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A Multi – Timing Perturbation Analysis ofthe Deformation and Dynamic Buckling Of A Viscously Damped Toroidal Shell Segment Stressed By a Step Load

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Abstract:
This paper is concerned with analytical determination of (a) the normal displacement and Airy stress function of an imperfect viscously damped toroidal shell segment subjected to a step load and (b) the dynamic buckling load of the structure using perturbation technique in asymptotic procedures. The continuously differentiable imperfection is assumed in the form of a two – term Fourier series expansion while homogeneous initial and boundary conditions are assumed. The formulation contains a small parameter depicting the amplitude of imperfection and upon which a multi – timing regular perturbation procedure is initiated, while the light viscous damping is of some order of imperfection. Simply – supported boundary conditions are assumed and in the final analysis, a uniformly valid asymptotic formula for determining the dynamic buckling load is determined nontrivially. The dynamic buckling load is related to the corresponding static buckling load and the relationship is independent of imperfection parameter. Besides, the dynamic load is found to depend, among other things, on the Fourier coefficients while the formula for determining the dynamic buckling load is implicit in the load parameter.

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I. Introduction

As in the case of elastic cylindrical shell segments, elastic toroidal segments are imperfection – sensitive structural materials that don't seem to attract as much attention as other imperfection – sensitive materials such as columns, plates and even spherical shells. To our knowledge, strictly analytical investigations into the dynamic behavior of toroidal shell segments appear rather scanty and far between. Perhaps, one of the earliest investigations into toroidal shell segments was that by Stein and Mc Elman [1] whereby the buckling of toroidal shells (static buckling) was discussed. Later, Hutchinson [2] studied the initial postbuckling behavior of toroidal shell segment. In yet another investigation, Oyesanya [3] investigated the asymptotic analysis of imperfection – sensitivity of toroidal segment with modal imperfection and further extended his study [4] to investigate and analyze the influence of extra terms on asymptotic analysis of imperfection – sensitivity of toroidal shell segment with random imperfection. In a similar investigation, Ette et al. [5] analyzed the normal response and buckling of a toroidal shell segment pressurized by a static load, while Ette et al. [6], in the like manner, studied the deformation and static buckling of a toroidal shell segment using a two – term Fourier series imperfection.

II. Dynamic buckling Criteria

Globally, the Budiansky / Roth criterion [8] and the Budiansky / Hutchinson criterion [9] appear to be popular and of wider applicability.In the first case (ieBudiansky / Roth criterion), the response (displacement) is plotted against the applied load and the particular load that initiates a sudden jump in the displacement is regarded as the dynamic buckling load. This criterion easily lends itself to easy application of numerical techniques such as Finite element method and to easy computer application. However, in the Budiansky / Hutchinson criterion, the dynamic buckling load is defined as the largest load parameter for the solution of the problem to remain bounded and the condition [9] for dynamic buckling is

$$\frac{d\lambda}{dU_a} = 0 \quad (1)$$

where,λ is the load parameter and U_a is the maximum displacement. This second criterion easily lends itself to easy application, through phase plane portriate, of the dynamic buckling analyses of some simple elastic one – dimensional materials under step load. It also lends itself to dynamic buckling investigations of some much more complicated multi – dimensional elastic structures such as cylindrical, spherical and even toroidal shells, hence its preference in this investigation.

In order to utilize equation (1), our initial aim will be to apply a two – timing regular perturbation technique and obtain a uniformly valid asymptotic expansion of the maximum displacement U_a subsequent upon which the invocation of (1) is initiated to obtain the dynamic buckling load. Similar perturbation techniques were initiated by Kervokian [10], Kuzmak [11], Luke [12], Li and Kervokian [13], Danielson [14] and Lockhart and Amazigo [15]. Mention must also be made of investigations by Reda and Forbes [16], Priyardarsini et al. [17], Mc Shane et al. [18], Kubiak [19] and Kolakowski and Mania [20].

III. Formulation of the Problem

By adjusting the associated Karman – Donnell equilibrium and compartibility equations in [3, 4], to the dynamic setting, the equations satisfied by the normal displacement W(X, Y, T) and Airy stress function F(X, Y, T) of the undamped toroidal shell segments are respectively

$$\rho W_{,TT} + D\nabla^4 W + \frac{1}{a}F_{,XX} + \frac{1}{b}F_{,YY} + P\left[\frac{1}{2}(W+\bar{W})_{,XX} + \left(1-\frac{1}{2}\frac{a}{b}\right)(W+\bar{W})_{,YY}\right] = \hat{S}(W+\bar{W},F) \qquad (2)$$

and

$$\frac{1}{Eh}\nabla^4 F - \frac{1}{a}W_{,XX} - \frac{1}{b}F_{,YY} = -\frac{1}{2}\hat{S}(W+\bar{W},W) \qquad (3)$$

$$W = W_{,XX} = F = F_{,XX} = 0 \; at X = 0, L \qquad (4)$$

$$W(X,Y,0) = W_{,T}(X,Y,0) = 0 \qquad (5)$$

$$0 < X < L, \quad 0 < Y < a \qquad (6)$$

Here, equations (2) and (3) are the equations of motion and compartibility respectively, X, Y and T are the axial, circumferential and time variables respectively while ρ is the mass per unit area.Simiarly, E and h are the Young's modulus and thickness respectively, while a and b are the two radii of the toroidal shell. P(T) is the step load, while $D = \frac{Eh^3}{12(1-\vartheta^2)}$ is the bending stiffness where ϑ is the Poisson's ratio. $\nabla^4$is the two – dimensional biharmonic operator defined as $\nabla^4 \equiv \left(\frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial Y^2}\right)^2$, while $\hat{S}$ is a symmetric bilinear differential functional defined as

$$\hat{S}(P,Q) = P_{,XX}Q_{,YY} + P_{,YY}Q_{,YY} - 2P_{,XY}Q_{,XY}.$$

In addition, $\bar{W}$ is a time – independent stress – free continuously differentiable normal displacement acting as the imperfection and a subscript following a comma denotes partial differentiation.

IV. Nondimensionalization of relevant equations

As in [3, 4] and [15], we now let

$$P(t) = \begin{cases} \bar{p}, & T > 0 \\ 0, & T < 0 \end{cases}, \quad x = \frac{\pi X}{L}, \quad y = \frac{2\pi Y}{a}, \; \epsilon\bar{w} = \frac{\bar{W}}{h},$$
$$w = \frac{W}{h}, \; \lambda g(t) = \frac{L^2 a\bar{P}}{\pi^2 D}, \; g(t) = \begin{cases} 1, & t > 0 \\ 0, & t < 0 \end{cases}, \quad A = \frac{L^2\sqrt{12(1-\vartheta^2)}}{\pi^2 ah},$$
$$H = \frac{h}{a}, \; \xi = \frac{L^2}{(\pi a)^2}, \quad K(\xi) = -\left(\frac{A}{1+\xi}\right)^2, \quad t = \frac{T\pi^2\left(D/\rho\right)^{1/2}}{L^2}, \quad 0 < \epsilon \ll 1,$$
$$g(t) = 1, \; t = 0 \qquad (7)$$

