IOSR Journal of Mathematics (IOSR-JM)
e-ISSN: 2278-5728, p-ISSN: 2319-765X. Volume 16, Issue 4 Ser. III (Jul.–Aug. 2020), PP 01-13
www.iosrjournals.org

Construction of a Family of Stable One-Block Methods Using Linear Multi-Step Quadruple

[1]Ajie, I.J.; [2]Durojaye M.O.; [1]Utalor, K and [1]Onumanyi, P
[1]*(Mathematics Programme, National Mathematical Centre, Abuja, Nigeria)*
[2]*(Department of Mathematics, University of Abuja, Abuja, Nigeria)*

Abstract
Background: *This paper deals with the construction of a family of implicit one-block methods for the solution of stiff problems using four different linear multistep methods.*
Method: *This is done by applying shift operator on the quadruple: Reversed Generalized Adams Moulton (RGAM), Generalized Backward Differentiation Formula (GBDF), Top Order Method (TOM) and Backward Differentiation Formula (BDF).*
Results: *The application of the shift operator on the quadruples is done in such a manner that the resultant one-block methods are self-starting and forms a family. Orders four and seven are L-stable.*
Conclusion: *Numerical experiments carried out using orders four, seven and ten of the family show that the methods are good for solving stiff initial value problems.*
Keywords: *Stiff initial value problem; One-block methods; Self-starting; quadruple and shift operator.*

Date of Submission: 17-07-2020 Date of Acceptance: 01-08-2020

I. Introduction

This paper deals with the construction of methods for finding the numerical solution $y(t)$ to the stiff initial value problems (sivp) in ode

$$\begin{aligned} &y'(t) = f(t, y(t)); \quad y(t_0) = y_0; \quad t \in [a,b]; \\ &f : \Re \times \Re^m \to \Re^m; \quad y : \Re \to \Re^m \end{aligned} \tag{1.1}$$

The problem in (1.1) can only be handled adequately by high order A-Stable methods. These high order A-Stable methods are difficult to come by due to the severe restrictions imposed by Dalquist order barrier theorem [7]. To circumvent this barrier, unconventional means were adopted by many researchers to achieve high order numerical integrators to handle (1.1). These include but not limited to: boundary value methods [3, 4]; second derivative methods [6, 8]; implicit two points numerical integration formula [9], general linear methods [5], second derivative general linear methods [14] and rational one-step numerical integrators [16].

The use of collocation and interpolation in the construction of some linear multistep formulas for solving ordinary differential equations has been mostly with two-point boundary value problems until [6] showed the connection to the backward differentiation formula (BDF). Current trend following Onumanyi et al [15] have extended this connections to many families of traditional linear multistep methods, including boundary value methods (BVMs) [4] possessing good stability properties suitable for efficient solutions of (1.1). Notwithstanding these desirable developments, the cumbersomeness in the construction process is a drawback and need to be eliminated. This paper will approach the construction of continuous linear multistep formulas from the perspective of the order definition. The already known families that will be used in this paper are: Reversed Adams Moulton (RAM) methods, Generalized Backward Differentiation Formulas (GBDF), Top Order Methods (TOM) and Backward Differentiation Formulas (BDF). These four families will be used to demonstrate both the construction of the continuous linear multistep formulas and the new family of methods which this paper is proposing.

This paper is divided into five sections, sections one and two deal with the introduction and construction processes respectively while three and four deal with stability analysis and numerical experiments respectively. Conclusion is made in section five.

II. Construction of Linear Multistep Formula (LMF)

Consider the classical LMF given by

$$\sum_{r=0}^{k}\alpha_r y_{n+r} = h_n\sum_{r=0}^{k}\beta_r f(t_{n+r}, y_{n+r}) \tag{2.1}$$

where $k>1$ is the step number and $h_n = t_{n+1} - t_n$ is a variable step length, $\{\alpha_r\}_{r=0}^{k}$ and $\{\beta_r\}_{r=0}^{k}$ are real constants and both not zero. Making use of Taylor's series expansion on linear operator $L[y(t);h]$ associated with the difference equation (2.1) defined as

$$L[y(t);h] = \sum_{r=0}^{k}\alpha_r y(t+rh) - h_n\sum_{r=0}^{k}\beta_r y'(t+rh) \tag{2.2}$$

where $y(t)$ is a solution which is continuously differentiable on the interval [a,b]. Expanding the function $y(t+rh)$ and its derivative $y'(t+rh)$ about t, and collecting terms gives

$$L[y(t);h] = C_0 y(t) + C_1 h y'(t) + \ldots + C_p h^p y^{(p)} + \ldots \tag{2.3}$$

where

$$\left.\begin{aligned} &C_0 = \alpha_0 + \alpha_1 + \alpha_2 + \ldots + \alpha_k \\ &C_1 = (\alpha_1 + 2\alpha_2 + \ldots + k\alpha_k) - (\beta_0 + \beta_1 + \beta_2 + \ldots + \beta_k) \\ &and \\ &C_p = \frac{1}{p!}(\alpha_1 + 2^p\alpha_2 + \ldots + k^p\alpha_k) - \frac{1}{(p-1)!}(\beta_1 + 2^{(p-1)}\beta_2 + \ldots + k^{(p-1)}\beta_k) \end{aligned}\right\} \tag{2.4}$$

p = 1, 2, 3, …, are constants [13]. The constants in (2.4) play important role in the determination of the order, error constants and the coefficients of the methods.

***Definition* 2.1: *Order of a LMF* (*see)*[6]**

The LMF (2.1) is said to be of order p if in (2.3) and (2.4)

$C_0 = C_1 = C_2 = \ldots = C_p = 0$ *and* $C_{p+1} \neq 0$. *The* $C_{p+1} \neq 0$ *is the error constant of the LMF (2.1).*

The order definition 2.1 will be used to determine the coefficients of (2.1). A method is defined by the choice of the coefficients to be determined.

