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Picard Sequence and Fixed Point Results on G-Metric Space

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Abstract
In this paper we introduced picard sequence and fixed point result in G-metric spaces. We have utilized these concepts to deduce certain fixed point theorems in G-metric space . our theorem extend and improve the results of Sumitra and Ranjeth kumar [3], B. Singh and S. Jain [4,5,6,7] and Urmila Mishra et al.[10] in the settings of G-metric space.
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I. Introduction

The concept of metric spaces has been generalized in many directions. The notion of parametric metric spaces being a natural generalization of metric spaces was recently introduced and studied by Hussain et al. [1]. Hussain et al. [2] introduced a new type of generalized metric space, called parametric G-metric space, as a generalization of both metric and b-metric spaces. For more details on parametric metric space, parametric G-metric spaces and related results we refer the reader to [8].

In this Paper, we deal with the study of fixed point theorems in parametric G-metric spaces. This paper is composed into three sections namely 1, 2 and 3. Section 1 is introductory, while in Section 2, we give a brief introduction of parametric G-metric spaces and the work already done. In Section 3, we obtain some fixed point

results single valued mappings with rational expression in the setting of a parametric G-metric space. These results improve and generalize some important known results in literature. Some related results and illustrative some examples to highlight the realized improvements are also furnished.

II. Preliminaries

Throughout this paper R and R^+ will represents the set of real numbers and nonnegative real numbers, respectively.

Recently, Hussain et al. [2] introduced the concept of parametric b-metric space.

Definition 2.1 Let X be a nonempty set, $s \geq 1$ be a real number and $\mathrm{P} : X \times X \times (0,+\infty) \to [0,+\infty)$ be a function. We say P is a parametric G-metric on X if,

(1) $\mathrm{P}(x, y, t) = 0$ for all $t > 0$ if and only if $x = y$,

(2) $\mathrm{P}(x, y, t) = \mathrm{P}(y, x, t)$ for all $t > 0$,

(3) $\mathrm{P}(x, y, t) \leq s[\mathrm{P}(x, z, t) + \mathrm{P}(z, y, t)]$ for all $x, y, z \in X$ and all $t > 0$, where $s \geq 1$.

and one says the pair (X, P, s) is a parametric metric space with parameter $s \geq 1$.

Obviously, for $s = 1$, parametric G-metric reduces to parametric metric.

The following definitions will be needed in the sequel which can be found in [2, 8].

Definition 2.2 Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in a parametric G-metric space (X, P, s).

1. $\{x_n\}_{n=1}^{\infty}$ is said to be convergent to $x \in X$, written as $\lim_{n\to\infty} x_n = x$ for all $t > 0$, if $\lim_{n\to\infty} \mathrm{P}(x_n, x, t) = 0$.

2. $\{x_n\}_{n=1}^{\infty}$ is said to be a Cauchy sequence in X, if for all $t>0$, if $\lim_{n,m\to\infty} \mathrm{P}(x_n, x_m, t) = 0$

3. $(X,\mathrm{P},s,)$ is said to be complete if every Cauchy sequence is a convergent sequence.

Example 2.2 [8] Let $X=[0,+\infty)$ and define $\mathrm{P}: X\times X\times(0,+\infty)\to[0,+\infty)$ by

$$\mathrm{P}(x,y,t)=t(x-y^p)$$

Then P is a parametric G-metric with constant $s=2^p$. In fact, we only need to prove (3) in Definition 2.1 as follows: let $x,y,z\in X$ and set $u=x-z, v=z-y,$ so $u+v=x-y$ From the inequality

$$(a+b)^p \le (2\max\{a,b\})^p \le 2^p\left(a^p+b^p\right), \forall a,b\ge 0,$$

We have

$$\begin{aligned}\mathrm{P}(x,y,t)&=t(x-y^p)\\&=t(u+v)^p\\&\le 2^p t(u^p+v^p)\\&=2^p\left(t(x-z)^p+t(z-y)^p\right)\\&=s(\mathrm{P}(x,z,t)+\mathrm{P}(z,y,t))\end{aligned}$$