Here, we have neglected the axial and circumferential inertia and have similarly neglected the boundary – layer effect by assuming that the pre – buckling deflection is constant so that

$$F = -\frac{1}{2}\bar{p}a\left(X^2 + \frac{1}{2}\alpha Y^2\right) + \frac{Eh^2L^2}{\pi^2 a(1+\xi)^2}\tilde{f} \qquad (8)$$

$$W = \frac{\bar{p}a^2(1-\alpha v)}{Eh} + hw \qquad (9)$$

The first terms in (8) and (9) are the pre – buckling approximations, while the parameter $\alpha = 1$ if pressure contributes to axial stress through and plates otherwise $\alpha = 0$ if pressure acts laterally. By introducing the light viscous damping term, namely $2\epsilon^2 w_{,t}$, to the equation of motion, the resultant nondimensional equations are

$$w_{,tt} + 2\epsilon^2 w_{,t} + \bar{\nabla}^4 w - K(\xi)\left(\tilde{f}_{,xx} + \xi r\tilde{f}_{,yy}\right) + \lambda\left[\frac{\alpha}{2}(w+\epsilon\bar{w})_{,xx} + \xi\left(1-\frac{\alpha}{2}\right)(w+\epsilon\bar{w})_{,yy}\right] = -K(\xi)HS\left(\tilde{f}, w+\epsilon\bar{w}\right), \quad t > 0 \qquad (10)$$

and

$$\bar{\nabla}^4\tilde{f} - (1+\xi)^2\left(w_{,xx} + \xi r w_{,yy}\right) = -\frac{1}{2}H(1+\xi)^2 S(w+\epsilon\bar{w}, w) \qquad (11)$$

$$w = w_{,xx} = \tilde{f} = \tilde{f}_{,xx} = 0 \; at x = 0, \pi, \quad 0 < x < \pi, \quad 0 < y < 2\pi, \quad t > 0, \qquad r = \frac{a}{b} \qquad (12)$$

$$w(x,y,0) = w_{,t}(x,y,0) = 0 \qquad (13)$$

Here,

$S(p,q) = p_{,xx}q_{,yy} + p_{,yy}q_{,xx} - 2p_{,xy}q_{,xy}$ and $\bar{\nabla}^4 = \left(\frac{\partial^2}{\partial x^2} + \xi\frac{\partial^2}{\partial y^2}\right)^2$

V. Classical Theory

The necessary equations in this case are obtained from (9) and (10) by neglecting all the time related terms and all forms of nonlinearity as well as imperfection. The relevant equations are

$$\bar{\nabla}^4 w - K(\xi)\left(\tilde{f}_{,xx} + \xi r\tilde{f}_{,yy}\right) + \lambda\left[\frac{\alpha}{2}w_{,xx} + \xi\left(1-\frac{\alpha}{2}\right)w_{,yy}\right] = 0 \qquad (14)$$

$$\bar{\nabla}^4 f - (1+\xi)^2\left(w_{,xx} + \xi r w_{,yy}\right) = -\frac{1}{2}H(1+\xi)S(w+\epsilon\bar{w}, w) \qquad (15)$$

$$w = w_{,xx} = \tilde{f} = \tilde{f}_{,xx} = 0 \; at x = 0, \pi$$

The classical buckling load λ_C is sought by letting

$$\left(w, \quad \tilde{f}\right) = (a_{mk}, \quad b_{mk}) sinmxsin(ky + \emptyset_{mk}) \qquad (16)$$

where $\emptyset_{mk}$ is an inconsequential phase and where $(a_{mk}, \; b_{mk}) \neq (0,0)$. On substituting (16) in (15), yields

$$b_{mk} = \frac{-(1+\xi)^2 m^2 a_{mk}}{(m^2+\xi k^2)^2 + (1+\xi)^2\xi r k^2} \qquad (17a)$$

On substituting (17a) into (15) and simplifying, yields

$$(m^2+\xi k^2)^2 - \lambda\left\{\frac{\alpha m^2}{2} + \xi k^2\left(1-\frac{\alpha}{2}\right)\right\} - \frac{K(\xi)(m^2+\xi k^2 r)(1+\xi)^2}{(m^2+\xi k^2)^2 + (1+\xi)^2\xi r k^2} = 0$$

The classical buckling load λ_C is determined from the maximization (assuming that k varies continuously) $\frac{d\lambda}{dk} = 0$. Thus, if $k = n$ is the value of k at classical buckling, we get

$$\lambda_c = \frac{(m^2+\xi n^2)^2 - \frac{K(\xi)(1+\xi)^2(m^2+\xi n^2 r)}{(m^2+\xi n^2)^2+(1+\xi)^2\xi r n^2}}{\frac{\alpha m^2}{2} + \left(1-\frac{\alpha}{2}\right)\xi r n^2} \qquad (17b)$$

On substituting for $K(\xi)$ from (7) into (17b) and letting $m = 1, \zeta = \xi n^2$, we get

$$\lambda_c = +\frac{(1+\zeta)^2 + \frac{A^2(1+\zeta r)}{(1+\zeta)^2+(1+\xi)^2\zeta r}}{\frac{\alpha}{2} + \left(1-\frac{\alpha}{2}\right)\zeta r} \qquad (18a)$$

It follows that

$$(w, \tilde{f}) = \left(1, \quad \frac{-(1+\xi)^2}{(1+\zeta)^2+(1+\xi)^2\zeta r}\right) a_{1n} sinxsin(ny + \emptyset_{1n}) \qquad (18b)$$

VI. Perturbation Solution of the dynamic Problem

Let

$$\tau = \epsilon^2 t, \quad w(x,y,t,\epsilon) = U(x,y,t,\tau,\epsilon), \ \tilde{f}(x,y,t,\epsilon) = f(x,y,t,\tau,\epsilon) \qquad (19a)$$

$$\therefore \ w_{,t} = U_{,t} + \epsilon^2 U_{,\tau}, \qquad w_{,tt} = U_{,tt} + 2\epsilon^2 U_{,t\tau} + \epsilon^4 U_{,\tau\tau} \qquad (19b)$$

Further let

$$\begin{pmatrix} U(x,y,t,\tau,\epsilon) \\ \tilde{f}(x,y,t,\tau,\epsilon) \end{pmatrix} = \sum_{i=1}^{\infty} \begin{pmatrix} U^{(i)} \\ f^{(i)} \end{pmatrix} \epsilon^i \qquad (20)$$

Substituting (19a, b) and (20) into (10) and (11), yields

$$O(\epsilon)\begin{cases} U^{(1)}_{,tt} + \overline{\nabla}^4 U^{(1)} - K(\xi)\left(f^{(1)}_{,xx} + \xi r f^{(1)}_{,yy}\right) + \lambda\left[\frac{\alpha}{2}\left(U^{(1)}+\overline{w}\right)_{,xx} + \xi\left(1-\frac{\alpha}{2}\right)\left(U^{(1)}+\overline{w}\right)_{,yy}\right] \\ = 0 \qquad (21) \\ \overline{\nabla}^4 f^{(1)} - (1+\xi)^2\left(U^{(1)}_{,xx} + \xi r U^{(1)}_{,yy}\right) = 0 \qquad (22) \end{cases}$$

$$O(\epsilon^2)\begin{cases} U^{(2)}_{,tt} + \overline{\nabla}^4 U^{(2)} - K(\xi)\left(f^{(2)}_{,xx} + \xi r f^{(2)}_{,yy}\right) + \lambda\left[\frac{\alpha}{2}U^{(2)}_{,xx} + \xi\left(1-\frac{\alpha}{2}\right)U^{(2)}_{,yy}\right] \\ = -K(\xi)H\left[S\left(f^{(1)},U^{(1)}\right) + S\left(f^{(1)},\overline{w}\right)\right] \qquad (23) \\ \overline{\nabla}^4 f^{(2)} - (1+\xi)^2\left(w^{(2)}_{,xx} + \xi r w^{(2)}_{,yy}\right) = -\frac{1}{2}H(1+\xi)\left[s\left(w^{(1)},w^{(1)}\right) + s\left(w^{(1)},\overline{w}\right)\right] \ (24) \end{cases}$$

$$O(\epsilon^3)\begin{cases} U^{(3)}_{,tt} + \overline{\nabla}^4 U^{(3)} - K(\xi)\left(f^{(3)}_{,xx} + \xi r f^{(3)}_{,yy}\right) + \lambda\left[\frac{\alpha}{2}U^{(3)}_{,xx} + \xi\left(1-\frac{\alpha}{2}\right)U^{(3)}_{,yy}\right] \\ = -K(\xi)H\left[S\left(f^{(1)},U^{(2)}\right) + S\left(f^{(2)},U^{(1)}\right) + S\left(f^{(2)},\overline{w}\right)\right] - 2\left(U^{(1)}_{,t\tau} + U^{(1)}_{,t}\right) \qquad (25) \\ \overline{\nabla}^4 f^{(3)} - (1+\xi)^2\left(U^{(3)}_{,xx} + \xi r U^{(3)}_{,yy}\right) = -\frac{1}{2}H(1+\xi)^2\left[S\left(U^{(1)},U^{(2)}\right) + S\left(U^{(2)},U^{(1)}\right) + S\left(U^{(2)},\overline{w}\right)\right] \ (26) \end{cases}$$

etc.