If the constant coefficients $\{\alpha_r\}_{r=0}^{k}$ and $\{\beta_r\}_{r=0}^{k}$ in the general k-step LMF given in (2.1), are replaced by the variable coefficients $\{\alpha_r(t)\}_{r=0}^{k}$ and $\{\beta_r(t)\}_{r=0}^{k}$, we have a continuous LMF given as

$$\sum_{r=0}^{k}\alpha_r(t) y_{n+r} = h_n\sum_{r=0}^{k}\beta_r(t) f(x_{n+r}, y_{n+r}) \tag{2.5}$$

where $0 \le t \le k$. We now consider the derivation of the formula for the coefficients $\alpha_r(t)$ and $\beta_r(t)$ in (2.5) using the order definition 2.1 as oppose to collocation and interpolation approach which makes use of basis function.

2.1 Coefficients Determination

Consider the Top Order Methods (TOM) which was first considered by Dahlquist [7], but was presented as unstable methods. The methods are of form

$$\sum_{i=0}^{k}\alpha_i y_{n+i} = h\sum_{i=0}^{k}\beta_i f_{n+i}\,;\ \alpha_i = -\alpha_{k-i}\ ;\ \beta_i = \beta_{k-i} \tag{2.6}$$

That is

$$\sum_{i=0}^{k}\alpha_i(y_{n+i} - y_{n+k-i}) = h\sum_{i=0}^{k}\beta_i(f_{n+i} + f_{n+k-i});$$

The coefficients are determined in order to have the maximum possible order p = 2k for a k-step method. Making use of definition 2.1 and (2.4), the coefficients in (2.6) can be determine by the following

Matrices:

$$\begin{pmatrix} 1 & 1 & 1 & 1.\ .\ .1 \\ 0 & 1 & 2 & 3.\ .\ .k \\ 0 & 1 & 2^2 & 3^2.\ .\ .k^2 \\ . & & & \\ . & & & \\ . & & & \\ 0 & 1 & 2^q & 3^q.\ ..k^q \end{pmatrix}\begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ . \\ . \\ . \\ \alpha_k \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & 0\ .\ .\ . & 0 \\ 1 & 1 & 1 & 1\ .\ .\ . & 1 \\ 0 & 1(2) & 2(2) & 3(2)\ .\ .\ . & k(2) \\ . & & & & \\ . & & & & \\ . & & & & \\ 0 & q & q(2^{(q-1)}) & q(3^{(q-1)}).\ ..q(k^{(q-1)}) & \end{pmatrix}\begin{pmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ . \\ . \\ . \\ \beta_k \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ . \\ . \\ . \\ 0 \end{pmatrix} \quad (2.7)$$

. in addition to the fact that

$$\beta_0 + \beta_1 + \beta_2 + \beta_3 + \ldots + \beta_k = 1$$

(2.8)

The order of the resultant matrix is $q = 2k$. This family has no continuous scheme. For any k-step, there is only one method. For instance, k = 3, (2.7) and (2.8) give the following matrix equation:

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & 2^2 & 3^2 \\ 0 & 1 & 2^3 & 3^3 \\ 0 & 1 & 2^4 & 3^4 \\ 0 & 1 & 2^5 & 3^5 \\ 0 & 1 & 2^6 & 3^6 \end{pmatrix}\begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & 0 \\ -1 & -1 & -1 & -1 \\ 1 & 1 & 1 & 1 \\ 0 & 2(1) & 2(2) & 2(3) \\ 0 & 3(1) & 3(2^2) & 3(3^2) \\ 0 & 4(1) & 4(2^3) & 4(3^3) \\ 0 & 5(1) & 5(2^4) & 5(3^4) \\ 0 & 6(1) & 6(2^5) & 6(3^5) \end{pmatrix}\begin{pmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

(2.9)

The coefficients in (2.6) for k = 3 are determined by solving for α_i and β_i in (2.9). The coefficients, and error constants for k = 2, 3, 4, 5 and 6 are given in the tables 2.1 and 2.2 below

Table 2.1: The values of α_i *for* $k = 2,\ldots,6$ in (2.6)

K	α_0	α_1	α_2	α_3	α_4	α_5	α_6	C_{p+1}
2	$\frac{-1}{2}$	0	$\frac{1}{2}$					$\frac{-1}{180}$
3	$\frac{-11}{60}$	$\frac{-9}{20}$	$\frac{9}{20}$	$\frac{11}{60}$				$\frac{-1}{2800}$
4	$\frac{-5}{84}$	$\frac{-8}{21}$	0	$\frac{8}{21}$	$\frac{5}{84}$			$\frac{-1}{44100}$
5	$\frac{-137}{7560}$	$\frac{-325}{1512}$	$\frac{-50}{189}$	$\frac{50}{189}$	$\frac{325}{1512}$	$\frac{137}{7560}$		$\frac{-1}{698544}$
6	$\frac{-7}{1320}$	$\frac{-1}{10}$	$\frac{-25}{88}$	0	$\frac{25}{88}$	$\frac{1}{10}$	$\frac{7}{1320}$	$\frac{-1}{11099088}$

Table 2.2: The values of β_i *for* $k = 2,\ldots,6$ in (2.6)

K	β_0	β_1	β_2	β_3	β_4	β_5	β_6	*Order* p
2	$\frac{1}{6}$	$\frac{2}{3}$	$\frac{1}{6}$					4

3	$\frac{1}{20}$	$\frac{9}{20}$	$\frac{9}{20}$	$\frac{1}{20}$				6
4	$\frac{1}{70}$	$\frac{8}{35}$	$\frac{18}{35}$	$\frac{8}{35}$	$\frac{1}{70}$			8
5	$\frac{1}{252}$	$\frac{25}{252}$	$\frac{25}{63}$	$\frac{25}{63}$	$\frac{25}{252}$	$\frac{1}{252}$		10
6	$\frac{1}{924}$	$\frac{3}{77}$	$\frac{75}{308}$	$\frac{100}{231}$	$\frac{75}{308}$	$\frac{3}{77}$	$\frac{1}{924}$	12