With $s=2^p>1.$

Definition 2.3 Let (X,P,s) be a parametric G-metric space and $T: X\to X$ be a mapping. We say T is a continuous mapping at X in X, if for any sequence $\{x_n\}_{n=1}^{\infty}$ in X such that $\lim_{n\to\infty} x_n = x$ then $\lim_{n\to\infty} Tx_n = Tx$

In general, a parametric G-metric function for $s>1$ is not jointly continuous in all its Variables

Lemma 2.4 Let (X,P,s) be a G-metric space with the coefficient $s\ge 1$ and let $\{x_n\}_{n=1}^{\infty}$ be a sequence in X .if $\{x_n\}_{n=1}^{\infty}$ converges to x and also $\{x_n\}_{n=1}^{\infty}$ converges to y ,then $x=y$. That is the limit of $\{x_n\}_{n=1}^{\infty}$ is unique.

Lemma 2.5 Let (X,P,s) be a G-metric space with the coefficient $s\ge 1$ and let $\{x_n\}_{n=1}^{\infty}$ be a sequence in X .if $\{x_n\}_{n=1}^{\infty}$ converges to x . Then

$$\frac{1}{s}\mathrm{P}(x,y,t)\le \lim_{n\to+\infty}\mathrm{P}(x_n,y,t)\le s\mathrm{P}(x,y,t)$$

$\forall y\in X$ and all $t>0$.

Lemma 2.6 Let (X,P,s) be a G-metric space with the coefficient $s\ge 1$ and let $\{x_k\}_{k=1}^{n}\subset X$ Then

$$\mathrm{P}(x_n,x_0,t)\le s\mathrm{P}(x_0,x_1,t)+s^2\mathrm{P}(x_2,x_3,t)+\dots\dots\dots + s^{n-1}\mathrm{P}(x_{n-2},x_{n-1},t)+s^n\mathrm{P}(x_{n-1},x_n,t)$$

Lemma 2.7 Let (X,P,s) be a parametric space with the coefficient $s\ge 1$

1.Let $\{x_n\}_{n=1}^{\infty}$ be a sequence of point of X such that

$$\mathrm{P}(x_{n,}x_{n+1,}t)\le \lambda\mathrm{P}(x_{n-1,}x_{n,}t)$$

Where $\lambda\in\left[0,\frac{1}{s}\right)$ and $n=1,2,\dots\dots.$ then $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence in (X,P,s)

3. Main Result

Let (X,P,s) be a a parametric G-metric space, let $x_0\in X$ and let $f: X\to X$ be a given mapping. The sequence $\{x_n\}_{n=1}^{\infty}$ with $x_n = f^n x_0 = fx_{n-1}$ for all $n\in N$ is called a Picard sequence of initial point x_0.

The following fixed point theorem is our first main result.

Theorem 3.1 Let (X,P,s) be a complete parametric b-metric space with the Coefficient $s\geq 1$ and let $f:X\to X$ be a mapping such that

$$s\mathrm{P}(fx,fy,t)\leq \frac{\mathrm{P}(x,fy,t)+\mathrm{P}(fx,y,t)}{\mathrm{P}(x,fx,t)+\mathrm{P}(y,fy,t)+\iota(t)}\mathrm{P}(x,y,t) \tag{3.1}$$

$\forall x,y\in X$ and all $t>0$, where $\iota(0,\infty)\to(0,\infty)$ is a function . Then

(i) T has at least one fixed point $x_1\in X$,

(ii) every Picard sequence of initial point $x_0\in X$ converges to a fixed point of f ,

(iii) if $x_1,x_1\in X$ are two distinct fixed points of f , then $(x_1,x_{2,}t)\geq \frac{s}{2}$ for all $t\geq 0$.