However, $\left(U^{(1)}, U^{(2)}\right) = \left(U^{(2)}, U^{(1)}\right)$.

$$\begin{cases} U^{(i)}(x,y,0,0) = f^{(i)}(x,y,0,0) = 0, \ \ i = 1,2,3,\ldots \\ U^{(k)}_{,t}(x,y,0,0) = f^{(k)}_{,t}(x,y,0,0) = 0, \qquad k = 1,2. \\ U^{(r)}_{,t}(x,y,0,0) + U^{(r-2)}_{,\tau}(x,y,0,0) = 0, \qquad r = 3,4,\ldots \end{cases} \qquad (27)$$

$$U^{(i)} = U^{(i)}_{,xx} = f^{(i)} = f^{(i)}_{,xx} = 0 \ \ atx = 0,\pi, i = 1,2,3,\ldots \qquad (28)$$

In line with the boundary conditions, we shall let

$$\overline{w} = \left(\bar{a}cosny + \bar{b}sinny\right)sinmx \qquad (29)$$

Generally, the solution of equations of any order of perturbation will be of the form

$$\begin{pmatrix} U^{(i)} \\ f^{(i)} \end{pmatrix} = \sum_{p=1,q=1}^{\infty}\left[\begin{pmatrix} U^{(i)}_1 \\ f^{(i)}_1 \end{pmatrix} cosqy + \begin{pmatrix} U^{(i)}_2 \\ f^{(i)}_2 \end{pmatrix} sinqy\right] sinpx \qquad (30)$$

Hence, using (30), the following will be of general applicability

$$U_{,tt}^{(\mathrm{i})} + \overline{\nabla}^4 U^{(\mathrm{i})} - K(\xi)\left(f_{,xx}^{(i)} + \xi r f_{,yy}^{(i)}\right) + \lambda\left[\frac{\alpha}{2}U_{,xx}^{(\mathrm{i})} + \xi\left(1-\frac{\alpha}{2}\right)U_{,yy}^{(i)}\right]$$
$$\equiv \sum_{p=1,q=1}^{\infty}\left[\left\{U_{,tt}^{(\mathrm{i})} + (p^2+\xi q^2)^2 U_1^{(\mathrm{i})} + (p^2 K(\xi) - q^2 r\xi) f_1^{(\mathrm{i})} - \lambda\left(\frac{\alpha p^2}{2} + \left(1-\frac{\alpha}{2}\right)\xi q^2\right)U_1^{(\mathrm{i})}\right\} sinpxcosqy + \left\{U_{,tt}^{(\mathrm{i})} + (p^2+\xi q^2)^2 U_2^{(\mathrm{i})} + (p^2 K(\xi) - q^2 r\xi) f_2^{(\mathrm{i})} - \lambda\left(\frac{\alpha p^2}{2} + \left(1-\frac{\alpha}{2}\right)\xi q^2\right)U_2^{(\mathrm{i})}\right\} sinpxsinqy\right] \quad (31)$$

and

$$\overline{\nabla}^4 f^{(\mathrm{i})} - (1+\xi)^2\left(U_{,xx}^{(3)} + \xi r U_{,yy}^{(3)}\right) \equiv \sum_{p=1,q=1}^{\infty}\left[\left\{(p^2+\xi q^2)^2 f_1^{(\mathrm{i})} + (1+\xi)^2(q^2 r\xi - p^2)U_1^{(\mathrm{i})}\right\} sinpxcosqy + \left\{(p^2+\xi q^2)^2 f_2^{(\mathrm{i})} + (1+\xi)^2(q^2 r\xi - p^2)U_2^{(\mathrm{i})}\right\} sinpxsinqy\right] \quad (32)$$

Any integration with respect to x will have $0\ and\ \pi$ as the lower and upper limits respectively while a similar integration with respect to y has 0 and 2π as the lower and upper limits respectively.

Solution of Equations of order ϵ

Substituting (30) into (22) and first multiplying through by $cosnysinmx$, and next, by $sinnysinmx$ and for $p = m$ and $q = n$ in each case, we get

$$f_1^{(1)} = \frac{-(1+\xi)^2(n^2 r\xi - m^2)w_1^{(1)}}{(m^2+\xi r n^2)^2}, \qquad f_2^{(1)} = \frac{-(1+\xi)^2(n^2 r\xi - m^2)w_2^{(1)}}{(m^2+\xi r n^2)^2} \quad (33)$$

Next, substituting (30) into (21), noting (31) and multiplying, first by $cosnysinmx$ and next, by $sinnysinms$, it is noted that for $p = m, q = n$, the following resultant equations are obtained (after substituting for $f_1^{(1)}$ and $f_2^{(1)}$ from (33))

$$U_{1,tt}^{(1)} + \varphi^2 U_1^{(1)} = \bar{a}B^{(1)}; \qquad U_{2,tt}^{(1)} + \varphi^2 U_2^{(1)} = \bar{b}B^{(1)} \quad (34a)$$

$$B^{(1)} = \lambda\left\{\frac{\alpha m^2}{2} + \left(1-\frac{\alpha}{2}\right)\xi n^2\right\} \quad (34b)$$

$$U_1^{(1)}(0,0) = 0,\quad U_{1,t}^{(1)}(0,0) = 0\,; \qquad U_2^{(1)}(0,0) = U_{2,t}^{(1)}(0,0) = 0 \quad (34c)$$

$$\varphi^2 = \left[(m^2+\xi n^2)^2 + \left\{\left(\frac{mA}{1+\xi}\right)^2 + n^2 r\xi\right\}(1+\xi)^2\frac{(n^2 r\xi - m^2)}{(m^2+\xi n^2)^2} - \lambda\left\{\frac{\alpha m^2}{2} + \left(1-\frac{\alpha}{2}\right)\xi n^2\right\}\right] > 0\ \forall m,n. \quad (35)$$

The solutions of (34a) – (35) are

$$U_1^{(1)} = \delta_1^{(1)}(\tau)cos\varphi t + \beta_1^{(1)}(\tau)sin\varphi t + \bar{a}B \quad (36a)$$

$$B = \frac{B^{(1)}}{\varphi^2}, \qquad \delta_1^{(1)}(0) = -\bar{a}B, \quad \beta_1^{(1)}(0) = 0 \quad (36b)$$

$$U_2^{(1)} = \delta_2^{(1)}(\tau)cos\varphi t + \beta_2^{(1)}(\tau)sin\varphi t + \bar{b}B \quad (37a)$$

$$\delta_2^{(1)}(0) = -\bar{b}B, \quad \beta_2^{(1)}(0) = 0 \quad (37b)$$

So far, it is clear that in the final analysis, we shall have

$$\begin{pmatrix} U^{(1)} \\ f^{(1)} \end{pmatrix} = \begin{pmatrix} 1 \\ -\varphi_0 \end{pmatrix}\left(U_1^{(1)} cosny + U_2^{(1)} sinny\right) sinmxsinny$$

where,

$$\varphi_0 = (1+\xi)^2\left\{\frac{n^2 r\xi - m^2}{(m^2+n^2 r\xi)^2}\right\} \quad (38)$$

