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Characterization of methods for solving the Linear Programming problems with interval coefficients

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Abstract: *We define the primal and dual linear programming problems involving interval numbers as the way of traditional linear programming problems. We discuss the solution concepts of primal and dual linear programming problems involving interval numbers without converting them to classical linear programming problems. By introducing arithmetic operations between interval numbers, we prove the weak and strong duality theorems. Complementary slackness theorem is also proved. A numerical example is provided to illustrate the theory developed in this paper.*

Keywords: *Closed intervals, Linear Programming, Weak Duality, Strong Duality, Complementary Slackness.*

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I. Introduction

Linear programming is a most widely and successfully used decision tool in the quantitative analysis of practical problems where rational decisions have to be made. In order to solve a Linear Programming Problem, the decision parameters of the model must be fixed at crisp values. But to model real-life problems and perform computations we must deal with uncertainty and inexactness. These uncertainty and inexactness are due to measurement inaccuracy, simplification of physical models, variations of the parameters of the system, computational errors etc. Interval analysis is an efficient and reliable tool that allows us to handle such problems effectively.

Linear programming problems with interval coefficients have been studied by several authors, such as Atanu Sengupta et al. [5, 6], Bitran [8], Chanas and Kuchta [9], Nakahara et al. [23], Steuer [29] and Tong Shaocheng [34]. Numerous methods for comparison of interval numbers can be found as in Atanu Sengupta and Tapan Kumar Pal [5, 6], Ganesan and Veeramani [11, 12] etc.

By taking maximum value range and minimum value range inequalities as constraint conditions, Tong Shaocheng [34] reduced the interval linear programming problem in to two classical linear programming problems and obtained an optimal interval solution to it. Ramesh and Ganesan [27] proposed a method for solving interval number linear programming problems without converting them to classical linear programming problems.

The duality theory for inexact linear programming problems was proposed by Soyster [30–31] and Thuente [33]. Falk [10] provided some properties on this problem. However, Pomerol [26] pointed out some drawbacks of Soyster‟s results and provided some mild conditions to improve them. Masahiro Inuiguchia [17] et al has studied the duality of interval number linear programming problems through fuzzy linear programming problems. Bector and Chandra [7] introduced a pair of linear primal-dual problems under fuzzy environment and established the duality relationship between them. Hsien-Chung Wu [15,16] introduced the concept of scalar product for closed intervals in the objective and inequality constraints of the primal and dual linear programming problems with interval numbers. He introduced a solution concept that is essentially similar to the notion of nondominated solution in multiobjective programming problems by imposing a partial ordering on the set of all closed intervals. He then proved the weak and strong duality theorems for linear programming problems with interval numbers. Rohn [28] also discussed the duality in a interval linear programming problem with real-valued objective function. In this paper, we attempt to develop the duality theory for interval linear programming problems without converting them to classical linear programming problems.

The rest of this paper is organized as follows: In section 2, we recall the definitions of interval number linear programming, interval numbers and some related results of interval arithmetic on them. In section 3, we define the interval number primal and dual linear programming problems as the way of traditional linear programming problems. We then prove the weak and strong duality theorems. Complementary Slackness theorem is also proved. In section 4, a numerical example is provided to illustrate the theory developed in this paper.

II. Preliminaries

The aim of this section is to present some notations, notions and results which are of useful in our further consideration.

2.1. Arithmetics of Closed Intervals

Let us denoted by $\mathfrak{T}$ the class of all closed intervals in $\mathbb{R}$. If A is closed interval, we also adopt the notation $\bar{A} = [a^L, a^U]$, where a^L and a^U means the lower and upper bounds of $\bar{A}$ respectively.

For any two intervals $\bar{A} = [a^L, a^U]$ and $\bar{B} = [b^L, b^U]$ and for $* \in \{+, -, \times, \div\}$, the arithmetic operations on A and B are defined as:

$$\bar{A} * \bar{B} = [a^L, a^U] * [b^L, b^U]$$
$$= [Min\{a^L * b^L, a^L * b^U, a^U * b^L, a^U * b^U\}, \quad Max\{a^L * b^L, a^L * b^U, a^U * b^L, a^U * b^U\}]$$

If $\bar{A} = a^L = a^U = a$, then $\bar{A} = [a, a] = a$.

In particular

(i) Addition: $\bar{A} + \bar{B} = [a^L + b^L, a^U + b^U]$

(ii) Subtraction: $\bar{A} - \bar{B} = [a^L - b^U, a^U - b^L]$

(iii) Multiplication:

$$\bar{A} \times \bar{B} = [Min\{a^L \times b^L, a^L \times b^U, a^U \times b^L, a^U \times b^U\}, \quad Max\{a^L \times b^L, a^L \times b^U, a^U \times b^L, a^U \times b^U\}]$$

(iv) Division:

$$\bar{A} \div \bar{B} = [a^L, a^U] \times \left[\frac{1}{b^U}, \frac{1}{b^L}\right] = \left[Min\left\{\frac{a^L}{b^U}, \frac{a^L}{b^L}, \frac{a^U}{b^U}, \frac{a^U}{b^L}\right\}, \ Max\left\{\frac{a^L}{b^U}, \frac{a^L}{b^L}, \frac{a^U}{b^U}, \frac{a^U}{b^L}\right\}\right]$$ with $\bar{B} \neq [0, 0]$.

2.2. Solution Concepts

For $\bar{A} = [a^L, a^U]$ and $\bar{B} = [b^L, b^U]$, we write $\bar{A} \preccurlyeq \bar{B}$, if and only if $a^L \le b^L$ and $a^U \le b^U$.

This means that $\bar{A}$ is inferior to $\bar{B}$ or $\bar{B}$ is superior to $\bar{A}$. It is easy to see that $\preccurlyeq$ is a partial ordering on $\mathfrak{T}$. We also define $\bar{A} \preccurlyeq \bar{B}$ if and only if $\bar{B} \succcurlyeq \bar{A}$.

Now, we define $\bar{A} \prec \bar{B}$ if and only if $\bar{A} \preccurlyeq \bar{B}$ and $\bar{A} \neq \bar{B}$. We also define $\bar{A} \prec \bar{B}$ if and only if $\bar{B} \succ \bar{A}$.

Equivalently $\bar{A} \prec \bar{B}$ if and only if $a^L < b^L$, $a^U = b^U$ or $a^L = b^L$, $a^U < b^U$ or $a^L < b^L$, $a^U < b^U$.

III. Materials and Methods

This part is devoted to the study of the simplex method. This method is the main tool for solving linear programming problems. It consists of following a certain number of stages before obtaining the solution of a given problem. It is an iterative algebraic method which allows to find the exact solution of a linear programming problem in a finite number of steps.