Proof Let $x_0\in X$ be an arbitrary point, and let $\{x_n\}_{n=1}^{\infty}$ be a Picard sequence of initial point x_0, that is,

$$x_n=f^n x_0=f^n x_{n-1} \text{ for all } n\in N .$$

If $x_{n_0}=x_{n_0-1}$ for some $n_0\in N$, then x_{n_0} is fixed point of fixed point of f and so $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence.

If $x_{n_0}\neq x_{n_0-1}$ for all $n\in N$ form (3.1), we have

$$\begin{aligned} s\mathrm{P}(x_n,x_{n+1},t)&=s\mathrm{P}(fx_{n-1},fx_n,t) \\ &\leq \frac{\mathrm{P}(x_{n-1},fx_n,t)+\mathrm{P}(x_n,fx_{n-1},t)}{\mathrm{P}(x_{n-1},fx_{n-1},t)+\mathrm{P}(x_n,fx_n,t)+\iota(t)}\mathrm{P}(x_{n-1},x_n,t) \\ &\leq \frac{\mathrm{P}(x_{n-1},x_{n+1},t)}{\mathrm{P}(x_{n-1},x_n,t)+\mathrm{P}(x_n,x_{n+1},t)+\iota(t)}\mathrm{P}(x_{n-1},x_n,t) \\ &\leq \frac{s[\mathrm{P}(x_{n-1},x_n,t)+\mathrm{P}(x_n,x_{n+1},t)]}{\mathrm{P}(x_{n-1},x_n,t)+\mathrm{P}(x_n,x_{n+1},t)+\iota(t)}\mathrm{P}(x_{n-1},x_n,t) \end{aligned} \tag{3.2}$$

The last inequality given us

$$\mathrm{P}(x_n,x_{n+1},t)\leq \frac{\mathrm{P}(x_{n-1},x_n,t)+\mathrm{P}(x_n,x_{n+1},t)}{\mathrm{P}(x_{n-1},x_n,t)+\mathrm{P}(x_n,x_{n+1},t)+\iota(t)}\mathrm{P}(x_{n-1},x_n,t) \tag{3.3}$$

From (3.3), we deduce that the sequence $\{\mathrm{P}(x_{n-1},x_n,t)\}$ is decreasing for all $t>0$. Thus there exists a nonnegative real number λ such that $\lim_{n\to\infty}\mathrm{P}(x_{n-1},x_n,t)=\lambda$. Then we claim that λ = 0. If λ > 0, on taking limit as $n\to+\infty$ on both sides of (3.3),
weobtain

$$\lambda\leq\frac{\lambda+\lambda}{\lambda+\lambda+\iota(t)}\lambda<\lambda$$

Which is contradiction. If follows that $\lambda=0$. Now we prove that $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence. Let $\delta\in\left[0,\frac{1}{s}\right[$. Since $\lambda=0$, then there exists $n(\delta)\in N$ such that for all $t>0$,

$$\frac{\mathrm{P}(x_{n-1},x_n,t)+\mathrm{P}(x_n,x_{n+1},t)}{\mathrm{P}(x_{n-1},x_n,t)+\mathrm{P}(x_n,x_{n+1},t)+\iota(t)}\leq\delta,\forall\ n\geq n(\delta) \tag{3.4}$$

This implies that

$$\mathrm{P}(x_n,x_{n+1},t)\leq\delta\mathrm{P}(x_{n-1},x_n,t),\forall\ n\geq n(\delta) \tag{3.5}$$

For all $t>0$. Repeating (3.5) n- times , we get

$$\mathrm{P}(x_n, x_{n+1}, t) \le \delta \mathrm{P}(x_0, x_1, t), \forall\ n \ge n(\delta) \tag{3.6}$$