Solution of equations of Order ϵ^2

On simplifying the right hand sides of (23) and (24) and equally simplifying their left hand sides using (31) and (32) and noting (30), we have, for $i = 2$,

$$\sum_{p=1,q=1}^{\infty}\Bigg[\left\{U_{1,tt}^{(2)}+(p^2+\xi q^2)^2U_1^{(2)}+(p^2K(\xi)-q^2r\xi)f_1^{(2)}-\lambda\left(\frac{\alpha p^2}{2}+\left(1-\frac{\alpha}{2}\right)\xi q^2\right)U_1^{(2)}\right\}sinpxcosqy$$
$$+\left\{U_{2,tt}^{(2)}+(p^2+\xi q^2)^2U_2^{(2)}+(p^2K(\xi)-q^2r\xi)f_2^{(2)}\right.$$
$$\left.-\lambda\left(\frac{\alpha p^2}{2}+\left(1-\frac{\alpha}{2}\right)\xi q^2\right)U_2^{(2)}\right\}sinpxsinqy\Bigg]$$
$$=-K(\xi)H(mn)^2\varphi_0\Big[\left({U_1^{(1)}}^2+{U_2^{(1)}}^2+\bar{a}U_1^{(1)}+\bar{b}U_2^{(1)}\right)cos2mx$$
$$+\left({U_2^{(1)}}^2+{U_1^{(1)}}^2+\bar{b}U_2^{(1)}-\bar{a}U_1^{(1)}\right)cos2ny$$
$$-\left(2U_1^{(1)}U_2^{(1)}+\bar{a}U_2^{(1)}+\bar{b}U_1^{(1)}\right)sin2ny\Big] \qquad (39)$$

and

$$\sum_{p=1,q=1}^{\infty}\Big[\left\{(p^2+\xi q^2)^2f_1^{(2)}+(1+\xi)^2(q^2r\xi-p^2)U_1^{(2)}\right\}sinpxcosqy$$
$$+\left\{(p^2+\xi q^2)^2f_2^{(2)}+(1+\xi)^2(q^2r\xi-p^2)U_2^{(2)}\right\}sinpxsinqy\Big]$$
$$=-\frac{1}{2}H(1+\xi)^2(mn)^2\Big[-\left({U_1^{(1)}}^2+{U_2^{(1)}}^2+\bar{a}U_1^{(1)}+\bar{b}U_2^{(1)}\right)cos2mx$$
$$+\left(-{U_2^{(1)}}^2+{U_1^{(1)}}^2-\bar{b}U_2^{(1)}+\bar{a}U_1^{(1)}\right)cos2ny$$
$$+\left(2U_1^{(1)}U_2^{(1)}+\bar{a}U_2^{(1)}+\bar{b}U_1^{(1)}\right)sin2ny\Big] \qquad (40)$$

$$U_1^{(2)}(0{,}0)=U_{1,t}^{(2)}(0{,}0)=U_2^{(2)}(0{,}0)=U_{2,t}^{(2)}(0{,}0)=0 \qquad (41)$$

For the solution of (40), we multiply both sides, first by $cos2nysinmx$and afterwards by sin$2nysinmx$. In the first multiplication, we get, for $p=m,q=2n$

$$f_1^{(2)}=\frac{-(1+\xi)^2(4n^2r\xi-m^2)U_1^{(2)}+\frac{2H(1+\xi)^2mn^2}{\pi}\left({U_1^{(1)}}^2-{U_2^{(1)}}^2+\bar{a}U_1^{(1)}-\bar{b}U_2^{(1)}\right)}{(m^2+4n^2\xi)^2} \qquad (42a)$$

This can further be written as

$$f_1^{(2)}=-\varphi_7U_1^{(2)}+\varphi_8\left({U_1^{(1)}}^2-{U_2^{(1)}}^2+\bar{a}U_1^{(1)}-\bar{b}U_2^{(1)}\right) \qquad (42b)$$

where,

$$\varphi_7=\frac{(1+\xi)^2(4n^2r\xi-m^2)}{(m^2+4n^2\xi)^2}, \qquad \varphi_8=\frac{2H(1+\xi)^2mn^2}{\pi(m^2+4n^2\xi)^2} \qquad (42c)$$

The second multiplication gives, for$=m,q=2n$,

$$f_2^{(2)}=\frac{-(1+\xi)^2(4n^2r\xi-m^2)U_2^{(2)}+\frac{2H(1+\xi)^2mn^2}{\pi}\left(2U_1^{(1)}U_2^{(1)}-\bar{a}U_2^{(1)}+\bar{b}U_1^{(1)}\right)}{(m^2+4n^2\xi)^2} \qquad (43a)$$

This can further be written as

$$f_2^{(2)}=-\varphi_7U_2^{(2)}+\varphi_8\left(2U_1^{(1)}U_2^{(1)}-\bar{a}U_2^{(1)}+\bar{b}U_1^{(1)}\right) \qquad (43b)$$

Next, multiplying (39), first by $cos2nysinmx$and after, by sin$2nysinmx$ and for $p=m,q=2n$, the result (in the first multiplication) is

$$U_{1,tt}^{(2)}+\Omega^2U_1^{(2)}=\varphi_3\left({U_1^{(1)}}^2-{U_2^{(1)}}^2+\bar{a}U_1^{(1)}-\bar{b}U_2^{(1)}\right)$$
$$+\varphi_4\left({U_2^{(1)}}^2-{U_1^{(1)}}^2+\bar{b}U_2^{(1)}-\bar{a}U_2^{(1)}\right) \qquad (44a)$$

$$U_1^{(2)}(0,0)=0, \qquad U_{1,t}^{(2)}(0,0)=0 \qquad (44b)$$

where,

$$\Omega^2=\left[(m^2+4\xi n^2)^2+\frac{\left\{\left(\frac{mA}{1+\xi}\right)^2+4n^2r\xi\right\}(1+\xi)^2(4n^2r\xi-m^2)}{(m^2+\xi n^2)^2}-\lambda\left\{\frac{\alpha m^2}{2}+\left(1-\frac{\alpha}{2}\right)4\xi n^2\right\}\right]$$
$$>0\ \forall m,n. \qquad (44c)$$

$$\varphi_3 = \frac{-2H(1+\xi)^2 mn^2}{\pi(m^2+4n^2\xi)^2}, \qquad \varphi_4 = \frac{4\pi H mn^2 \varphi_0 A^2}{(1+\xi)^2} \qquad (44d)$$

Equation (44a) can be further written as

$$U_{1,tt}^{(2)} + \Omega^2 U_1^{(2)} = (\varphi_3 - \varphi_4)\left({U_1^{(1)}}^2 - {U_2^{(1)}}^2 + \left(\bar{a}U_1^{(1)} - \bar{b}U_2^{(1)}\right)\right) \qquad (45)$$

The second multiplication by $sin2nysinmx$ in (39) yields

$$U_{2,tt}^{(2)} + \Omega^2 U_2^{(2)} = (\varphi_3 + \varphi_4)\left(2U_1^{(1)}U_2^{(1)} + \bar{a}U_2^{(1)}U_1^{(1)}\right) \qquad (46a)$$

$$U_2^{(2)}(0,0) = 0, \qquad U_{2,t}^{(2)}(0,0) = 0 \qquad (46b)$$

The following simplifications are necessary in the solutions of (45) and (46a) that soon follow.

$$ {U_1^{(1)}}^2 = \left[\left\{\frac{1}{2}\left({\delta_1^{(1)}}^2 + {\beta_1^{(1)}}^2\right) + \bar{a}^2B^2\right\} + 2\bar{a}B\left(\delta_1^{(1)}cos\varphi t + \beta_1^{(1)}sin\varphi t\right) + \delta_1^{(1)}\beta_1^{(1)}sin2\varphi t + \frac{1}{2}\left({\delta_1^{(1)}}^2 - {\beta_1^{(1)}}^2 cos2\varphi t\right)\right] \qquad (47a)$$

$$ {U_2^{(1)}}^2 = \left[\left\{\frac{1}{2}\left({\delta_2^{(1)}}^2 + {\beta_2^{(1)}}^2\right) + \bar{b}^2B^2\right\} + 2\bar{b}B\left(\delta_2^{(1)}cos\varphi t + \beta_2^{(1)}sin\varphi t\right) + \delta_2^{(1)}\beta_2^{(1)}sin2\varphi t + \frac{1}{2}\left({\delta_2^{(1)}}^2 - {\beta_2^{(1)}}^2 cos2\varphi t\right)\right] \qquad (47b)$$

$$2U_1^{(1)}U_2^{(1)} = 2\left[\frac{1}{2}\left\{\delta_1^{(1)}\delta_2^{(1)} + \beta_1^{(1)}\beta_2^{(1)}\right\} + B\left(\bar{b}\delta_1^{(1)} + \bar{a}\delta_2^{(1)}\right)cos\varphi t + B\left(\bar{a}\beta_2^{(1)} + \bar{b}\beta_1^{(1)}\right)sin\varphi t + \frac{1}{2}\left(\delta_1^{(1)}\delta_2^{(1)} - \beta_1^{(1)}\beta_2^{(1)}\right)cos2\varphi t + \frac{1}{2}\left(\delta_1^{(1)}\beta_2^{(1)} + \delta_2^{(1)}\beta_1^{(1)}\right)sin2\varphi t\right] \qquad (47c)$$