3.1. Mathematical Formulation of LP

3.1.1. The primal LP problem

Standard Form

Consider the following primal linear programming problem

$$Z(x_1, \dots, x_n) = c_1x_1 + \cdots + c_nx_n \longrightarrow Max$$
$$\textbf{(SF)}\text{Subject to} \begin{cases} a_{r1}x_1 + a_{r2}x_2 + \ldots + a_{rn}x_n \le b_r \ , 1 \le r \le p \\ a_{s1}x_1 + a_{s2}x_2 + \ldots + a_{sn}x_n \ge b_s \ , p+1 \le s \le m \\ x_j \ge 0, \ 1 \le j \le n \\ b_i > 0, \ 1 \le i \le m \end{cases}$$

in an equivalent way

$$Z(x_1, \dots, x_n) = c_1x_1 + \cdots + c_nx_n \longrightarrow Max$$
$$\textbf{(SF)}\ \text{Subject to} \begin{cases} a_{r1}x_1 + a_{r2}x_2 + \ldots + a_{rn}x_n \le b_r \ , 1 \le r \le p \\ -a_{s1}x_1 - a_{s2}x_2 - \ldots - a_{sn}x_n \le -b_s \ , p+1 \le s \le m \\ x_j \ge 0, \ 1 \le j \le n \\ b_i > 0, \ 1 \le i \le m \end{cases}$$

where a_{ij} , c_j , b_i, $x_j \in \mathbb{R}$, $i = 1,2, \dots, m$ and $j = 1,2, \dots, n$.

Canonical Form
We introduce a slack variable $x_{n+i} \geq 0$ (slack variable for i-th contraint) and write the canonical for

$$Z(x_1, \dots, x_n) = c_1x_1 + \cdots + c_nx_n + 0x_{n+1} + \cdots + 0x_{n+m} \longrightarrow Max$$

$$\textbf{(CF)}\text{Subject to} \begin{cases} a_{r1}x_1 + a_{r2}x_2 + \ldots + a_{rn}x_n + x_{n+r} = b_r \, , 1 \leq r \leq p \\ -a_{s1}x_1 - a_{s2}x_2 - \ldots - a_{sn}x_n + x_{n+s} = -b_s \, , p+1 \leq s \leq m \\ x_j \geq 0, \; 1 \leq j \leq n+m \\ b_i > 0, \; 1 \leq i \leq m \end{cases}$$

3.1.2. The dual LP problem
In accordance with the duality theory of linear programming the dual problem for **(SF)**is as follows:

$$W(x_1, \dots, x_n) = b_1y_1 + \cdots + b_py_p - b_{p+1}y_{p+1} - \cdots - b_my_m \longrightarrow Min$$

$$\textbf{(DLP)} \text{ Subject to} \begin{cases} a_{1j}\,y_1 + \cdots + a_{pj}\,y_p - a_{(p+1)j}\,y_{p+1} - \ldots - a_{mj}\,y_m \geq c_j \\ y_i \geq 0 \\ 1 \leq i \leq m \\ 1 \leq j \leq n \end{cases}$$

3.2. Simplex table: $\boldsymbol{T^{(s)}}$
We propose the simplex table model as follows:

Initial table of the simplex $\boldsymbol{T^{(0)}}$from **(CF)**

Basic variables $x_B^{(0)}$	Coefficients of basis in $Z(x)$: $C_B^{(0)}$	c_1	c_2	. . .	c_{n+m}	Current values $X_B^{(0)}$
		$A_1^{(0)}$	$A_2^{(0)}$	. . .	$A_{n+m}^{(0)}$	
x_{n+1}	0	a_{11}	a_{12}	. . .	$a_{1(n+m)}$	b_1
	⋮	⋮	⋮	⋮ ⋮ ⋮	⋮	⋮
x_{n+p}	0	a_{p1}	a_{p2}	. . .	$a_{p(n+m)}$	b_p
x_{n+p+1}	0	$a_{(p+1)1}$	$a_{(p+1)2}$	. . .	$a_{(p+1)(n+m)}$	$-b_{p+1}$
⋮	⋮	⋮	⋮	⋮ ⋮ ⋮	⋮	⋮
x_{n+m}	0	a_{m1}	a_{m2}	. . .	$a_{m(n+m)}$	$-b_m$
$Z_j^{(0)} = C_B^{(0)}A_j^{(0)}$		0	0	. . .	0	$Z(x) = C_B^{(0)}X_B^{(0)}$
$\Delta_j^{(0)} = Z_j^{(0)} - c_j$		$-c_1$	$-c_2$	. . .	0	

In the iteration (s) or in the s-th table called the simplex table $\boldsymbol{T^{(s)}}$.
In the simplex table $\boldsymbol{T^{(s)}}$ and for $1 \leq j \leq n+m$ we have: the basic variables column is $x_B^{(s)} = \{x_{J_1}, x_{J_2}, \dots, x_{J_m}\}$,the solution matrix is $X_B^{(s)} = \left(x_{J_1} = b_1^{(s)} \;\; x_{J_2} = b_2^{(s)} \; \dots \; x_{J_m} = b_m^{(s)}\right)^t$, the matrices of each column of the table are
$A_j^{(s)} = \left(a_{1j}^{(s)} a_{2j}^{(s)} \; \dots \; a_{mj}^{(s)}\right)^t$, $\;B_B^{-1(s)} = \left(A_{n+1}^{(s)} A_{n+2}^{(s)} \; \dots \; A_{n+m}^{(s)}\right), = \left(a_{ij}^{(s)}\right)_{\substack{1 \leq i \leq m \\ n+1 \leq j \leq n+m}}$ and the opportunity and marginal costs of each activity $Z_j^{(s)} = C_B^{(s)}A_j^{(s)}$ and $\Delta_j^{(s)} = Z_j^{(s)} - c_j$ The values of the functions F is: $Z(x) = C_B^{(s)}X_B^{(s)}$. Moreover $A_j^{(s)} = B_B^{-1(s)}A_j$ and $X_B^{(s)} = B_B^{-1(s)}b$.