The continuous Adams Moulton (AM) type methods are of the form:

$$y_{n+t} - y_{n+t-1} = h\sum_{i=0}^{k} \beta_i(t) f_{n+i} \qquad t = 1(1)k \tag{2.10}$$

When $t = k$, (2.10) gives the standard AM methods but when $t = 1$, (2.10) gives RAM which we made use of in this paper. Imposing the maximum order $p = k + 1$ on (2.10) leads to the following matrix equation:

$$\begin{pmatrix} 1 \\ \frac{t^2 - (t-1)^2}{2} \\ \frac{t^3 - (t-1)^3}{3} \\ . \\ . \\ . \\ \frac{t^{k+1} - (t-1)^{k+1}}{k+1} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1. \; . \; .1 \\ 0 & 1 & 2 & 3. \; . \; .k \\ 0 & 1 & 2^2 & 3^2. \; . \; .k^2 \\ . & . & . & . \; . \; . \; . \; . \\ . & . & . & . \; . \; . \; . \; . \\ . & . & . & . \; . \; . \; . \; . \\ 0 & 1 & 2^k & 3^k. \; . \, .k^k \end{pmatrix} \begin{pmatrix} \beta_0(t) \\ \beta_1(t) \\ \beta_2(t) \\ . \\ . \\ . \\ \beta_k(t) \end{pmatrix} \tag{2.11}$$

Solving for $\{\beta_i(t)\}_{i=0}^{k}$ in (2.11) gives the coefficients in 2.10 as a function of $t \in (0, k]$.
For example, for k = 5, we have the following continuous coefficients:

$$\beta_0(t) = \frac{1}{1440}(4227 - 7220t + 4620t^2 - 1420t^3 + 210t^4 - 12t^5);$$

$$\beta_1(t) = -\frac{2641}{480} + 15t - \frac{97t^2}{8} + \frac{307t^3}{72} - \frac{11t^4}{16} + \frac{t^5}{24};$$

$$\beta_2(t) = \frac{1}{720}(4991 - 14400t + 13440t^2 - 5300t^3 + 930t^4 - 60t^5);$$

$$\beta_3(t) = \frac{1}{720}(-3649 + 10800t - 10680t^2 + 4580t^3 - 870t^4 + 60t^5);$$

$$\beta_4(t) = \frac{959}{480} - 6t + \frac{49t^2}{8} - \frac{199t^3}{72} + \frac{9t^4}{16} - \frac{t^5}{24}; \quad \beta_5(t) = -\frac{95}{288} + t - \frac{25t^2}{24} + \frac{35t^3}{72} - \frac{5t^4}{48} + \frac{t^5}{120}$$

Taking t=1 in (2.11), we have the following coefficients and error constants for various k-step RAM displayed in the table 2.3 below:

Table 2.3: The coefficients and error constants of RAM for $k = 1, 2, \ldots, 7$

k	β_0	β_1	β_2	β_3	β_4	β_5	β_6	β_7	C_{p+1}
1	$\frac{1}{2}$	$\frac{1}{2}$							$-\frac{1}{12}$
2	$\frac{5}{12}$	$\frac{8}{12}$	$-\frac{1}{12}$						$\frac{1}{24}$
3	$\frac{9}{24}$	$\frac{19}{24}$	$-\frac{5}{24}$	$\frac{1}{24}$					$-\frac{19}{720}$
4	$\frac{251}{720}$	$\frac{646}{720}$	$-\frac{11}{30}$	$\frac{106}{720}$	$-\frac{19}{720}$				$\frac{3}{160}$
5	$\frac{95}{288}$	$\frac{1427}{1440}$	$\frac{133}{240}$	$\frac{241}{720}$	$-\frac{173}{1440}$	$\frac{3}{160}$			$-\frac{863}{6048}$
6	$\frac{19087}{60480}$	$\frac{2713}{2520}$	$-\frac{1548}{2016}$	$\frac{586}{945}$	$-\frac{6737}{2016}$	$\frac{263}{2520}$	$-\frac{863}{6048}$		$\frac{275}{24192}$
7	$\frac{5257}{17280}$	$\frac{139849}{120960}$	$-\frac{4511}{4480}$	$\frac{123133}{120960}$	$-\frac{88547}{120960}$	$\frac{1537}{4480}$	$-\frac{11351}{120960}$	$\frac{275}{24192}$	$\frac{33963}{3628800}$

The general formula for continuous BDF is

$$\sum_{i=0}^{k} \alpha_i(t) y_{n+i} = h\beta_t(t) f_{n+t} \qquad t = 1(1)k \tag{2.12}$$

Imposing order k on (2.12) and using definition 2.1 leads to the matrix equation:

$$\begin{pmatrix} 0 \\ 1 \\ 2t \\ . \\ . \\ . \\ kt^{k-1} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1. \; . \; .1 \\ 0 & 1 & 2 & 3. \; . \; .k \\ 0 & 1 & 2^2 & 3^2. \; . \; .k^2 \\ . & & & \\ . & & & \\ . & & & \\ 0 & 1 & 2^k & 3^k. \; . \; .k^k \end{pmatrix} \begin{pmatrix} \alpha_0(t) \\ \alpha_1(t) \\ \alpha_2(t) \\ . \\ . \\ . \\ \alpha_k(t) \end{pmatrix} \tag{2.13}$$

if $\beta_t(t) = 1$. The coefficients in (2.12) are uniquely determined by solving for $\{\alpha_i(t)\}_{i=1}^{k}$ in (2.13).

For a particular *k*, *k* different methods can be constructed using the continuous coefficients. Putting t = k, gives the standard BDF type methods, if t = (k+1)/2, we have GBDF. The coefficients of BDF and GBDF are respectively displayed in the tables below:

Consider the case *k* = 5, we have.