Substituting (47a, b) in (45) and simplifying, yields

$$U_{1,tt}^{(2)} + \Omega^2 U_1^{(2)} = (\varphi_3 - \varphi_4)(r_0 + r_1 cos\varphi t + r_2 sin\varphi t + r_3 sin2\varphi t + r_4 cos2\varphi t) \qquad (48)$$

where,

$$r_0 = \left\{\frac{1}{2}\left({\delta_1^{(1)}}^2 + {\beta_1^{(1)}}^2\right) + \bar{a}^2B^2\right\} - \left\{\frac{1}{2}\left({\delta_2^{(1)}}^2 + {\beta_2^{(1)}}^2\right) + \bar{b}^2B^2\right\} + B\left(\bar{a}^2 - \bar{b}^2\right) \qquad (49a)$$

$$r_1 = 2B\left(\bar{a} + \bar{b}\right)\left(\delta_1^{(1)} - \delta_2^{(1)}\right) + \left(\bar{a}\delta_1^{(1)} - \bar{b}\delta_2^{(1)}\right) \qquad (49b)$$

$$r_2 = 2B\left(\bar{a} + \bar{b}\right)\left(\beta_1^{(1)} - \beta_2^{(1)}\right) + \left(\bar{a}\beta_1^{(1)} - \bar{b}\beta_2^{(1)}\right) \qquad (49c)$$

$$r_3 = \left(\delta_1^{(1)}\beta_1^{(1)} - \delta_2^{(1)}\beta_2^{(1)}\right), \qquad r_4 = \frac{1}{2}\left\{\left({\delta_1^{(1)}}^2 - {\beta_1^{(1)}}^2\right) - \left({\delta_2^{(1)}}^2 - {\beta_2^{(1)}}^2\right)\right\} \qquad (49d)$$

$$r_0(0) = \left(\bar{a}^2 - \bar{b}^2\right)\left(\frac{3B^2}{2} + B\right), \quad r_1(0) = \left(\bar{a}^2 - \bar{b}^2\right)(2B^2 + B) \qquad (49e)$$

$$r_2(0) = r_3(0) = 0; \qquad r_4(0) = \frac{B^2}{2}\left(\bar{a}^2 - \bar{b}^2\right) \qquad (49f)$$

The solution of (48) and (49a – f), using (44b), is

$$U_1^{(2)}(t,\tau) = \delta_1^{(2)}(\tau)cos\Omega t + \beta_1^{(2)}(\tau)sin\Omega t + (\varphi_3 - \varphi_4)\left[\frac{r_0}{\Omega^2} + \left(\frac{1}{\Omega^2 - \varphi^2}\right)(r_1 cos\varphi t + r_2 sin\varphi t) + \left(\frac{1}{\Omega^2 - 4\varphi^2}\right)(r_3 sin2\varphi t + r_4 cos2\varphi t)\right] \qquad (50)$$

$$\delta_1^{(2)}(0) = -(\varphi_3 - \varphi_4)\left[\frac{r_0}{\Omega^2} + \left(\frac{r_1}{\Omega^2 - \varphi^2}\right) + \left(\frac{r_3}{\Omega^2 - 4\varphi^2}\right)\right]\Big|_{\tau=0} = (\varphi_3 - \varphi_4)\left(\bar{a}^2 - \bar{b}^2\right)\left[\frac{3B^2 + 2B}{2\Omega^2} + \frac{2B^2 + B}{\Omega^2 - \varphi^2} + \frac{B^2}{2(\Omega^2 - 4\varphi^2)}\right] \qquad (51a)$$

$$\beta_1^{(2)}(0) = 0 \qquad (51b)$$

Substituting (47c) into (46a) and simplifying, yields

$$U_{2,tt}^{(2)} + \Omega^2 U_2^{(2)} = (\varphi_3 + \varphi_4)(r_5 + r_6 cos\varphi t + r_7 sin\varphi t + r_8 cos2\varphi t + r_9 sin2\varphi t) \qquad (52)$$

where,

$$r_5 = \left\{\delta_1^{(1)}\delta_2^{(1)} + \beta_1^{(1)}\beta_2^{(1)} + 2\bar{a}\bar{b}B\right\}, \quad r_6 = \left\{2B\left(\bar{b}\delta_1^{(1)} + \bar{a}\delta_2^{(1)}\right) + \bar{a}\delta_2^{(1)} + \bar{b}\delta_1^{(1)}\right\}$$

$$r_7 = \left\{2B\left(\bar{a}\beta_2^{(1)} + \bar{b}\beta_1^{(1)}\right) + \bar{a}\beta_2^{(1)} + \bar{b}\beta_1^{(1)}\right\}, \quad r_8 = \left\{\delta_1^{(1)}\delta_2^{(1)} - \beta_1^{(1)}\beta_2^{(1)}\right\}$$

$$r_9 = \left(\delta_1^{(1)}\beta_2^{(1)} - \delta_2^{(1)}\beta_1^{(1)}\right), \quad r_5(0) = \bar{a}\bar{b}B^2 + 2\bar{a}\bar{b}B, \quad r_6(0) = 2\bar{a}\bar{b}(B^2 + B)$$

$$r_7(0) = 0, \qquad r_8(0) = \bar{a}\bar{b}B^2, \qquad r_9(0) = 0$$

The solution of (52), using (46b) yields

$$U_2^{(2)}(t,\tau) = \delta_2^{(2)}(\tau)cos\Omega t + \beta_2^{(2)}(\tau)sin\Omega t + (\varphi_3+\varphi_4)\left[\frac{r_5}{\Omega^2} + \left(\frac{1}{\Omega^2-\varphi^2}\right)(r_6 cos\varphi t + r_7 sin\varphi t) + \left(\frac{1}{\Omega^2-4\varphi^2}\right)(r_8 cos2\varphi t + r_9 sin2\varphi t)\right] \quad (53a)$$

where,

$$\delta_2^{(2)}(0) = -(\varphi_3+\varphi_4)\left[\frac{r_5}{\Omega^2} + \frac{r_6}{\Omega^2-\varphi^2} + \frac{r_7}{\Omega^2-4\varphi^2}\right]|_{\tau=0} = -(\varphi_3+\varphi_4)\bar{a}\bar{b}B^2\left[\frac{1}{2\Omega^2} + \frac{2}{\Omega^2-\varphi^2} + \frac{1}{2(\Omega^2-4\varphi^2)}\right] + O(B), \quad \beta_2^{(2)}(0) = 0 \quad (53b)$$