Simplex table $\boldsymbol{T^{(s)}}$:

Basic variables $x_B^{(s)}$	Coefficients of basis in $Z(x)$: $C_B^{(s)}$	c_1	c_2	. . .	c_{n+m}	Current values $X_B^{(s)}$
		$A_1^{(s)}$	$A_2^{(s)}$	. . .	$A_{n+m}^{(s)}$	
x_{J_1}	c_{J_1}	$a_{11}^{(s)}$	$a_{12}^{(s)}$	. . .	$a_{1(n+m)}^{(s)}$	$x_{J_1} = b_1^{(s)}$
x_{J_2}	c_{J_2}	$a_{21}^{(s)}$	$a_{22}^{(s)}$	. . .	$a_{2(n+m)}^{(s)}$	$x_{J_2} = b_2^{(s)}$
⋮	⋮	⋮	⋮	⋮ ⋮ ⋮	⋮	⋮
x_{J_m}	c_{J_m}	$a_{m1}^{(s)}$	$a_{m2}^{(s)}$	. . .	$a_{m(n+m)}^{(s)}$	$x_{J_m} = b_m^{(s)}$
$Z_j^{(s)} = C_B^{(s)}A_j^{(s)}$		$Z_1^{(s)}$	$Z_2^{(s)}$	. . .	$Z_{n+m}^{(s)}$	$Z(x) = C_B^{(s)}X_B^{(s)}$
$\Delta_j^{(s)} = Z_j^{(s)} - c_j$		$\Delta_1^{(s)}$	$\Delta_2^{(s)}$	. . .	$\Delta_{n+m}^{(s)}$	

Optimal solution:

If $\boldsymbol{T}^{(s)}$ is optimal, then the current basis is $x_B^{(s)} = \{x_{J_1}, x_{J_2}, \dots, x_{J_m}\}$and the corresponding solution is $x_B^* = \left\{x_{J_1} = b_1^{(s)}, x_{J_2} = b_2^{(s)}, \dots, x_{J_m} = b_m^{(s)}\right\}$. Moreover, the current nonbasic variables is$x_N^{(s)} = \left\{x_d,\ \ x_d \notin x_B^{(s)}\right\}$ and the corresponding solution is $x_N^* = \left\{x_d = 0,\ \ x_d \in x_N^{(s)}\right\}$. Hence the optimal solution to the problem can be written as $x^* = (x_1 x_2\ \dots\ x_n\ \dots\ x_{n+m})^t$with the associated value of the objective function $F(x^*) = cx^*$.

3.3. Iteration procedure

It is an iterative algebraic method which allows to find the exact solution of a linear programming problem in a finite number of steps.

Algorithm 1: Maximization Form

STEP (0) The problem is initially in canonical form with$m = r$ in **(CF)** and construct the initial table of the simplex $\boldsymbol{T}^{(\mathbf{0})}$

STEP (1) If $\Delta_j^{(s)} \geq 0$for $j = 1,2, \dots, n$ then *stop*; we are optimal $\boldsymbol{T}^{(\boldsymbol{s})}$.

If we continue thenthere exists some $\Delta_j^{(s)} < 0$.

STEP (2) Choose the column k to pivot in (i.e., the variable x_k to introduce into the basis) by $\Delta_k^{(s)} = \min_{d \in x_N^{(s)}} \left(\Delta_d^{(s)}\right)$.If $a_{ik}^{(s)} \leq 0$for $i = 1,2, \dots, m$ then *stop*; the primal problem is unbounded.

If we continue, then $a_{ik}^{(s)} > 0$for some $i = 1,2, \dots, m$.

STEP (3) Choose row $\ell$to pivot in (i.e., the variable x_ℓ to drop from the basis) by the ratio test:

$$\frac{b_\ell^{(s)}}{a_{\ell k}^s} = \min_{1 \leq i \leq m} \left(\frac{b_i^{(s)}}{a_{ik}^{(s)}}, a_{ik}^{(s)} > 0\right).$$

STEP (4) Replace the basic variable in row ℓ with variable $k$and re-establish the canonical form (i.e., pivot on the coefficient $a_{\ell k}^{(s)}$).

STEP (5) do
$$\begin{cases} L_\ell^{(s+1)} = \frac{L_\ell^{(s)}}{a_{\ell k}^{(s)}} \\ L_r^{(s+1)} = L_r^{(s)} - a_{rk}^{(s)} L_\ell^{(s+1)} \end{cases} \text{with } 1 \leq r \neq \ell \leq m$$

STEP (6) Go to step (1).

Algorithm 2: Minimization Form

STEP (0) The problem is initially in canonical form with $m = r$ in **(CF)** and construct the initial table of the simplex $\boldsymbol{T}^{(\mathbf{0})}$

STEP (1) If $\Delta_j^{(s)} \leq 0$for $j = 1,2, \dots, n$ then *stop*; we are optimal $\boldsymbol{T}^{(\boldsymbol{s})}$. If we continue then there exists some $\Delta_j^{(s)} > 0$.

STEP (2) Choose the column k to pivot in (i.e., the variable x_k to introduce into the basis) by $\Delta_k^{(s)} = \max_{d \in x_N^{(s)}} \left(\Delta_d^{(s)}\right)$.If $a_{ik}^{(s)} \leq 0$for $i = 1,2, \dots, m$ then *stop*; the primal problem is unbounded.

If we continue, then $a_{ik}^{(s)} > 0$for some $i = 1,2, \dots, m$.

STEP (3) Choose row $\ell$to pivot in (i.e., the variable x_ℓ to drop from the basis) by the ratio test:

$$\frac{b_\ell^{(s)}}{a_{\ell k}^s} = \min_{1 \leq i \leq m} \left(\frac{b_i^{(s)}}{a_{ik}^{(s)}}, a_{ik}^{(s)} > 0\right).$$

STEP (4) Replace the basic variable in row ℓ with variable $k$and re-establish the canonical form (i.e., pivot on the coefficient $a_{\ell k}^{(s)}$).

STEP (5) do
$$\begin{cases} L_\ell^{(s+1)} = \frac{L_\ell^{(s)}}{a_{\ell k}^{(s)}} \\ L_r^{(s+1)} = L_r^{(s)} - a_{rk}^{(s)} L_\ell^{(s+1)} \end{cases} \text{with} 1 \leq r \neq \ell \leq m$$

STEP (6) Go to step (1).

Algorithm 3: Maximization Form

STEP (0) The problem is initially in canonical form **(CF)** and construct the initial table of the simplex $\boldsymbol{T}^{(\mathbf{0})}$

STEP (1) If $\Delta_j^{(s)} \geq 0$for $j = 1,2, \dots, n$ and $b_i \geq 0,\ 1 \leq i \leq m$then *stop*; we are optimal $\boldsymbol{T}^{(\boldsymbol{s})}$. If we continue thenthere exists some $b_i^{(s)} < 0,\ 1 \leq i \leq m$.