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 & 5 \\ 0 & 1 & 2^2 & 3^2 & 4^2 & 5^2 \\ 0 & 1 & 2^3 & 3^3 & 4^3 & 5^3 \\ 0 & 1 & 2^4 & 3^4 & 4^4 & 5^4 \\ 0 & 1 & 2^5 & 3^5 & 4^5 & 5^5 \end{pmatrix} \begin{pmatrix} \alpha_0(t) \\ \alpha_1(t) \\ \alpha_2(t) \\ \alpha_3(t) \\ \alpha_4(t) \\ \alpha_5(t) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 2t \\ 3t^2 \\ 4t^3 \\ 5t^4 \end{pmatrix} \tag{2.14}$$

Solving for $\{\alpha_i(t)\}_{i=1}^{k}$ in (2.14) gives

$$\alpha_0(t)=-\frac{137}{60}+\frac{15t}{4}-\frac{17t^2}{8}+\frac{t^3}{2}-\frac{t^4}{24};\ \alpha_1(t)=5-\frac{77t}{6}+\frac{71t^2}{8}-\frac{7t^3}{3}+\frac{5t^4}{24}$$

$$\alpha_2(t)=-5+\frac{107t}{6}-\frac{59t^2}{4}+\frac{13t^3}{3}-\frac{5t^4}{12};\ \alpha_3(t)=\frac{10}{3}-13t+\frac{49t^2}{4}-4t^3+\frac{5t^4}{12}$$

$$\alpha_4(t)=-\frac{5}{4}+\frac{61t}{12}-\frac{41t^2}{8}+\frac{11t^3}{6}-\frac{5t^4}{24};\ \alpha_5(t)=\frac{1}{5}-\frac{5t}{6}+\frac{7t^2}{8}\frac{t^3}{3}+\frac{t^4}{24}$$

For BDF type methods take t = 5. For GBDF, take t = (5+1)/2. The coefficients and error constants of BDF and GBDF for k = 1, 2, 3, . . ., 7 are respectively displayed in the tables 2.4 and 2.5 below:

Table 2.4: The coefficients and error constants of BDF for $k=1,2,\ldots,7$

K	α_0	α_1	α_2	α_3	α_4	α_5	α_6	α_7	C_{p+1}
1	-1	1							$-\frac{1}{2}$
2	$\frac{1}{2}$	-2	$\frac{3}{2}$						$-\frac{1}{3}$
3	$-\frac{1}{3}$	$\frac{3}{2}$	-3	$\frac{11}{6}$					$-\frac{1}{4}$
4	$\frac{1}{4}$	$-\frac{4}{3}$	3	-4	$\frac{25}{12}$				$-\frac{1}{5}$
5	$-\frac{1}{5}$	$\frac{5}{4}$	$-\frac{10}{3}$	5	-5	$\frac{137}{60}$			$-\frac{1}{6}$
6	$\frac{1}{6}$	$-\frac{6}{5}$	$\frac{15}{4}$	$-\frac{20}{3}$	$\frac{15}{2}$	-6	$\frac{49}{20}$		$-\frac{1}{7}$
7	$-\frac{1}{7}$	$\frac{7}{6}$	$-\frac{21}{5}$	$\frac{35}{4}$	$-\frac{35}{3}$	$\frac{21}{2}$	-7	$\frac{363}{140}$	$-\frac{1}{8}$

Table 2.5: The coefficients and error constants of GBDF for $k=1,2,\ldots,7$

K	α_0	α_1	α_2	α_3	α_4	α_5	α_6	α_7	C_{p+1}
1	-1	1							-1
2	$\frac{1}{2}$	-2	$\frac{3}{2}$						$-\frac{1}{3}$
3	$\frac{1}{6}$	-1	$\frac{1}{2}$	$\frac{1}{3}$					$\frac{1}{12}$
4	$-\frac{1}{12}$	$\frac{1}{2}$	$-\frac{3}{2}$	$\frac{5}{6}$	$\frac{1}{4}$				$\frac{1}{20}$
5	$-\frac{1}{30}$	$\frac{1}{4}$	-1	$\frac{1}{3}$	$\frac{1}{2}$	$-\frac{1}{20}$			$-\frac{1}{60}$
6	$\frac{1}{60}$	$-\frac{2}{15}$	$\frac{1}{2}$	$-\frac{4}{3}$	$\frac{7}{12}$	$\frac{2}{5}$	$-\frac{1}{30}$		$-\frac{1}{105}$
7	$\frac{1}{140}$	$\frac{-1}{15}$	$\frac{3}{10}$	-1	$\frac{1}{4}$	$\frac{3}{5}$	$-\frac{1}{10}$	$\frac{1}{105}$	$\frac{1}{280}$

2.2 Construction of the block methods

The methodology for the construction of the methods is well explained in the proposition giving in Ajie et al [1, 2].

Given four different families of k-step LMF completely defined by the characteristics polynomial $\left\{\rho_k^{[j]}(R),\sigma_k^{[j]}(R)\right\}_{j=1,k=1}^{4,K}$ *and specified by*

$$\rho_k^{[j]}(E)y_n = h\sigma_k^{[j]}(E)f_n\,;\;\; j=1(1)4,\;\; k=1(1)K \qquad (2.15)$$

with $\left\{\rho_k^{[j]},\sigma_k^{[j]}\right\}$ *for a fixed j forming a family of variable order* $p_{k,j}$ *of variable step number k. The the resultant system of composite LMF*

$$E^i\rho_k^{[j]}(E)y_n = hE^i\sigma_k^{[j]}(E)f_n\,;\;\; i=0(1)k-l;\;\; j=1,2,3,4 \qquad (2.16)$$

arising from the E-operator transformation of (2.16) *can be composed as the block method*

$$A_1Y_{n+1} + A_0Y_n = h(B_1F_{n+1} + B_0F_n)\,;\; \det(A_1)\neq 0 \qquad (2.17)$$

if k is chosen such that l is an integer given as

$$l = \frac{2(2+k)}{3}\;;\; k\geq 4 \text{ and } k-l\geq 0 \qquad (2.18)$$

where Y_{n+1}, Y_n ; F_{n+1} *and* F_n ; $n=0,1,2,\ldots$ *are as defined in (2.19) and* A_1, A_0, B_1, B_0 *are square matrices defined in* (2.20)*. The construction of this family is possible ofor the following integer values of k: 4, 7, 10, 13, . . .*

