$. Then K_1 and K_2 are R-I-closed sets such that $x \in K_1,\ y \in K_2,\ y \notin K_1,\ x \notin K_2$ and $K_1 \cup K_2 = X$.

(iii) ⇒ (i)

This is the statement of theorem 4.10.

Theorem 4.12. An ideal topological space is R-I-R_1 if and only if $x \in X - R - I - Cl(\{y\})$ implies that x and y have disjoint R-I-open neighbourhoods.

Proof. Let (X, τ, I) be R-I-R_1 and let $x \in X - R - I - Cl(\{y\})$. Then $R - I - Cl(\{x\}) \neq R - I - Cl(\{y\})$. Then x and y have disjoint R-I-open neighbourhoods.

Assume the converse. First, we prove that (X, τ, I) is R-I-R_0. Let U be a R-I-open set containing x. Suppose $y \notin U$. Then $R - I - Cl(\{y\}) \cap U = \varphi$. Also, $x \notin R - I - Cl(\{y\})$. Then there exists disjoint R-I-open sets V_1 and V_2 such that $x \in V_1$ and $y \in V_2$. Then $R - I - Cl(\{x\}) \subset R - I - Cl(V_1)$ and $R - I - Cl(\{x\}) \cap V_2 \subset R - I - Cl(V_1) \cap V_2 = \varphi$. Thus $y \notin R - I - Cl(\{x\})$. Hence $R - I - Cl(\{x\}) \subset U$ and (X, τ, I) is R-I-R_0. Now suppose $R - I - Cl(\{x\}) \neq R - I - Cl(\{y\})$. Then there exists an element $w \in R - I - Cl(\{x\})$ and $w \notin R - I - Cl(\{y\})$. By assumption there exists disjoint R-I-open sets W_1 and W_2 such that $w \in W_1$ and $y \in W_2$. Since $w \in R - I - Cl(\{x\}),\ x \in W_1$. Since (X, τ, I) is R-I-R_0, $R - I - Cl(\{x\}) \subset W_1$ and $R - I - Cl(\{y\}) \subset W_2$. Thus (X, τ, I) is R-I-R_1.

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