Let $m > n$. It follows that

$$\begin{aligned}\mathrm{P}(x_n, x_m, t) &\le s\mathrm{P}(x_n, x_{n+1}, t) + s^2\mathrm{P}(x_{n+1}, x_{n+2}, t) + \dots\dots + s^{m-n}\mathrm{P}(x_{m-1}, x_m, t) \\ &\le \left(s\delta^n + s^2\delta^{n+1} + \dots\dots\dots . + s^{m-n}\delta^{m-1}\right)\mathrm{P}(x_0, x_1, t) \\ &\le s\delta^n\left(1 + s\delta + \dots\dots\dots + (s\delta)^{m-n-1}\right)\mathrm{P}(x_0, x_1, t) \\ &\le \frac{s\delta^n}{1 - s\delta^n}\mathrm{P}(x_0, x_1, t)\end{aligned} \tag{3.7}$$

For all $t > 0$. Since $s\delta < 1$. Assume that $\mathrm{P}(x_0, x_1, t) > 0$. By taking limit as $m, n \to +\infty$ in above inequality we get

$$\lim_{n,m\to\infty} \mathrm{P}(x_n, x_m, t) = 0 \tag{3.8}$$

Therefore, $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence in X. Also, if $\mathrm{P}(x_0, x_1, t) = 0$ then $\mathrm{P}(x_n, x_m, t) = 0$ for all $m > n$ and we deduce again that $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence in X. Since X is a complete parametric G-metric space, the sequence $\{x_n\}_{n=1}^{\infty}$ converges to exists $x \in X$.

Now, we shall prove that x_0 is fixed point of f Using (3.1) with $x = x_n, y = x_1$ and all $t > 0$, we obtain

$$\begin{aligned}s\mathrm{P}(x_{n+1}, fx_1, t) &= s\mathrm{P}(x_n, fx_1, t) \\ &\le \frac{\mathrm{P}(x_n, fx_1, t) + \mathrm{P}(x_1, fx_n, t)}{\mathrm{P}(x_n, fx_n, t) + \mathrm{P}(x_1, fx_1, t) + \ell(t)}\mathrm{P}(x_n, x_1, t) \\ &\le \frac{\mathrm{P}(x_n, fx_1, t) + \mathrm{P}(x_1, fx_n, t)}{\mathrm{P}(x_n, fx_{n+1}, t) + \mathrm{P}(x_1, fx_1, t) + \ell(t)}\mathrm{P}(x_n, x_1, t)\end{aligned} \tag{3.9}$$

Moreover, form

$$\mathrm{P}(x_1, fx_1, t) \le s\left[\mathrm{P}(x_1, x_n, t) + \mathrm{P}(x_n, fx_1, t)\right]$$

We have

$$\begin{aligned}\mathrm{P}(x_1, fx_1, t) - s\mathrm{P}(x_n, x_1, t) &\le s\mathrm{P}(x_n, fx_1, t) \\ &\le s^2\left[\mathrm{P}(x_n, fx_1, t) + \mathrm{P}(x_1, fx_1, t)\right]\end{aligned} \tag{3.10}$$

As $n \to +\infty$, we deduce that

$$\begin{aligned}\mathrm{P}(x_1, fx_1, t) &\le \lim_{n\to\infty} \inf_{t>0} s\mathrm{P}(x_n, fx_1, t) \\ &\le \lim_{n\to\infty} \sup_{t>0} s\mathrm{P}(x_n, fx_1, t) \\ &\le s^2\mathrm{P}(x_1, fx_1, t)\end{aligned} \tag{3.11}$$

0n letting lim inf , as $n \to +\infty$, on both sides of (3.11) and using (3.9) we obtain

$$\mathrm{P}(x_1, fx_1, t) \le \lim_{n\to\infty} \inf_{t>0} s\mathrm{P}(x_{n+1}, fx_1, t)$$