Here,

$$A_0 = \left(\begin{array}{c|c} & \alpha_0^{[1]} \\ & \alpha_0^{[2]} \\ & \alpha_0^{[3]} \\ & \alpha_0^{[4]} \\ & \\ O & 0 \\ & . \\ & . \\ & . \\ & 0 \end{array}\right)_{(2k-l)\times(2k-l)} ; \quad B_0 = \left(\begin{array}{c|c} & \beta_0^{[1]} \\ & \beta_0^{[2]} \\ & \beta_0^{[3]} \\ & \beta_0^{[4]} \\ & \\ O & 0 \\ & . \\ & . \\ & . \\ & 0 \end{array}\right)_{(2k-l)\times(2k-l)} \qquad (2.19)$$

$$A_1 = \left(\begin{array}{ccccccccccccccccc}
\alpha_1^{[1]} & \alpha_2^{[1]} & \cdot & \cdot & \alpha_k^{[1]} & 0 & 0 & 0 & \cdot & \cdot & \cdot & & & & & & 0 \\
\alpha_1^{[2]} & \alpha_2^{[2]} & \cdot & \cdot & \alpha_k^{[2]} & 0 & 0 & 0 & \cdot & \cdot & \cdot & & & & & & 0 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & & & & & & & & 0 \\
\alpha_1^{[4]} & \alpha_2^{[4]} & \cdot & \cdot & \alpha_k^{[4]} & 0 & \cdot & \cdot & & & & & & & & & 0 \\
\alpha_0^{[1]} & \alpha_1^{[1]} & \cdot & \cdot & \cdot & \alpha_k^{[1]} & \cdot & \cdot & & & & & & & & & 0 \\
\cdot & \cdot & \cdot & \cdot & & \vdots & 0 & & & & & & & & & & 0 \\
\vdots & \vdots & \cdot & \cdot & \cdot & \cdot & 0 & & & & & & & & & & \cdot \\
\alpha_0^{[4]} & \alpha_1^{[4]} & & & & \alpha_k^{[4]} & 0 & & & & & & & & & & \cdot \\
0 & \alpha_0^{[1]} & \alpha_1^{[1]} & & & & \alpha_k^{[1]} & & & & & & & & & & \cdot \\
0 & \cdot & \cdot & & & \cdot & \cdot & & & & & & & & & & \\
0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & & & & & & & & & \\
0 & \alpha_0^{[4]} & \alpha_1^{[4]} & \cdot & \cdot & \cdot & \alpha_k^{[4]} & 0 & & & & & & & & & \\
0 & 0 & \cdot & & & & \cdot & & & & & & & & & & \\
\cdot & .0 & \cdot & & & & \cdot & & & & & & & \cdot & \cdot & \cdot & \\
\cdot & \cdot & \cdot & & & & \cdot & & & & & \cdot & \cdot & & & & 0 \\
\cdot & \cdot & & & & & & & & & & \cdot & \alpha_0^{[1]} & & & & \alpha_k^{[1]} \\
0 & \cdot & & & & & & & & & & \cdot & \vdots & & & & \vdots \\
0 & .0 & \cdot & \cdot & & & & & & & & 0 & \alpha_0^{[4]} & \cdot & \cdot & \cdot & \alpha_k^{[4]}
\end{array}\right)_{(2k-l)\times(2k-l)}$$

$$B_1 = \left(\begin{array}{ccccccccccccccccc}
\beta_1^{[1]} & \cdot & \cdot & \cdot & \beta_k^{[1]} & 0 & 0 & 0 & \cdot & \cdot & \cdot & & & & & & 0 \\
\vdots & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & & & & & & & & 0 \\
\beta_1^{[4]} & \cdot & \cdot & \cdot & \beta_k^{[4]} & 0 & \cdot & \cdot & & & & & & & & & 0 \\
\beta_0^{[1]} & \beta_1^{[1]} & \cdot & \cdot & \cdot & \beta_k^{[1]} & \cdot & \cdot & & & & & & & & & 0 \\
\cdot & \cdot & \cdot & \cdot & & \vdots & 0 & & & & & & & & & & 0 \\
\vdots & \vdots & \cdot & \cdot & \cdot & \cdot & 0 & & & & & & & & & & \cdot \\
\beta_0^{[4]} & \beta_1^{[4]} & & & & \beta_k^{[4]} & 0 & & & & & & & & & & \cdot \\
0 & \beta_0^{[1]} & \beta_1^{[1]} & & & \beta_{k-1}^{1} & \beta_k^{[1]} & & & & & & & & & & \cdot \\
0 & \cdot & \cdot & & & \vdots & \cdot & & & & & & & & & & \\
 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & & & & & & & & & \\
 & \beta_0^{[4]} & \beta_1^{[4]} & \cdot & \cdot & \beta_{k-1}^{[4]} & \beta_k^{[4]} & 0 & & & & & & & & & \\
 & \cdot & \cdot & & & & \cdot & & & & & & & & & & \\
\cdot & \cdot & \cdot & & & & \cdot & & & & & & & \cdot & \cdot & \cdot & \\
\cdot & \cdot & \cdot & & & & \cdot & & & & & \cdot & \cdot & & & & 0 \\
\cdot & & & & & & & & & & & 0 & \beta_0^{[1]} & & & & \beta_k^{[1]} \\
 & & & & & & & & & & & \cdot & \vdots & & & & \vdots \\
0 & \cdot & \cdot & \cdot & & & & & & & & 0 & \beta_0^{[4]} & \cdot & \cdot & \cdot & \beta_k^{[4]}
\end{array}\right)_{(2k-l)\times(2k-l)}$$

while the solution and function block vectors are given as

$$\left.\begin{array}{l} Y_{n+1} = (y_{n+1}\, y_{n+2} \cdots y_{n+2k-l})^T;\;\; Y_n = (y_{n-2k+l+1}\, y_{n-2k+l+2} \cdots y_{n-1}\, y_n)^T; \\ F_{n+1} = (f_{n+1}\, f_{n+2} \cdots f_{n+2k-l})^T;\;\; F_n = (f_{n-2k+l+1}\, f_{n-2k+l+2} \cdots, f_{n-1}\, f_n)^T \\ n = 0, 1, 2, \ldots \end{array}\right\} \quad (2.20)$$