STEP (2) Choose row $\ell$to pivot in (i.e., the variable x_ℓ to drop from the basis) by

$$b_\ell^{(s)} = \min_{1\le i\le m}\left(b_i^{(s)}\right)$$

STEP (3) Choose the column k to pivot in (i.e., the variable x_k to introduce into the basis) by the ratio test:

$$\frac{\Delta_k^{(s)}}{a_{\ell k}^{(s)}} = \max_{d\in x_N^{(s)}}\left(\frac{\Delta_d^{(s)}}{a_{\ell d}^{(s)}}, a_{\ell d}^{(s)} < 0\right)$$

If $a_{\ell d}^{(s)} \ge 0$for $d \in x_N^{(s)}$ then *stop*; the primal problem is unbounded.
If we continue, then $a_{\ell d}^{(s)} < 0$for some $d \in x_N^{(s)}$.
STEP (4) Replace the basic variable in row ℓ with variable $k$and re-establish the canonical form (i.e., pivot on the coefficient $a_{\ell k}^{(s)}$).

STEP (5) do $\begin{cases} L_\ell^{(s+1)} = \frac{L_\ell^{(s)}}{a_{\ell k}^{(s)}} \\ L_r^{(s+1)} = L_r^{(s)} - a_{rk}^{(s)} L_\ell^{(s+1)} \end{cases}$ with $1 \le r \ne \ell \le m$

STEP (6) Go to step (1).
STEP (7) For some $\Delta_j^{(s)} < 0$and $b_i \ge 0,\ 1 \le i \le m$ then Go to **Algorithm 1.**

Algorithm 4: Minimization Form
STEP (0) The problem is initially in canonical form **(CF)** and construct the initial table of the simplex $\boldsymbol{T}^{(\mathbf{0})}$
STEP (1) If $\Delta_j^{(s)} \le 0$for $j = 1,2,\dots,n$ and $b_i \ge 0,\ 1 \le i \le m$ then *stop*; we are optimal $\boldsymbol{T}^{(s)}$. If we continue then there exists some$b_i^{(s)} < 0,\ 1 \le i \le m$.
STEP (2) Choose row $\ell$to pivot in (i.e., the variable x_ℓ to drop from the basis) by

$$b_\ell^{(s)} = \min_{1\le i\le m}\left(b_i^{(s)}\right)$$

STEP (3) Choose the column k to pivot in (i.e., the variable x_k to introduce into the basis) by the ratio test:

$$\frac{\Delta_k^{(s)}}{a_{\ell k}^{(s)}} = \min_{d\in x_N^{(s)}}\left(\frac{\Delta_d^{(s)}}{a_{\ell d}^{(s)}}, a_{\ell d}^{(s)} < 0\right)$$

If $a_{\ell d}^{(s)} \ge 0$for $d \in x_N^{(s)}$ then *stop*; the primal problem is unbounded.
If we continue, then $a_{\ell d}^{(s)} < 0$for some $d \in x_N^{(s)}$.
STEP (4) Replace the basic variable in row ℓ with variable $k$and re-establish the canonical form (i.e., pivot on the coefficient $a_{\ell k}^{(s)}$).

STEP (5) do $\begin{cases} L_\ell^{(s+1)} = \frac{L_\ell^{(s)}}{a_{\ell k}^{(s)}} \\ L_r^{(s+1)} = L_r^{(s)} - a_{rk}^{(s)} L_\ell^{(s+1)} \end{cases}$ with $1 \le r \ne \ell \le m$

STEP (6) Go to step (1).
STEP (7) For some $\Delta_j^{(s)} > 0$and $b_i \ge 0,\ 1 \le i \le m$ then Go to **Algorithm 2.**

IV. Main Results

Now we are in a position to prove interval analogue of some important relationships between the primal and dual linear programming problems. We consider the primal and dual linear programming problems involving interval numbers as follows:

4.1. Mathematical Formulation of LPproblem involving interval numbers

4.1.1. The primal LP problem involving interval numbers

Standard Form

Consider the following linear programming problem involving interval numbers

$$\bar{Z}(\overline{x_1},\dots,\overline{x_n}) \approx \overline{c_1}\overline{x_1} + \cdots + \overline{c_n}\overline{x_n} \longrightarrow Max$$

(SFI) Subject to $\begin{cases} \overline{a_{r1}x_1} + \overline{a_{r2}x_2} + \ .\ .\ . + \overline{a_{rn}x_n} \preccurlyeq \overline{b_r}\ , 1 \le r \le p \\ \overline{a_{s1}x_1} + \overline{a_{s2}x_2} + \ .\ .\ . + \overline{a_{sn}x_n} \succcurlyeq \overline{b_s}\ , p+1 \le s \le m \\ \overline{x_j} \succcurlyeq 0,\ 1 \le j \le n \\ \overline{b_i} \succcurlyeq 0,\ 1 \le i \le m \end{cases}$

in an equivalent way

$$\bar{Z}(\overline{x_1},\dots,\overline{x_n}) \approx \overline{c_1}\overline{x_1} + \cdots + \overline{c_n}\overline{x_n} \longrightarrow Max$$

$$\textbf{(SFI)}\ \text{Subject to}\begin{cases} \overline{a_{r1}x_1}+\overline{a_{r2}x_2}+\ldots+\overline{a_{rn}x_n}\preccurlyeq\overline{b_r}\ ,1\le r\le p\\ -\overline{a_{s1}x_1}-\overline{a_{s2}x_2}-\ldots-\overline{a_{sn}x_n}\preccurlyeq-\overline{b_s}\ ,p+1\le s\le m\\ \overline{x_j}\succcurlyeq 0,\ 1\le j\le n\\ \overline{b_i}\succcurlyeq 0,\ 1\le i\le m\end{cases}$$

where $\overline{a_{ij}}, \overline{c_j}, \overline{b_i}, \overline{x_j} \in \mathbb{R}$, $i = 1,2,\ldots,m$ and $j = 1,2,\ldots,n$.

We call the above problem **(SFI)** as the primal interval linear programming problem, and it can be rewritten as **(SFI)**: $Max\ \bar{Z} \approx \bar{c}\bar{x}$subject to $\bar{A}\bar{x} \preccurlyeq \bar{b}$ and $\bar{x} \succcurlyeq 0$, where $\bar{A}$, $\bar{b}$,$\bar{c}$,$\bar{x}$ are $(m\times n)$, $(m\times 1)$,$(1\times n)$, $(n\times 1)$ matrices involving interval numbers.