(3.12)

$$\begin{aligned}&\le \frac{s^2\mathrm{P}(x_1, fx_1, t)}{\mathrm{P}(x_1, fx_1, t) + \ell(t)} \lim_{n\to\infty} \inf_{t>0} s\mathrm{P}(x_n, fx_1, t) \\ &= 0\end{aligned}$$

This implies that $\mathrm{P}(x_1, fx_1, t) = 0$ for all $t > 0$, that is, $fx_1 = x_1$ and hence x_1 is a fixed point of f. Thus (i) and (ii) hold if $x_1 \in X$ with $x_1 \ne x_2$, is another fixed point of f, then using (3.1) with $x = x_1$ and $y = x_2$, we get

$$\begin{aligned} sP(x_1, fx_2, t) &\le \frac{P(x_1, fx_2, t) + P(x_2, fx_1, t)}{P(x_1, fx_1, t) + P(x_2, fx_2, t) + \ell(t)} P(x_1, x_2, t) \\ &\le [P(x_1, fx_2, t) + P(x_2, fx_1, t)]P(x_1, x_2, t) \\ &= [P(x_1, x_2, t) + P(x_2, x_1, t)]P(x_1, x_2, t) \\ &= 2P^2(x_1, x_2, t) \end{aligned}$$

And hence $P(x_1, x_2, t) \ge \frac{s}{2}$; that is, (iii) holds.

If we take $s = 1$ in Theorem 3.1, we obtain following:

Corollary 3.2 (Theorem 16, [12]) Let (X, P) be a complete parametric metric space and let $f : X \to X$ be a mapping such that

$$P(fx, fy, t) \le \frac{P(x, fy, t) + P(y, fx, t)}{P(x, fx, t) + P(y, fy, t) + \ell(t)} P(x, y, t)$$

(3.13)

$\forall x, y \in X$ and all $t > 0$, where $\ell(0, \infty) \to (0, \infty)$ is a function. Then

(i) T has at least one fixed point $x_1 \in X$,

(ii) every Picard sequence of initial point $x_0 \in X$ converges to a fixed point Of f ;

(iii) if $x_1, x_2 \in X$ are two distinct fixed points of f , then $P(x_1, x_2, t) \ge \frac{1}{2}$ for all $t > 0$.

In the following result we consider a weak contractive condition.

Theorem 3.3 Let (X, P, s) be a complete parametric G-metric space with the coefficient $s \ge 1$ and let $f : X \to X$ be a mapping such that

$$P(fx, fy, t) \le \frac{P(x, fy, t) + P(y, fx, t)}{P(x, fx, t) + P(y, fy, t) + \ell(t)} P(x, y, t) + \mu P(y, fx, t) \tag{3.14}$$

$\forall x, y \in X$ and all $t > 0$, where $\ell(0, \infty) \to (0, \infty)$ is a function and $\mu < s$ is a nonnegative real number. Then

(i). f has at least one fixed point $x_1 \in X$;

(ii). every Picard sequence of initial point $x_0 \in X$ converges to a fixed point of f ;

(iii). if $x_1, x_2 \in X$ are two distinct fixed points of f ,then $P(x_1, x_2, t) \ge \max\left\{0, \frac{(s-\mu)}{2}\right\}$ for all $t > 0$.

Proof Let $x_0 \in X$ be an arbitrary point, and let $\{x_n\}_{n=1}^{\infty}$ be a Picard sequence of initial point x_0, that is,

$$x_n = f^n x_0 = f^n x_{n-1} \text{ for all } n \in N .$$

If $x_{n_0} = x_{n_0 - 1}$ for some $n_0 \in N$, then x_{n_0} is fixed point of fixed point of f and so $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence.