The resultant methods for orders 4 and 7 when written in the form of (2.17) have the following coefficient matrices:

For order 4

$$A_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{-3}{2} & \frac{5}{6} & \frac{1}{4} \\ 0 & \frac{1}{2} & 0 & 0 \\ -\frac{4}{3} & 3 & -4 & \frac{25}{12} \end{pmatrix};\; A_0 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -\frac{1}{12} \\ 0 & 0 & 0 & -\frac{1}{2} \\ 0 & 0 & 0 & \frac{1}{4} \end{pmatrix};\; B_1 = \begin{pmatrix} \frac{19}{24} & -\frac{5}{24} & \frac{1}{24} & 0 \\ 0 & 0 & 1 & 0 \\ \frac{2}{3} & \frac{1}{6} & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix};\; B_0 = \begin{pmatrix} 0 & 0 & 0 & \frac{3}{8} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{6} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

For order 7

$$A_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{15} & \frac{3}{10} & -1 & \frac{1}{4} & \frac{3}{5} & -\frac{1}{10} & \frac{1}{105} & 0 \\ -\frac{8}{21} & 0 & \frac{8}{21} & \frac{5}{84} & 0 & 0 & 0 & 0 \\ \frac{7}{6} & -\frac{21}{5} & \frac{35}{4} & -\frac{35}{3} & \frac{21}{2} & -7 & \frac{363}{140} & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{140} & -\frac{1}{15} & \frac{3}{10} & -1 & \frac{1}{4} & \frac{3}{5} & -\frac{1}{10} & \frac{1}{105} \\ -\frac{5}{84} & -\frac{8}{21} & 0 & \frac{8}{21} & \frac{5}{84} & 0 & 0 & 0 \\ -\frac{1}{7} & \frac{7}{6} & -\frac{21}{5} & \frac{35}{4} & -\frac{35}{3} & \frac{21}{2} & -7 & \frac{363}{140} \end{pmatrix};$$

$$B_1 = \begin{pmatrix} \frac{2713}{2520} & -\frac{15487}{20160} & \frac{586}{945} & -\frac{6737}{20160} & \frac{263}{2520} & -\frac{863}{60480} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ \frac{8}{35} & \frac{18}{35} & \frac{8}{35} & \frac{1}{70} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ \frac{19087}{60480} & \frac{2713}{2520} & -\frac{15487}{20160} & \frac{586}{945} & -\frac{6737}{20160} & \frac{263}{2520} & -\frac{863}{60480} & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ \frac{1}{70} & \frac{8}{35} & \frac{18}{35} & \frac{8}{35} & \frac{1}{70} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix};$$

$$A_0 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{140} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{5}{84} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{7} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix};\; B_0 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{19087}{60480} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{70} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

III. Stability Analysis of the Methods

When (2.17) is applied to the test equation

$$y' = \lambda y\,,\ \mathrm{Re}(\lambda) < 0 \tag{3.1}$$

it yields the characteristics equation.

$$\pi(w,z) = \det(A_1 w + A_0 - z(B_1 w + B_0))\,, \quad z = \lambda h \tag{3.2}$$

The region of absolute stability R_A associated with (2.17) is the set

$$R_A = \{z \in \mathbb{C} : \left|w_j(z)\right| \le 1,\ j = 1(1)(4\frac{(k-1)}{3})\} \tag{3.3}$$

For order 4 above $w_j(z),\ j = 1(1)4$ are given as

$$\{w \to 0\}, \{w \to 0\}, \{w \to 0\}, \{w \to \frac{2364 + 4326z + 3151z^2 + 953z^3}{2364 - 5130z + 4759z^2 - 2259z^3 + 468z^4}\}$$

For order 7 $w_j(z),\ j = 1(1)8$ are given as

$$\{\{w \to 0\}, \{w \to 0\}, \{w \to 0\}, \{w \to 0\}, \{w \to 0\}, \{w \to 0\}, \{w \to 0\},\ \{w = \frac{3495902320592640 + 10746415327105080z + 12863275360428960z^2 + 4509468608610360z^3 - 6717280910860608z^4 - 10946604461093677z^5 - 7527040403669756z^6 - 2228006201403288z^7}{3495902320592640 - 17220803237636040z + 38760827002552800z^2 - 52828441831364040z^3 + 48438336555493632z^4 - 31131036972511789z^5 + 13948525963167340z^6 - 4010892550923608z^7 + 521283129074400z^8}\}\}$$

The only non-zero value of $w(z)$ for this family of methods are given as a rational function $T(z) = \dfrac{P(z)}{Q(z)}$. where $P(z)$ and $Q(z)$ are polynomials. From the above for order $p = 4$,

$$T(z) = \frac{2364 + 4326z + 3151z^2 + 953z^3}{2364 - 5130z + 4759z^2 - 2259z^3 + 468z^4} \tag{3.4}$$

For order p = 7,

$$T(z) = \frac{3495902320592640 + 10746415327105080z + 12863275360428960z^2 + 4509468608610360z^3 - 6717280910860608z^4 - 10946604461093677z^5 - 7527040403669756z^6 - 2228006201403288z^7}{3495902320592640 - 17220803237636040z + 38760827002552800z^2 - 52828441831364040z^3 + 48438336555493632z^4 - 31131036972511789z^5 + 13948525963167340z^6 - 4010892550923608z^7 + 521283129074400z^8}$$

These values of $T(z)$ tend to zero as z tends to infinity.