Canonical Form

We introduce a slack variable $\overline{x_{n+i}} \ge 0$ (slack variable for i-th contraint) and write the canonical form:

$$\bar{Z}(\overline{x_1},\ldots,\overline{x_n}) \approx \overline{c_1}\overline{x_1}+\cdots+\overline{c_n}\overline{x_n}+\bar{0}\overline{x_{n+1}}+\cdots+\bar{0}\overline{x_{n+m}} \longrightarrow Max$$

$$\textbf{(CFI)}\ \text{Subject to}\begin{cases} \overline{a_{r1}x_1}+\overline{a_{r2}x_2}+\ldots+\overline{a_{rn}x_n}+\overline{x_{n+r}}\approx\overline{b_r}\ ,1\le r\le p\\ -\overline{a_{s1}x_1}-\overline{a_{s2}x_2}-\ldots-\overline{a_{sn}x_n}+\overline{x_{n+s}}\approx-\overline{b_s}\ ,p+1\le s\le m\\ \overline{x_j}\succcurlyeq 0,\ 1\le j\le n\\ \overline{b_i}\succcurlyeq 0,\ 1\le i\le m\end{cases}$$

4.1.2. The dual LP problem involving interval numbers

In accordance with the duality theory of linear programming the dual problem involving interval numbers for**(SFI)**is as follows:

$$\bar{W}(\overline{y_1},\ldots,\overline{y_m}) \approx \overline{b_1}\overline{y_1}+\cdots+\overline{b_p}\overline{y_p}-\overline{b_{p+1}}\overline{y_{p+1}}-\cdots-\overline{b_m}\overline{y_m} \longrightarrow Min$$

$$\textbf{(DLPI)}\ \text{Subject to}\begin{cases} \overline{a_{1j}}\,\overline{y_1}+\cdots+\overline{a_{pj}}\,\overline{y_p}-\overline{a_{(p+1)j}}\,\overline{y_{p+1}}-\ldots-\overline{a_{mj}}\,\overline{y_m}\succcurlyeq\overline{c_j}\\ \overline{y_i}\succcurlyeq 0\\ 1\le i\le m\\ 1\le j\le n\end{cases}$$

We call the above problem **(DLPI)** as the dual interval linear programming problem of the primal problem **(SFI)** , and it can be rewritten as **(DLPI)**: $Min\ \bar{W} \approx \bar{b}\bar{y}$subject to $\bar{A}\bar{y} \succcurlyeq \bar{c}$ and $\bar{y} \succcurlyeq 0$, where $\bar{A}$, $\bar{b}$,$\bar{c}$,$\bar{y}$ are $(m\times n)$, $(1\times m)$, $(n\times 1)$, $(m\times 1)$ matrices involving interval numbers.

Theorem 1. Consider $\bar{A}\bar{x} \approx \bar{b}$, where $\bar{A} = (\bar{a}_{ij})_{m\times n}$, $\bar{a}_{ij} \in \mathbb{R}$. Then $\bar{X}_B^{(s)} = \bar{B}_B^{-1(s)}\bar{b}$ is a solution of $\bar{A}\bar{x} \approx \bar{b}$.

Theorem 2. (Weak duality theorem) If $\bar{x} = (\overline{x_1}, \overline{x_2}, \ldots, \overline{x_n})^t$ is any feasible solution to the primal interval linear programming problem **(SFI)** and $\bar{y} = (\overline{y_1}, \overline{y_2}, \ldots, \overline{y_m})$ is any feasible solution to the dual interval linear programming problem **(DLPI)**, then $\bar{c}\bar{x} \preccurlyeq \bar{b}\bar{y}$ or $\sum_{j=1}^{n} \overline{c_j}\overline{x_j} \preccurlyeq \sum_{i=1}^{m} \overline{b_i}\overline{y_i}$.

Theorem 3. (Strong duality theorem) If $\bar{x} = (\overline{x_1}, \overline{x_2}, \ldots, \overline{x_n})^t$ is an optimal solution to the primal problem **(SFI)**, then there exit a feasible solution $\bar{y} = (\overline{y_1}, \overline{y_2}, \ldots, \overline{y_m})$ to the dual problem **(DLPI)** such that $\bar{c}\bar{x} \approx \bar{b}\bar{y}$ or $\sum_{j=1}^{n} \overline{c_j}\overline{x_j} \approx \sum_{i=1}^{m} \overline{b_i}\overline{y_i}$.

Theorem 4. (Complementary Slackness theorem)) If $\overline{x^*} = (\overline{x_1}, \overline{x_2}, \ldots, \overline{x_n})^t$ is a feasible solution to the primal problem **(SFI)** and$\overline{y^*} = (\overline{y_1}, \overline{y_2}, \ldots, \overline{y_m})$is a feasible solution to the dual problem **(DLPI)**, then they must satisfy the so-called complementary slackness conditions:

(i) If$\overline{y_i^*} \succ \bar{0}$, then$\sum_{j=1}^{n} \overline{a_{ij}}\overline{x^*_j} \approx \overline{b_i}$ (ii) If $\sum_{j=1}^{n} \overline{a_{ij}}\overline{x^*_j} \prec \overline{b_i}$, then $\overline{y_i^*} \approx \bar{O}$.

(iii) If $\overline{x_j^*} \succ \bar{0}$, then $\sum_{i=1}^{m} \overline{a_{ij}}\overline{y^*_i} \approx \overline{c_j}$ (iv) If $\sum_{i=1}^{m} \overline{a_{ij}}\overline{y^*_i} \prec \overline{c_j}$, then $\overline{x_j^*} \approx \bar{O}$.

4.1.3. Table$\bar{T}^*$ optimal for linear programming problem involving interval numbers

If $\bar{T}^*$ is optimal, then the current basis is $\bar{x}_B^{(s)} = \{\bar{x}_{J_1}, \bar{x}_{J_2}, \ldots, \bar{x}_{J_m}\}$and the corresponding solution is $\bar{x}_B^* = (\bar{x}_{J_1}\bar{x}_{J_2}\ \ldots\ \bar{x}_{J_m})^t = \bar{B}_B^{-1(s)}\bar{b}$ and $\bar{X}_B^{(s)} = \bar{B}_B^{-1(s)}\bar{b}$ with $\bar{B}_B^{-1(s)} = (\bar{a}_{ij}{}^{(s)})_{\substack{1\le i\le m\\ n+1\le j\le n+m}}$

Moreover, the current nonbasic variables is$\bar{x}_N^{(s)} = \{\bar{x}_d,\ \ \bar{x}_d \notin \bar{x}_B^{(s)}\}$and the corresponding solution is $\bar{x}_N^* = \{\bar{x}_d \approx \bar{0},\ \ x_d \in x_N^{(s)}\}$. Hence the optimal solution to the problem can be written as $x^* = (\bar{x}_1\bar{x}_2\ \ldots\ \bar{x}_n\ \ldots\ \bar{x}_{n+m})^t$with the associated value of the objective function$\bar{Z}(\bar{x}^*) \approx \bar{c}\bar{x}^*$.