If $x_n \ne x_{n-1}$ for all $n \in N$ form (3.14), we have

$$\begin{aligned} sP(x_n, x_{n+1}, t) &= sP(fx_{n-1}, fx_n, t) \\ &\le \frac{P(x_{n-1}, fx_n, t) + P(x_n, fx_{n-1}, t)}{P(x_{n-1}, fx_{n-1}, t) + P(x_n, fx_n, t) + \iota(t)} P(x_{n-1}, x_n, t) + \mu P(x_{n-1}, x_n, t) \\ &\le \frac{P(x_{n-1}, x_{n+1}, t)}{P(x_{n-1}, x_n, t) + P(x_n, x_{n+1}, t) + \iota(t)} P(x_{n-1}, x_n, t) \end{aligned} \tag{3.15}$$

$$\leq \frac{s[P(x_{n-1}, x_n, t) + P(x_n, x_{n+1}, t)]}{P(x_{n-1}, x_n, t) + P(x_n, x_{n+1}, t) + \iota(t)} P(x_{n-1}, x_n, t)$$

The last inequality given us

$$P(x_n, x_{n+1}, t) \leq \frac{P(x_{n-1}, x_n, t) + P(x_n, x_{n+1}, t)}{P(x_{n-1}, x_n, t) + P(x_n, x_{n+1}, t) + \iota(t)} P(x_{n-1}, x_n, t)$$

(3.16)

From (3.16), we deduce that the sequence $\{P(x_{n-1}, x_n, t)\}$ is decreasing for all $t > 0$. Thus there exists a nonnegative real number λ such that $\lim_{n\to\infty} P(x_{n-1}, x_n, t) = \lambda$. Then we claim that $\lambda = 0$. If $\lambda > 0$, on taking limit as $n \to +\infty$ on both sides of (3.14),

weobtain

$$\lambda \leq \frac{\lambda + \lambda}{\lambda + \lambda + \iota(t)} \lambda < \lambda \tag{3.17}$$

Which is contradiction. If follows that $\lambda = 0$. Now we prove that $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence. Let $\delta \in \left[0, \frac{1}{s}\right[$. Since $\lambda = 0$, then there exists $n(\delta) \in N$ such that for all $t > 0$,

$$\frac{P(x_{n-1}, x_n, t) + P(x_n, x_{n+1}, t)}{P(x_{n-1}, x_n, t) + P(x_n, x_{n+1}, t) + \iota(t)} \leq \delta, \forall\ n \geq n(\delta) \tag{3.18}$$

This implies that

$$P(x_n, x_{n+1}, t) \leq \delta P(x_{n-1}, x_n, t), \forall\ n \geq n(\delta) \tag{3.19}$$

For all $t > 0$. Repeating (3.5) n- times , we get

$$P(x_n, x_{n+1}, t) \leq \delta P(x_0, x_1, t), \forall\ n \geq n(\delta) \tag{3.20}$$

Now,it is easy to show $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence in X . The completeness of X ensures that the sequence $\{x_n\}_{n=1}^{\infty}$ converges to some $x_1 \in X$.

Now,we shall prove that x_1 is a fixed point of f . Using (3.14) with $x = x_n, y = x_1$ and all $t > 0$,
We obtain

$$sP(x_{n+1}, fx_1, t) = sP(x_n, fx_1, t) \tag{3.21}$$

$$\leq \frac{P(x_n, fx_1, t) + P(x_1, fx_n, t)}{P(x_n, fx_n, t) + P(x_1, fx_1, t) + \ell(t)} P(x_n, x_1, t) + \mu P(x_1, fx_n, t)$$

$$\leq \frac{P(x_n, fx_1, t) + P(x_1, fx_n, t)}{P(x_n, fx_{n+1}, t) + P(x_1, fx_1, t) + \ell(t)} P(x_n, x_1, t) + \mu P(x_1, fx_n, t)$$