Definition 3.1: A block method is said to be pre-stable if the roots of $Q(z)$ are contained in C^+ (see [4, 12]). The roots of $Q(z)$ for order 4 are

$$\{z \to 1.0082125369126547 - 1.164677955507617i\},$$
$$\{z \to 1.0082125369126547 + 1.164677955507617i\},$$
$$\{z \to 1.405249001548883 - 0.39236814273553433i\},$$
$$\{z \to 1.405249001548883 + 0.39236814273553433i\}\}$$

While the roots of $Q(z)$ for order 7 are

$$\{\{z \to 0.34310043332258827 - 1.4797687914142503i\},$$
$$\{z \to 0.34310043332258827 + 1.4797687914142503i\},$$
$$\{z \to 0.7613416549353051 - 0.8312425541243531i\},$$
$$\{z \to 0.7613416549353051 + 0.8312425541243531i\},$$
$$\{z \to 0.8910672693695683 - 0.344548117684201i\},$$
$$\{z \to 0.8910672693695683 + 0.344548117684201i\},$$
$$\{z \to 0.8912380378302361\}, \{z \to 2.812012113903553\}\}$$

The entire roots are contained in C^+.

Definition 3.2: A one block method is *A*-stable if and only if it is stable on the imaginary axis (*I*- stable)[12]:
That is if $T(iy) \le 1$ for all $y \in \Re$, and T(z) is analytic for $z < 0$ (i.e. $Q(z)$ does not have roots with negative or zero real parts). *I*-stability is equivalent to the fact that the Norsett polynomial defined by

$$G(y) = \left|Q(iy)\right|^2 - \left|P(iy)\right|^2 = Q(iy)Q(-iy) - P(iy)P(-iy) \tag{3.5}$$

satisfies $G(y) > 0$ for all $y \in \Re$ (see [12]).

Definition 3.3: A block method is said to be $L-Stable$ if it is $A-Stable$ and also $T(z) \to 0$ as $z \to \infty$ [10].

Orders 4 and 7 of this family satisfy the conditions in definitions 3.1, 3.2, 3.3 and equation (3.5). Therefore they are $L-Stable$.

IV. Numerical Experiments

In this section, three problems were considered to test the effectiveness of the methods in solving stiff initial value problems.

Problem 4.1: Linear equations (cf: see [4, 13]),

$$y' = \begin{pmatrix} -21 & 19 & -20 \\ 19 & -21 & 20 \\ 40 & -40 & -40 \end{pmatrix} y; \quad y(0) = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

$$y(t) = \frac{1}{2}\begin{pmatrix} e^{-2t} + e^{-40t}(\cos(40t) + \sin(40t)) \\ e^{-2t} - e^{-40t}(\cos(40t) + \sin(40t)) \\ 2e^{-40t}(\sin(40t) - \cos(40t)) \end{pmatrix}$$

Problem 4.2: Van der Pol problem (cf: see [4])

$$y_1' = y_2$$

$$y_2' = -y_1 + \mu y_2(1 - y_1^2); \; y_1(0) = 2, \; y_2(0) = 0, \mu = 200$$

Problem 4.3: A Chemical Kinetics Problem (cf: see [10])

$$\frac{dy}{dt} = f(y); \; t \in [0, T]$$

The function f is defined by

$$f(y) = \begin{pmatrix} -k_1 y_1 + k_2 y_2 y_3 \\ k_1 y_1 - k_2 y_2 y_3 - k_3 y_2^2 \\ k_3 y_2^2 \end{pmatrix}; \; y(0) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$k_1 = 0.04; \; k_2 = 10^4; \; k_3 = 3.10^7$$

Problem 4.1 is computed using order 4 of the methods. The graph of the computed solutions and the exact solutions are plotted and shown in figure 4.1.

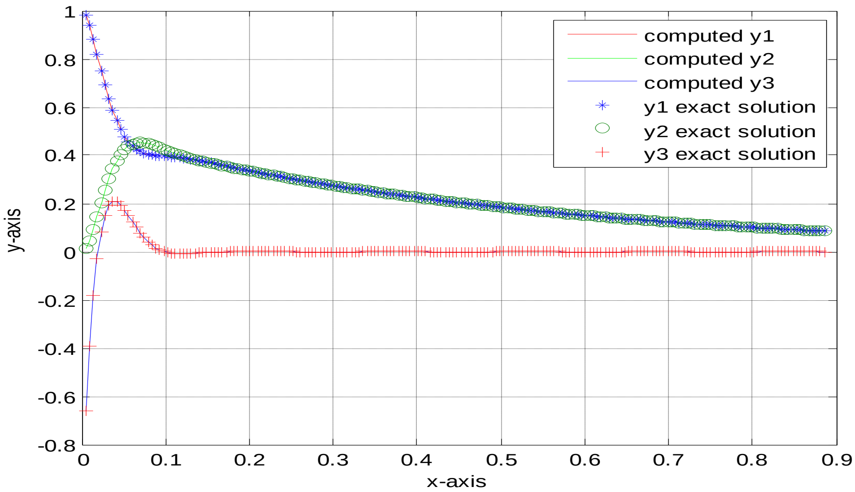

Figure 4.1: *Solution of problem 4.1 using order p = 4*

The phase diagram of the computed solutions of problem 4.2 using order 7 of the methods and ode15s is plotted and shown in figure 4.2. The red line is that of order 7, while the blue dotted line is plotted using ode 15s.