Solution of equations of Order ϵ^3

After simplifying the right hand sides of (25) and (26), and substituting on their left hand sides, using (31) and (32) for $i = 3$, the resultant equations are

$$\sum_{p=1,q=1}^{\infty}\left[\left\{U_{1,tt}^{(3)} + (p^2+\xi q^2)^2 U_1^{(3)} + (p^2K(\xi) - q^2 r\xi)f_1^{(3)} - \lambda\left(\frac{\alpha p^2}{2} + \left(1-\frac{\alpha}{2}\right)\xi q^2\right)U_1^{(3)}\right\}sinpxcosqy + \left\{U_{2,tt}^{(3)} + (p^2+\xi q^2)^2 U_2^{(3)} + (p^2K(\xi) - q^2 r\xi)f_2^{(3)} - \lambda\left(\frac{\alpha p^2}{2} + \left(1-\frac{\alpha}{2}\right)\xi q^2\right)U_2^{(3)}\right\}sinpxsinqy\right] =$$
$$- K(\xi)H(mn)^2\left[\frac{5}{4}\left\{\left(f_1^{(1)}U_1^{(2)} + f_2^{(1)}U_2^{(2)}\right)cosny + \left(f_1^{(1)}U_2^{(2)} - f_2^{(1)}U_1^{(2)}\right)sinny + \left(f_1^{(1)}U_2^{(2)} + f_2^{(1)}U_1^{(2)}\right)sin3ny + \left(f_1^{(1)}U_1^{(2)} - f_2^{(1)}U_1^{(2)}\right)cos3ny\right\}(1-cos2mx)\right.$$
$$- \left\{\left(f_2^{(1)}U_2^{(2)} + f_1^{(1)}U_1^{(2)}\right)cosny + \left(f_1^{(1)}U_2^{(2)} - f_2^{(1)}U_1^{(2)}\right)sinny + \left(f_2^{(1)}U_2^{(2)} - f_1^{(1)}U_2^{(2)}\right)cos3ny - \left(f_2^{(1)}U_1^{(2)} + f_1^{(1)}U_2^{(2)}\right)sin3ny\right\}(1+cos2mx)$$
$$+ \frac{5}{4}\left\{\left(f_1^{(2)}U_1^{(1)} + f_2^{(2)}U_2^{(1)}\right)cosny + \left(f_2^{(2)}U_1^{(1)} - f_1^{(2)}U_2^{(1)}\right)sinny + \left(f_2^{(2)}U_1^{(1)} + f_1^{(2)}U_2^{(1)}\right)sin3ny + \left(f_1^{(2)}U_1^{(1)} - f_2^{(2)}U_2^{(1)}\right)cos3ny\right\}(1-cos2mx)$$
$$- \left\{\left(f_2^{(2)}U_2^{(1)} + f_1^{(2)}U_1^{(1)}\right)cosny - \left(f_1^{(2)}U_2^{(1)} - f_2^{(2)}U_1^{(1)}\right)sinny + \left(f_2^{(2)}U_2^{(1)} - f_1^{(2)}U_1^{(1)}\right)cos3ny - \left(f_1^{(2)}U_2^{(1)} + f_2^{(2)}U_1^{(1)}\right)sin3ny\right\}(1+cos2mx)$$
$$+ \frac{5}{4}\left\{\left(\bar{a}f_1^{(2)} + \bar{b}f_2^{(2)}\right)cosny + \left(\bar{a}f_2^{(2)} - \bar{b}f_1^{(2)}\right)sinny + \left(\bar{a}f_2^{(2)} + \bar{b}f_1^{(2)}\right)sin3ny + \left(\bar{a}f_1^{(2)} - \bar{b}f_2^{(2)}\right)cos3ny\right\}(1-cos2mx)$$
$$\left.- \left\{\left(\bar{b}f_2^{(2)} + \bar{a}f_1^{(2)}\right)cosny - \left(\bar{b}f_1^{(2)} - \bar{a}f_2^{(2)}\right)sinny + \left(\bar{b}f_2^{(2)} - \bar{a}f_1^{(2)}\right)cos3ny - \left(\bar{b}f_1^{(2)} + \bar{a}f_2^{(2)}\right)sin3ny\right\}(1+cos2mx)\right] - 2\left(U_{1,t}^{(1)} + U_{1,t\tau}^{(1)}\right)cosnysinmx - 2\left(U_{2,t}^{(1)} + U_{2,t\tau}^{(1)}\right)sinnysinmx \quad (54)$$

$$U_1^{(3)}(0,0) = 0, \qquad U_{1,t}^{(3)}(0,0) + U_{1,\tau}^{(1)}(0,0) = 0 \quad (55)$$

and

$$\sum_{p=1,q=1}^{\infty}\left[\left\{(p^2+\xi q^2)^2 f_1^{(3)}+(1+\xi)^2(q^2 r\xi-p^2)U_1^{(3)}\right\}sinpxcosqy\right.$$
$$\left.+\left\{(p^2+\xi q^2)^2 f_2^{(3)}+(1+\xi)^2(q^2 r\xi-p^2)U_2^{(3)}\right\}sinpxsinqy\right]$$
$$=-H(1+\xi)^2(mn)^2\left[\frac{5}{4}\left\{\left(U_1^{(1)}U_1^{(2)}+U_2^{(1)}U_2^{(2)}\right)cosny+\left(U_1^{(1)}U_2^{(2)}-U_2^{(1)}U_1^{(2)}\right)sinny\right.\right.$$
$$+\left(U_1^{(1)}U_2^{(2)}+U_2^{(1)}U_1^{(2)}\right)sin3ny+\left.\left(U_1^{(1)}U_1^{(2)}-U_2^{(1)}U_1^{(2)}\right)cos3ny\right\}(1-cos2mx)$$
$$-\left\{\left(U_2^{(1)}U_2^{(2)}+U_1^{(1)}U_1^{(2)}\right)cosny+\left(U_1^{(1)}U_2^{(2)}-U_2^{(1)}U_1^{(2)}\right)sinny\right.$$
$$+\left(U_2^{(1)}U_2^{(2)}-U_1^{(1)}U_2^{(2)}\right)cos3ny-\left.\left(U_2^{(1)}U_1^{(2)}+U_1^{(1)}U_2^{(2)}\right)sin3ny\right\}(1+cos2mx)$$
$$+\frac{1}{2}\left\{\frac{5}{4}\left\{\left(\bar{a}U_1^{(2)}+\bar{b}U_2^{(2)}\right)cosny+\left(\bar{a}U_2^{(2)}-\bar{b}U_1^{(2)}\right)sinny+\left(\bar{a}U_2^{(2)}+\bar{b}U_1^{(2)}\right)sin3ny\right.\right.$$
$$-\left.\left(\bar{b}U_1^{(2)}-\bar{a}U_2^{(2)}\right)cos3ny\right\}(1-cos2mx)$$
$$-\left\{\left(\bar{b}U_2^{(2)}+\bar{a}U_1^{(2)}\right)cosny-\left(\bar{b}U_1^{(2)}-\bar{a}U_2^{(2)}\right)sinny+\left(\bar{b}U_1^{(2)}-\bar{a}U_2^{(2)}\right)cos3ny\right.$$
$$-\left(\bar{b}U_1^{(2)}+\bar{a}U_2^{(2)}\right)csinny\Big\}(1$$
$$\left.\left.+cos2mx)\right\}\right\}\Bigg] \qquad (56)$$

$U_2^{(3)}(0,0)=0, \quad U_{2,t}^{(3)}(0,0)+U_{2,\tau}^{(1)}(0,0)=0$ (57)

A careful observation of (54) and (56) shows that there will be buckling modes in the shapes of $cosnysinmx$, sin$nysinmx$, cos3$nysinmx$and sin3$nysinmx$. To determine the associated Airy stress function in the shape of $cosnysinmx$, we multiply (56) by $cosnysinmx$ and for $p=m, q=n$, we get

$$f_{1(\mathrm{m,n})}^{(3)}=-\frac{1}{(m^2+n^2\xi)^2}\left[(1+\xi)^2(n^2r\xi-m^2)U_{1(\mathrm{m,n})}^{(3)}\right.$$
$$+\frac{4H(1+\xi)^2mn^2}{\pi}\left\{\left(U_1^{(1)}U_1^{(2)}+U_2^{(1)}U_2^{(2)}\right)\right.$$
$$+\frac{1}{2}\left(\bar{a}U_1^{(2)}\right.$$
$$\left.\left.\left.+\bar{b}U_2^{(2)}\right)\right\}\right] \qquad (58)$$