Moreover, form

$$P(x_1, fx_1, t) \leq s[P(x_1, x_n, t) + P(x_n, fx_1, t)]$$

We have

$$P(x_1, fx_1, t) - sP(x_n, x_1, t) \leq sP(x_n, fx_1, t) \tag{3.22}$$

$$\leq s^2 [P(x_n, fx_1, t) + P(x_1, fx_1, t)]$$

As $n \to +\infty$, we deduce that

$$P(x_1, fx_1, t) \leq \lim_{n\to\infty} \inf_{t>0} sP(x_n, fx_1, t) \tag{3.23}$$

$$\leq \lim_{n\to\infty} \sup_{t>0} s\mathrm{P}(x_n, fx_1, t)$$
$$\leq s^2\mathrm{P}(x_1, fx_1, t)$$

0n letting lim inf , as $n \to +\infty$, on both sides of (3.23) and using (3.21) we obtain

$$\mathrm{P}(x_1, fx_1, t) \leq \lim_{n\to\infty} \inf_{t>0} s\mathrm{P}(x_{n+1}, fx_1, t)$$

(3.24)

$$\leq \frac{s^2\mathrm{P}(x_1, fx_1, t)}{\mathrm{P}(x_1, fx_1, t) + \ell(t)} \lim_{n\to\infty} \inf_{t>0} s\mathrm{P}(x_n, fx_1, t)$$
$$= 0$$

This implies that $\mathrm{P}(x_1, fx_1, t) = 0$ for all $t > 0$, that is, $fx_1 = x_1$ and hence x_1 is a fixed point of f . Thus (i) and (ii) hold

Now we shall prove uniqueness $x_1 \in X$ with $x_1 \neq x_2$, is another fixed point of f, then using (3.14) with $x = x_1$ and $y = x_2$, we get

$$s\mathrm{P}(x_1, fx_2, t) \leq \frac{\mathrm{P}(x_1, fx_2, t) + \mathrm{P}(x_2, fx_1, t)}{\mathrm{P}(x_1, fx_1, t) + \mathrm{P}(x_2, fx_2, t) + \ell(t)} \mathrm{P}(x_1, x_2, t) + \mu \mathrm{P}(x_1, fx_2, t)$$

(3.25)

$$\leq [\mathrm{P}(x_1, fx_2, t) + \mathrm{P}(x_2, fx_1, t)]\mathrm{P}(x_1, x_2, t) + \mu\mathrm{P}(x_1, x_2, t)$$
$$= [\mathrm{P}(x_1, x_2, t) + \mathrm{P}(x_2, x_1, t)]\mathrm{P}(x_1, x_2, t) + \mu\mathrm{P}(x_1, x_2, t)$$
$$= 2\mathrm{P}^2(x_1, x_2, t) + \mu\mathrm{P}(x_1, x_2, t)$$

And hence $\mathrm{P}(x_1, x_2, t) \geq \max\left\{0, \frac{(s-\mu)}{2}\right\}$ for all $t > 0$,that is, (iii) holds.

If we take $s = 1$, then we have the following corollary .

Corollary 3.4 Let (X, P, s) be a complete parametric G-metric space with the coefficient $s \geq 1$ and let $f : X \to X$ be a mapping such that

$$\mathrm{P}(fx, fy, t) \leq \frac{\mathrm{P}(x, fy, t) + \mathrm{P}(y, fx, t)}{\mathrm{P}(x, fx, t) + \mathrm{P}(y, fy, t) + \ell(t)} \mathrm{P}(x, y, t) + \mu\mathrm{P}(y, fx, t) \tag{3.26}$$

$\forall x, y \in X$ and all $t > 0$, where $\ell(0,\infty) \to (0,\infty)$ is a function and $\mu > 1$ is a nonnegative real number. Then

(i). f has at least one fixed point $x_0 \in X$;

(ii). every Picard sequence of initial point $x_0 \in X$ converges to a fixed point of f ;

(iii).if $x_1, x_2 \in X$ are two distinct fixed points of f ,then $\mathrm{P}(x_1, x_2, t) \geq \max\left\{0, \frac{(1-\mu)}{2}\right\}$ for all $t > 0$.

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