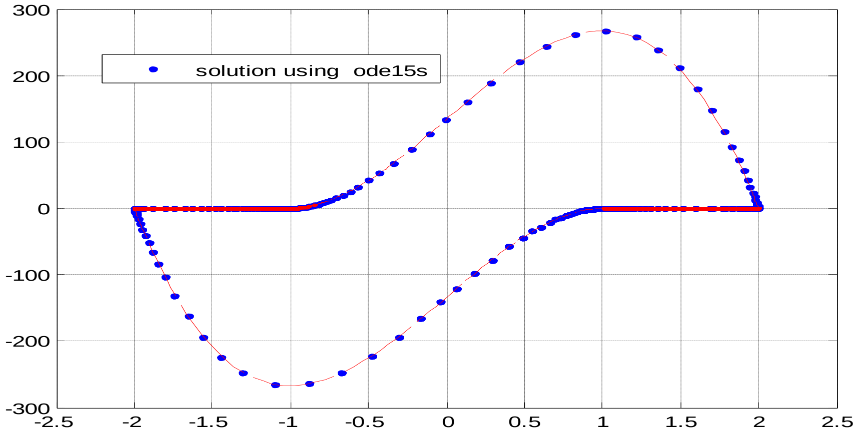

Figure 4.2: The phase diagram of problem 4.2 computed with order 7 of the method ode15s

Problem 4.3 is solved using order 10 and constant step size h = 0.0001. The errors in the table 4.1 are the maximum absolute values of the difference between our approximate solution and that of MATLAB ODE15s (which we assumed to be the exact solution of the problem). The solution computed by the two methods (ODE15s and order p = 10 of our methods are plotted in figure 4.3.

***Table 4.1**: Errors* $\|e\|_2$ *from order p=10 using ode 15s as exact*

T	2.00	5.00	7.5	10.00
Errors	2.30e-006	4.20e-006	4.41e-005	7.19e-005

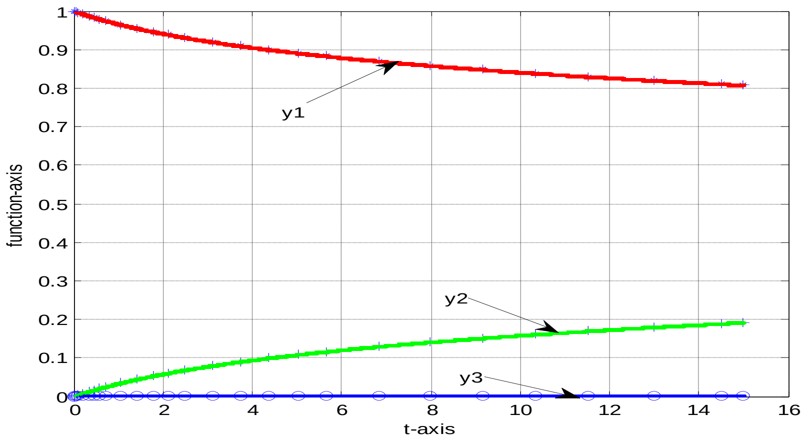

***Figure 4.3:** Graph of solutions computed by order, p = 10 and ODE15s for Problem 4.3*

V. Conclusion

This paper has demonstrated how self starting block methods can be constructed using four different k-step linear multistep formulas. The family constructed using the quadruple: RAM, GBDF, TOM and BDF is L-stable up to order 7. This paper also showed how continuous coefficients linear multistep methods can be constructed using the order definition. The numerical experiments performed using orders 4, 7 and 10 of the family of the methods on stiff initial value problem show that the methods are effective.

References

[1]. Ajie, I. J., Ikhile, M. N. and Onumanyi, P.: Variable Order Implicit $L(\alpha)$-Stable Block Methods for Stiff Ordinary Differential Equations. Journal of Mathematical Sciences, 2014, vol. 3 No 1: National Mathematical Centre (NMC), Abuja, Nigeria.

[2]. Ajie, I.J., Utalor, K. and Onumanyi, P. A Family of High Order One-Block Methods for the Solution of Stiff Initial Value Problems. Journal of Advances in Mathematics and Computer Science. 2019, 31(6), 1-14.

[3]. Amodio, P. and Mazzia, F. (1995). A boundary value approach to the numerical solution of ODE by multistep methods, J. Difference Equations Appl.vol.1 pp. 353-367.

[4]. Brugnano, L. and Trigiante, D., Solving differential problems by multistep initial and boundary value methods. Published by Gordon and Beach Science Publishers, 1998.
[5]. Butcher, J. C. and Jackiewicz, Z.. Construction of GLM with Runge-Kutta stability properties Numer Alg., 2004, 36, pp. 53-72.
[6]. Cash, J. On the integration of stiff systems of ODE using extended BDF, Numer. Math., 1980, Vol. 34, pp. 235-246.
[7]. Dahlquist, G., Convergence and stability in the numerical integration of ODE. Math. Scandinavica, 1956, 4, pp. 33-53.
[8]. Enright, W. Optimal second derivative methods for stiff systems.In Stiff Differential Systems, R. Willoughby(ed.), Plenum Press, New York, 1974.
[9]. Fatunla, S.O., Numerical methods for IVPs in ODEs. Academic press, USA, 1989.
[10]. Hairer, E and Wanner, G., *SolvingODE II.*Stiff and differential-algebraic problems. 2nd revised edition. SpringerVerlag, Springer series in Computation Mathematics, 1996.
[11]. Henrici, P., *Discrete Variable Methods ODE*. John Wiley, New York, 1962.
[12]. Jackiewicz, Z., General linear methods for ordinary differential equation. A John wiley& sons, inc., publication, 2009.
[13]. Lambert, J. D., Computational Methods in Ordinary Differential Equations. New York: JohnWiley, Sons, Inc., 1991.
[14]. Okuonghae, R.I. and Ikhile, M.N.O., Some GLM for the solution of non-stiff IVP in ODE. Journal of algorithm and computational technology, 2013, vol. 7, number 1.
[15]. [15]Onumanyi, P., Awoyemi, D. O., Jator, S. N. and Sirisena, U. W., ‘‘New LMM with Continuous Coefficients for First Order IVP” *Journal of The Nigerian Mathematical Society*, 1994, vol. 13.
[16]. Otunta, F.O. and Ikhile, M.N.O.: Efficient rational one-step numerical integrators for IVP in ODE. International journal of computer mathematics, 1999, vol. 72(1), pp. 49-61
[17]. Stetter, H. J.. Cyclic finite difference methods for ODE. Lecture notes in math. 1974, 363, pp. 134-143.

Ajie, I.J, et. al. " Construction of a Family of Stable One-Block Methods Using Linear Multi-Step Quadruple." *IOSR Journal of Mathematics (IOSR-JM)*, 16(4), (2020): pp. 01-13.