Maximization form:$\overline{y_i^*} \approx \overline{\Delta_{n+i}}$, $\overline{y_{m+j}{}^*} \approx \overline{\Delta_j}$, $i = 1,2,\ldots,m$ and $j = 1,2,\ldots,n$.

Minimization form: $\overline{y_i^*} \approx [|\overline{\Delta^U{}_{n+i}}|,\ |\overline{\Delta^L{}_{n+i}}|]$, $\overline{y_{m+j}{}^*} \approx [|\overline{\Delta^U{}_j}|,\ |\overline{\Delta^L{}_j}|]$.

4.2. Numerical Examples

Example 1. Consider the following interval number linear programming problem:

$$\bar{Z}(\overline{x_1},\overline{x_2},\overline{x_3}) \approx [29,31]\overline{x_1} + [22,24]\overline{x_2} + [28,30]\overline{x_3} \longrightarrow Max$$

$$\textbf{(P)}\ \text{Subject to}\begin{cases} 6\,\overline{x_1} + 5\,\overline{x_2} + \ 3\,\overline{x_3} \preccurlyeq [25,27] \\ 4\,\overline{x_1} + 2\,\overline{x_2} + \ 5\,\overline{x_3} \preccurlyeq [6,8] \\ \overline{x_1},\overline{x_2},\overline{x_3} \succcurlyeq 0 \end{cases}$$

We call the above problem as the primal problem. Then the corresponding dual problem is given by

$$\bar{W}(\overline{y_1},\overline{y_2}) \approx [25,27]\overline{y_1} + [6,8]\overline{y_2} \longrightarrow Min$$

$$\textbf{(D)}\ \text{Subject to}\begin{cases} 6\,\overline{y_1} + 4\,\overline{y_2} \succcurlyeq [29,31] \\ 5\,\overline{y_1} + 2\,\overline{y_2} \succcurlyeq [22,24] \\ 3\,\overline{y_1} + 5\,\overline{y_2} \succcurlyeq [28,30] \\ \overline{y_1},\overline{y_2} \succcurlyeq 0 \end{cases}$$

Resolution 1: Optimal solution to the primal interval number linear programming problem

Let us apply the interval version of simplex algorithm and the interval arithmetic to solve the primal problem. We convert the primal problem **(P)**to its canonical form by adding slack variables$\overline{x_{n+i}} \succcurlyeq \bar{0}$ as follows:

$$\bar{Z}(\overline{x_1},\overline{x_2},\overline{x_3}) \approx [29,31]\overline{x_1} + [22,24]\overline{x_2} + [28,30]\overline{x_3} + 0\overline{x_4} + 0\overline{x_5} \longrightarrow Max$$

$$\textbf{(P)}\ \text{Subject to}\begin{cases} 6\,\overline{x_1} + 5\,\overline{x_2} + \ 3\,\overline{x_3} + \overline{x_4} + 0\overline{x_5} \approx [25,27] \\ 4\,\overline{x_1} + 2\,\overline{x_2} + \ 5\,\overline{x_3} + 0\overline{x_4} + \overline{x_5} \approx [6,8] \\ \overline{x_1},\overline{x_2},\overline{x_3},\overline{x_4},\overline{x_5} \succcurlyeq 0 \end{cases}$$

Initial iteration (Algorithm 1, $\boldsymbol{T^{(s=0)}}$): Initial basic feasible solution

Basic variables $\bar{x}_B^{(0)}$	Coefficients of basis in $\bar{Z}(\bar{x})$: $\bar{C}_B^{(0)}$	$[29,31]$	$[22,24]$	$[28,30]$	0	0	Current values $\bar{X}_B^{(0)}$
		$\bar{A}_1^{(0)}$	$\bar{A}_2^{(0)}$	$\bar{A}_3^{(0)}$	$\bar{A}_4^{(0)}$	$\bar{A}_5^{(0)}$	
$\bar{x}_4$	0	6	5	3	1	0	$[25,27]$
$\bar{x}_5$	0	4	2	5	0	1	$[6,8]$
$\bar{Z}_j^{(0)} = \bar{C}_B^{(0)}\bar{A}_j^{(0)}$		0	0	0	0	0	$\bar{Z}(\bar{x}) = \bar{C}_B^{(0)}\bar{X}_B^{(0)} = 0$
$\bar{\Delta}_j^{(0)} = \bar{Z}_j^{(0)} - \bar{c}_j$		$[-31,-29]$	$[-24,-22]$	$[-30,-28]$	0	0	

First iteration (Algorithm 1, $\boldsymbol{T^{(s=1)}}$): Here $\overline{x_5}$ leaves the basis and $\overline{x_1}$enters in to the basis

Basic variables $\bar{x}_B^{(1)}$	Coefficients of basis in $\bar{Z}(\bar{x})$: $\bar{C}_B^{(1)}$	$[29,31]$	$[22,24]$	$[28,30]$	0	0	Current values $\bar{X}_B^{(1)}$
		$\bar{A}_1^{(1)}$	$\bar{A}_2^{(1)}$	$\bar{A}_3^{(1)}$	$\bar{A}_4^{(1)}$	$\bar{A}_5^{(1)}$	
$\bar{x}_4$	0	0	2	$\frac{-9}{2}$	1	$\frac{-3}{2}$	$[13,18]$
$\bar{x}_1$	$[29,31]$	1	$\frac{1}{2}$	$\frac{5}{4}$	0	$\frac{1}{4}$	$\left[\frac{3}{2},2\right]$
$\bar{Z}_j^{(1)} = \bar{C}_B^{(1)}\bar{A}_j^{(1)}$		$[29,31]$	$\left[\frac{29}{2},\frac{31}{2}\right]$	$\left[\frac{145}{4},\frac{155}{4}\right]$	0	$\left[\frac{29}{4},\frac{31}{4}\right]$	$\bar{Z}(\bar{x}) = \bar{C}_B^{(1)}\bar{X}_B^{(1)}$
$\bar{\Delta}_j^{(1)} = \bar{Z}_j^{(1)} - \bar{c}_j$		$[-31,-29]$	$\left[\frac{-19}{2},\frac{-13}{2}\right]$	$\left[\frac{25}{4},\frac{34}{4}\right]$	0	$\left[\frac{29}{4},\frac{31}{4}\right]$	