This is valid for m odd. Next, multiplying (56) by $sinnysinmx$ and for $p=m, q=n$, the Airy stress function is

$$f_{2(\mathrm{m,n})}^{(3)}=-\frac{1}{(m^2+n^2\xi)^2}\left[(1+\xi)^2(n^2r\xi-m^2)U_{2(\mathrm{m,n})}^{(3)}\right.$$
$$+\frac{4H(1+\xi)^2mn^2}{\pi}\left\{\left(U_1^{(1)}U_2^{(2)}-U_2^{(1)}U_1^{(2)}\right)\right.$$
$$+\frac{1}{2}\left(\bar{a}U_2^{(2)}\right.$$
$$\left.\left.\left.-\bar{b}U_1^{(2)}\right)\right\}\right] \qquad (59)$$

valid for m odd. Multiplying (56) by $cos3nysinmx$and for $p=m, q=3n$, we get

$$f_{1(\mathrm{m,3n})}^{(3)}=-\frac{1}{(m^2+9n^2\xi)^2}\left[(1+\xi)^2(9n^2r\xi-m^2)U_{1(\mathrm{m,3n})}^{(3)}\right.$$
$$+\frac{28H(1+\xi)^2mn^2}{3\pi}\left\{\left(U_1^{(1)}U_1^{(2)}-U_2^{(1)}U_2^{(2)}\right)\right.$$
$$+\frac{1}{2}\left(\bar{a}U_2^{(2)}\right.$$
$$\left.\left.\left.-\bar{b}U_1^{(2)}\right)\right\}\right] \qquad (60)$$

valid for m odd. Lastly, we multiply (56) by $sin3nysinmx$and for $p=m, q=3n$, and get

$$f^{(3)}_{2(\mathrm{m,3n})} = -\frac{1}{(m^2+9n^2\xi)^2}\left[(1+\xi)^2(9n^2r\xi - m^2)U^{(3)}_{2(\mathrm{m,3n})} + \frac{28H(1+\xi)^2mn^2}{3\pi}\left\{\left(U_1^{(1)}U_2^{(2)} + U_2^{(1)}U_1^{(2)}\right) + \frac{1}{2}\left(\bar{a}U_2^{(2)} + \bar{b}U_1^{(2)}\right)\right\}\right] \quad (61)$$

valid for m odd. Next, multiplying (54) by $cosnysinmx$, we observe that for $p = m, q = n$, we get (after substituting for $f^{(3)}_{1(\mathrm{m,n})}$ from (58))

$$U^{(3)}_{1,tt} + \varphi^2 U_1^{(3)} = -\varphi_5\left[U_1^{(1)}U_1^{(2)} + U_2^{(1)}U_2^{(2)} + \frac{1}{2}\left(\bar{a}U_1^{(2)} + \bar{b}U_2^{(2)}\right)\right] - \varphi_6\left\{\left(f_1^{(1)}U_1^{(2)} + f_2^{(1)}U_2^{(2)}\right) + \left(\bar{a}f_1^{(2)} + \bar{b}f_2^{(2)}\right) + \left(U_2^{(1)}f_2^{(2)} + U_1^{(1)}f_1^{(2)}\right)\right\} - 2\left(U^{(1)}_{1,t} + U^{(1)}_{1,t\tau}\right) \quad (62)$$

$$U_1^{(3)}(0,0) = 0, \qquad U^{(3)}_{1,t}(0,0) + U^{(1)}_{1,\tau}(0,0) = 0 \quad (63)$$

$$\varphi_5 = \frac{4H(1+\xi)^2mn^2\left\{\left(\frac{mA}{1+\xi}\right)^2 + n^2\xi\right\}}{\pi(m^2+n^2\xi)^2}, \qquad \varphi_6 = \frac{4HK(\xi)mn^2}{\pi} \quad (64)$$

Multiplying (54) by $sinnyxinmx$ and for $p = m, q = n$, we get

$$U^{(3)}_{2,tt} + \varphi^2 U_2^{(3)} = \left[\varphi_5\left(U_1^{(1)}U_2^{(2)} - U_2^{(1)}U_1^{(2)}\right) - \varphi_6\left(-\varphi_0 U_1^{(1)}U_2^{(2)}\right)\right] + \varphi_0 U_2^{(1)}U_1^{(2)} + U_1^{(1)}f_2^{(2)} - U_2^{(1)}f_1^{(2)} - 2\left(U^{(1)}_{2,t} + U^{(1)}_{2,t\tau}\right) \quad (65)$$

$$U_2^{(3)}(0,0) = 0, \qquad U^{(3)}_{2,t}(0,0) + U^{(1)}_{2,\tau}(0,0) = 0 \quad (66)$$

In (65), we have retained only the terms that are cubic in displacement on expansion. The following simplifications are necessary

$$U_1^{(2)}f_1^{(1)} = -\varphi_0 U_1^{(1)}U_1^{(2)}, \quad U_2^{(2)}f_2^{(1)} = -\varphi_0 U_2^{(1)}U_2^{(2)}, \quad U_2^{(1)}f_2^{(2)} = -\varphi_7 U_2^{(1)}U_2^{(2)} + 2\varphi_8 U_1^{(1)}{U_2^{(1)}}^2$$

$$U_1^{(1)}f_1^{(2)} = -\varphi_7 U_1^{(1)}U_1^{(2)} + \varphi_8\left({U_2^{(1)}}^3 - U_1^{(1)}{U_2^{(1)}}^2\right),$$

$${U_2^{(1)}}^3 = \left(\delta_1^{(1)}cos\varphi t + \beta_1^{(1)}sin\varphi t + \bar{a}B\right)^3 = r_{55} + r_{56}cos\varphi t + r_{57}sin\varphi t + r_{58}cos2\varphi t + r_{59}sin2\varphi t + r_{60}cos3\varphi t + r_{61}sin3\varphi t \quad (67a)$$

where,

$$r_{55} = \left(\frac{3{\delta_1^{(1)}}^2 B}{2} + \frac{3{\beta_1^{(1)}}^2 B}{2} + B^3\right), \quad r_{56} = \left(\frac{3{\delta_1^{(1)}}^3}{4} + \frac{3\delta_1^{(1)}{\beta_1^{(1)}}^2}{4} + 3\delta_1^{(1)}\mathrm{B}^2\right) \quad (67b)$$

$$r_{57} = \frac{3}{4}\left({\delta_1^{(1)}}^2\beta_1^{(1)} + \delta_1^{(1)}{\beta_1^{(1)}}^2\right) + 3\beta_1^{(1)}B^2, \quad r_{58} = \frac{3}{2}\left({\delta_1^{(1)}}^2 B - {\beta_1^{(1)}}^2 B\right) \quad (67c)$$

$$r_{59} = 3\delta_1^{(1)}\beta_1^{(1)}B, \quad r_{60} = \frac{1}{4}\left({\delta_1^{(1)}}^3 - 3\delta_1^{(1)}{\beta_1^{(1)}}^2\right), \quad r_{61} = \frac{1}{4}\left(3{\delta_1^{(1)}}^2\beta_1^{(1)} - {\beta_1^{(1)}}^3\right) \quad (67d)$$

$$r_{55}(0) = B^3\left(\frac{3\bar{a}^2}{2} + 1\right), \quad r_{56}(0) = 3B^3\left(\frac{3\bar{a}^3}{4} - \bar{a}\right), \quad r_{57}(0) = 0 \quad (67e)$$

$$r_{58}(0) = \frac{3\bar{a}^3B^3}{2}, \quad r_{59}(0) = 0, \quad r_{60}(0) = -\frac{\bar{a}^3B^3}{4}, \quad r_{61}(0) = 0 \quad (67f)$$