Second iteration (Algorithm 1, $\boldsymbol{T^{(s=2)}}$): Here$\overline{x_1}$leaves the basis and$\overline{x_2}$enters in to the basis

Basic variables $\bar{x}_B^{(2)}$	Coefficients of basis in $\bar{Z}(\bar{x})$: $\bar{C}_B^{(2)}$	$[29,31]$	$[22,24]$	$[28,30]$	0	0	Current values $\bar{X}_B^{(2)}$
		$\bar{A}_1^{(2)}$	$\bar{A}_2^{(2)}$	$\bar{A}_3^{(2)}$	$\bar{A}_4^{(2)}$	$\bar{A}_5^{(2)}$	
$\bar{x}_4$	0	-4	0	$\frac{-19}{2}$	1	$\frac{-5}{2}$	$[5,12]$
$\bar{x}_2$	$[22,24]$	2	1	$\frac{5}{2}$	0	$\frac{1}{2}$	$[3,4]$
$\bar{Z}_j^{(2)} = \bar{C}_B^{(2)}\bar{A}_j^{(2)}$		$[44,48]$	$[22,24]$	$[55,60]$	0	$[11,12]$	$\bar{Z}(\bar{x}) = \bar{C}_B^{(2)}\bar{X}_B^{(2)} = [66,96]$
$\bar{\Delta}_j^{(0)} = \bar{Z}_j^{(0)} - \bar{c}_j$		$[13,19]$	0	$[25,32]$	0	$[11,12]$	

If $\bar{\boldsymbol{T}}^* = \boldsymbol{T}^{(s=2)}$ is optimal, then the current basis is $\bar{x}_B^{(2)} = \{\bar{x}_5, \bar{x}_2\}$and the corresponding solution is

$$\bar{x}_B^* = \begin{pmatrix}\bar{x}_5\\ \bar{x}_2\end{pmatrix} = \bar{B}_B^{-1(2)}\bar{b} = \left(\bar{a}_{ij}{}^{(2)}\right)_{\substack{1\le i\le 2\\ 4\le j\le 5}}\bar{b} = \begin{pmatrix}1 & \frac{-5}{2}\\ 0 & \frac{1}{2}\end{pmatrix}\begin{pmatrix}[25,27]\\ [6,8]\end{pmatrix} = \begin{pmatrix}[5,12]\\ [3,4]\end{pmatrix}$$

Moreover, the current nonbasic variables is$\bar{x}_N^{(2)} = \{\bar{x}_1, \bar{x}_3, \bar{x}_4\}$ and the corresponding solution is $\bar{x}_N^* = \{\bar{x}_1 \approx \bar{0}, \bar{x}_3 \approx \bar{0}, \bar{x}_4 \approx \bar{0}\}$. Hence the optimal solution to the problem can be written as $x^* = (\bar{x}_1 \approx \bar{0}\bar{x}_2 \approx [5,12]\bar{x}_3 \approx \bar{0}\bar{x}_4 \approx \bar{0}\bar{x}_5 \approx [3,4])^t$with the associated value of the objective function $\bar{Z}(\bar{x}^*) \approx \bar{c}\bar{x}^* \approx [66,96]$.

Optimal solution of (D): Maximization form: $\overline{y_i{}^*} \approx \overline{\Delta_{3+i}}$, $\overline{y_{2+j}{}^*} \approx \bar{\Delta}_j$, $i = 1,2$ and $j = 1,2,3$.

$\overline{y_1{}^*} \approx \overline{\Delta_4} \approx \bar{0}$, $\overline{y_2{}^*} \approx \overline{\Delta_2} \approx [11,12]$, $\overline{y_3{}^*} \approx \overline{\Delta_1} \approx [13,19]$,$\overline{y_4{}^*} \approx \overline{\Delta_2} \approx \bar{0}$ and $\overline{y_5{}^*} \approx \overline{\Delta_3} \approx [25,32]$.

$Min\ \bar{W}(\overline{y_1{}^*}, \overline{y_2{}^*}) \approx [25,27] \times \bar{0} + [6,8] \times [11,12] = [66,96] \approx Max\ \bar{Z}(\bar{x}^*)$.

Resolution 2: Optimal solution to the dual interval number linear programming problem

We convert the primal problem **(D)**to its canonical form by adding slack variables $\overline{y_{n+i}} \geqslant \bar{0}$ as follows:

$$\bar{W}(\overline{y_1}, \overline{y_2}) \approx [25,27]\overline{y_1} + [6,8]\overline{y_2} + 0\overline{y_3} + 0\overline{y_4} + 0\overline{y_5} \longrightarrow Min$$

$$\textbf{(D)}\ \text{Subject to} \begin{cases} -6\,\overline{y_1} - 4\,\overline{y_2} + \overline{y_3} + 0\overline{y_4} + 0\overline{y_5} \approx [-31,-29]\\ -5\,\overline{y_1} - 2\,\overline{y_2} + 0\overline{y_3} + \overline{y_4} + 0\overline{y_5} \approx [-24,-22]\\ -3\,\overline{y_1} - 5\,\overline{y_2} + 0\overline{y_3} + 0\overline{y_4} + \overline{y_5} \approx [-30,-28]\\ \overline{y_1}, \overline{y_2}, \overline{y_3}, \overline{y_4}, \overline{y_5} \geqslant 0 \end{cases}$$

Initial iteration (Algorithm 4, $\boldsymbol{T}^{(s=0)}$): Initial basic feasible solution

Basic variables $\bar{y}_B^{(0)}$	Coefficients of basis in $\bar{W}(\bar{y})$: $\bar{C}_B^{(0)}$	$[25,27]$	$[6,8]$	0	0	0	Current values $\bar{Y}_B^{(0)}$
		$\bar{A}_1^{(0)}$	$\bar{A}_2^{(0)}$	$\bar{A}_3^{(0)}$	$\bar{A}_4^{(0)}$	$\bar{A}_5^{(0)}$	
$\bar{y}_3$	0	-6	-4	1	0	0	$[-31,-29]$
$\bar{y}_4$	0	-5	-2	0	1	0	$[-24,-22]$
$\bar{y}_5$	0	-3	-5	0	0	1	$[-30,-28]$
$\bar{Z}_j^{(0)} = \bar{C}_B^{(0)}\bar{A}_j^{(0)}$		0	0	0	0	0	$\bar{Z}(\bar{y}) = \bar{C}_B^{(0)}\bar{Y}_B^{(0)} = 0$
$\bar{\Delta}_j^{(0)} = \bar{Z}_j^{(0)} - \bar{c}_j$		$[-27,-25]$	$[-8,-6]$	0	0	0	

First iteration (Algorithm 4, $\boldsymbol{T}^{(s=1)}$): Here $\overline{y_3}$ leaves the basis and $\overline{y_2}$enters in to the basis

Basic variables $\bar{y}_B^{(1)}$	Coefficients of basis in $\bar{W}(\bar{y})$: $\bar{C}_B^{(1)}$	$[25,27]$	$[6,8]$	0	0	0	Current values $\bar{Y}_B^{(1)}$
		$\bar{A}_1^{(1)}$	$\bar{A}_2^{(1)}$	$\bar{A}_3^{(1)}$	$\bar{A}_4^{(1)}$	$\bar{A}_5^{(1)}$	
$\bar{y}_2$	$[6,8]$	$\frac{3}{2}$	1	$\frac{-1}{4}$	0	0	$\left[\frac{29}{4}, \frac{31}{4}\right]$
$\bar{y}_4$	0	-2	0	$\frac{-1}{2}$	1	0	$\left[\frac{-19}{4}, \frac{-13}{4}\right]$
$\bar{y}_5$	0	$\frac{9}{2}$	0	$\frac{-5}{4}$	0	1	$\left[\frac{25}{4}, \frac{43}{4}\right]$
$\bar{Z}_j^{(1)} = \bar{C}_B^{(1)}\bar{A}_j^{(1)}$		$[9,12]$	$[6,8]$	$\left[-2, \frac{-3}{2}\right]$	0	0	$\bar{Z}(\bar{y}) = \bar{C}_B^{(1)}\bar{Y}_B^{(1)}$
$\bar{\Delta}_j^{(1)} = \bar{Z}_j^{(1)} - \bar{c}_j$		$[-18,-13]$	0	$\left[-2, \frac{-3}{2}\right]$	0	0	

Second iteration (Algorithm 4, $\boldsymbol{T}^{(s=2)}$): Here $\overline{y_4}$leaves the basis and $\overline{y_3}$enters in to the basis

Basic variables $\bar{y}_B^{(2)}$	Coefficients of basis in $\bar{W}(\bar{y})$: $\bar{C}_B^{(2)}$	$[25,27]$	$[6,8]$	0	0	0	Current values $\bar{Y}_B^{(2)}$
		$\bar{A}_1^{(2)}$	$\bar{A}_2^{(2)}$	$\bar{A}_3^{(2)}$	$\bar{A}_4^{(2)}$	$\bar{A}_5^{(2)}$	
$\bar{y}_2$	$[6,8]$	$\frac{5}{2}$	1	0	$\frac{-1}{2}$	0	$[11,12]$
$\bar{y}_3$	0	4	0	1	-2	0	$[13,19]$

$\bar{y}_5$	0	$\frac{19}{2}$	0	0	$\frac{-5}{2}$	1	[25,32]
$\bar{Z}_j^{(2)} = \bar{C}_B^{(2)}\bar{A}_j^{(2)}$		[15, 20]	[6, 8]	0	[−4, −3]	0	$\bar{Z}(\bar{y}) = \bar{C}_B^{(2)}\bar{Y}_B^{(2)} = [66, 96]$
$\bar{\Delta}_j^{(2)} = \bar{Z}_j^{(2)} - \bar{c}_j$		[−12, −5]	0	0	[−4, −3]	0	

If $\bar{\boldsymbol{T}}^* = \boldsymbol{T}^{(s=2)}$ is optimal, then the current basis is $\bar{y}_B^{(2)} = \{\bar{y}_2, \bar{y}_3, \bar{y}_5\}$and the corresponding solution is

$$\bar{y}_B^* = \begin{pmatrix} \bar{y}_2 \\ \bar{y}_3 \\ \bar{y}_5 \end{pmatrix} = \bar{B}_B^{-1(2)}\bar{b} = \left(\bar{a}_{ij}{}^{(2)}\right)_{\substack{1\le i\le 3 \\ 3\le j\le 5}}\bar{b} = \begin{pmatrix} 0 & \frac{-1}{2} & 0 \\ 1 & -2 & 0 \\ 0 & \frac{-5}{2} & 1 \end{pmatrix}\begin{pmatrix} [-31, -29] \\ [-24, -22] \\ [-30, -28] \end{pmatrix} = \begin{pmatrix} [11,12] \\ [13,19] \\ [25,32] \end{pmatrix}$$

Moreover, the current nonbasic variables is$\bar{y}_N^{(2)} = \{\bar{y}_1, \bar{y}_4\}$ and the corresponding solution is $\bar{y}_N^* = \{\bar{y}_1 \approx \bar{0}, \bar{y}_4 \approx \bar{0}\}$. Hence the optimal solution to the problem can be written as $y^* = (\bar{y}_1 \approx \bar{0}\bar{y}_2 \approx [11, 12]\bar{y}_3 \approx [13,19]\bar{y}_4 \approx \bar{0}\bar{y}_5 \approx [25, 32])^t$with the associated value of the objective function $\bar{W}(\bar{y}^*) \approx \bar{b}\bar{y}^* \approx [66, 96]$.

Optimal solution of (P): Maximization form:
$\overline{x_i}^* \approx \left[\left|\overline{\Delta^U}_{2+i}\right|, \left|\overline{\Delta^L}_{2+i}\right|\right]$, $\overline{x_{3+j}}^* \approx \left[\left|\overline{\Delta^U}_j\right|, \left|\overline{\Delta^L}_j\right|\right]$, $i = 1,2,3$ and $j = 1,2$.
$\overline{x_1}^* \approx \overline{\Delta_3} \approx \bar{0}$, $\overline{x_2}^* \approx \overline{\Delta_4} \approx [3, 4]$, $\overline{x_3}^* \approx \overline{\Delta_5} \approx \bar{0}$,$\overline{x_4}^* \approx \overline{\Delta_1} \approx [5, 12]$ and $\overline{x_5}^* \approx \overline{\Delta_2} \approx \bar{0}$.
$Max\ \bar{Z}(\overline{x_1}, \overline{x_2}, \overline{x_3}) \approx 0\overline{x_1} + [22, 24][3, 4] + 0\ \overline{x_3} \approx [66, 96] \approx Min\ \bar{W}(\bar{y}^*)$.

V. Conclusions

We introduced the notation of linear programming problems involving interval numbers as the way of traditional linear programming problems. The solution concepts of linear programming problems involving interval numbers without converting them to classical linear programming problems is proposed. Under arithmetic operations between interval numbers. These results will be useful for post optimality analysis. A numerical example is provided to show that the problems have optimal solutions.

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