We also have

$$U_1^{(1)}{U_2^{(1)}}^2 = r_{37} + r_{38}cos\varphi t + r_{39}sin\varphi t + r_{40}cos2\varphi t + r_{41}sin2\varphi t + r_{42}cos3\varphi t + r_{43}sin3\varphi t \quad (68a)$$

$$r_{37} = \delta_2^{(1)}\delta_1^{(1)}\bar{b}B + \bar{b}B\beta_1^{(1)}\beta_2^{(1)} + \bar{a}B\left(\frac{1}{2}\left({\delta_2^{(1)}}^2 + {\beta_2^{(1)}}^2\right) + \left(\bar{b}B\right)^2\right) \quad (68b)$$

$$r_{38} = \delta_1^{(1)}\left(\frac{1}{2}\left({\delta_2^{(1)}}^2 + {\beta_2^{(1)}}^2\right) + \left(\bar{b}B\right)^2\right) + \frac{\delta_1^{(1)}}{4}\left({\delta_2^{(1)}}^2 - {\beta_2^{(1)}}^2\right) + \frac{1}{2}\delta_2^{(1)}\beta_1^{(1)}\beta_2^{(1)} + 2\bar{a}\bar{b}B^2\delta_2^{(1)} \quad (68c)$$

$$r_{39} = \frac{1}{2}\delta_1^{(1)}\delta_2^{(1)}\beta_2^{(1)} + \left(\frac{1}{2}\beta_1^{(1)}\left({\delta_2^{(1)}}^2 + {\beta_2^{(1)}}^2\right) + \left(\bar{b}B\right)^2\beta_1^{(1)}\right) - \frac{\beta_1^{(1)}}{4}\left({\delta_2^{(1)}}^2 - {\beta_2^{(1)}}^2\right) + 2\bar{a}\bar{b}B^2\beta_2^{(1)} \quad (68d)$$

$$r_{40} = \delta_1^{(1)}\delta_2^{(1)}\bar{b}B - \bar{b}B\beta_1^{(1)}\beta_2^{(1)} + \frac{\bar{a}B}{2}\left(\delta_2^{(1)^2} - \beta_2^{(1)^2}\right) \quad (68e)$$

$$r_{41} = \delta_1^{(1)}\beta_2^{(1)}\bar{b}B + \bar{b}B\beta_1^{(1)}\delta_2^{(1)} + (\bar{a}B)\delta_2^{(1)}\beta_2^{(1)} \quad (68f)$$

$$r_{42} = \frac{\delta_1^{(1)}}{4}\left(\delta_2^{(1)^2} - \beta_2^{(1)^2}\right) - \frac{1}{2}\delta_2^{(1)}\beta_1^{(1)}\beta_2^{(1)} \quad (68g)$$

$$r_{43} = \frac{1}{2}\delta_1^{(1)}\delta_2^{(1)}\beta_2^{(1)} + \frac{\beta_1^{(1)}}{4}\left(\delta_2^{(1)^2} - \beta_2^{(1)^2}\right) - \quad (68h)$$

$$r_{37}(0) = B^3\left(\frac{3\bar{a}\bar{b}^2 + 2\bar{a}^3}{2}\right), \quad r_{38}(0) = \frac{-11\bar{a}\bar{b}^2B^3}{4}, \quad r_{39}(0) = 0 \quad (68i)$$

$$r_{40}(0) = \frac{3\bar{a}\bar{b}^2B^3}{2}, \quad r_{41}(0) = 0, \quad r_{42}(0) = \frac{-\bar{a}\bar{b}^2B^3}{4}, \quad r_{43}(0) = 0 \quad (68j)$$

Similarly, we have

$$U_1^{(1)}U_1^{(2)} = r_{11} + r_{12}cos\varphi t + r_{13}sin\varphi t + r_{14}cos2\varphi t + r_{15}sin2\varphi t + r_{16}cos3\varphi t + r_{17}sin3\varphi t + r_{18}cos\Omega t + r_{19}sin\Omega t + r_{20}cos(\varphi - \Omega)t + r_{21}sin(\varphi - \Omega)t + r_{22}cos(\varphi + \Omega)t + r_{23}sin(\varphi + \Omega)t \quad (69a)$$

where,

$$r_{11} = (\varphi_3 - \varphi_4)\left[\frac{1}{\Omega^2 - \varphi^2}\left(\frac{r_1\delta_1^{(1)}}{2} + \frac{r_2\beta_1^{(1)}}{2}\right) + \frac{\bar{a}Br_0}{\Omega^2}\right] \quad (69b)$$

$$r_{12} = (\varphi_3 - \varphi_4)\left[\frac{r_0\delta_1^{(1)}}{\Omega^2} + \frac{r_4\delta_1^{(1)}}{2(\Omega^2 - 4\varphi^2)} + \frac{r_3\beta_1^{(1)}}{2(\Omega^2 - 4\varphi^2)} + \frac{\bar{a}B}{\Omega^2 - \varphi^2}\right] \quad (69c)$$

$$r_{13} = (\varphi_3 - \varphi_4)\left[\frac{r_3\delta_1^{(1)}}{\Omega^2 - 4\varphi^2} + \frac{r_0\beta_1^{(1)}}{\Omega^2} - \frac{r_4\beta_1^{(1)}}{2(\Omega^2 - 4\varphi^2)} + \frac{\bar{a}B}{\Omega^2 - \varphi^2}\right] \quad (69d)$$

$$r_{14} = (\varphi_3 - \varphi_4)\left[\frac{r_1\delta_1^{(1)}}{\Omega^2 - \varphi^2} - \frac{r_2\beta_1^{(1)}}{2(\Omega^2 - \varphi^2)} + \frac{\bar{a}Br_4}{\Omega^2 - 4\varphi^2}\right] \quad (69e)$$

$$r_{15} = (\varphi_3 - \varphi_4)\left[\frac{r_2\delta_1^{(1)}}{\Omega^2 - \varphi^2} + \frac{r_1\beta_1^{(1)}}{2(\Omega^2 - \varphi^2)} + \frac{\bar{a}Br_3}{\Omega^2 - 4\varphi^2}\right] \quad (69f)$$

$$r_{16} = (\varphi_3 - \varphi_4)\left[\frac{r_4\delta_1^{(1)}}{2(\Omega^2 - 4\varphi^2)} - \frac{r_3\beta_1^{(1)}}{2(\Omega^2 - 4\varphi^2)}\right] \quad (69g)$$

$$r_{17} = \frac{\varphi_3 - \varphi_4}{2(\Omega^2 - 4\varphi^2)}\left(r_3\delta_1^{(1)} + r_4\beta_1^{(1)}\right), \quad r_{18} = \bar{a}B\delta_1^{(2)}, \quad r_{19} = \bar{a}B\beta_1^{(2)} \quad (69h)$$

$$r_{20} = \frac{1}{2}\left(\delta_1^{(1)}\delta_1^{(2)} + \beta_1^{(1)}\beta_2^{(2)}\right), \quad r_{21} = \frac{1}{2}\left(\delta_1^{(2)}\beta_1^{(1)} - \delta_1^{(1)}\beta_1^{(2)}\right) \quad (69i)$$

$$r_{22} = \frac{1}{2}\left(\delta_1^{(1)}\delta_1^{(2)} - \beta_1^{(1)}\beta_2^{(2)}\right), \quad r_{23} = \frac{1}{2}\left(\delta_1^{(1)}\beta_1^{(2)} + \delta_1^{(2)}\beta_1^{(1)}\right) \quad (69j)$$

$$r_{11}(0) = R_{11}B^3 + O(B^2), \quad R_{11} = (\varphi_3 - \varphi_4)\left[\frac{\bar{a}\left(\bar{a}^2 - \bar{b}^2\right)}{\Omega^2 - \varphi^2} + \frac{2\bar{a}(\varphi_3 + \varphi_4)}{\Omega^2}\right] \quad (69k)$$

$$r_{12}(0) = R_{12}B^3 + O(B^2), \quad R_{12} = (\varphi_3 - \varphi_4)\left(\bar{a}^2 - \bar{b}^2\right)\left[\frac{3}{2\Omega^2} + \frac{2}{\Omega^2 - \varphi^2}\right], \quad r_{13}(0) = 0 \quad (69l)$$

$$r_{14}(0) = R_{14}B^3 + O(B^2), \quad R_{14} = (\varphi_3 - \varphi_4)\left(\bar{a}^2 - \bar{b}^2\right)\left[\frac{1}{2(\Omega^2 - 4\varphi^2)} - \frac{2}{\Omega^2 - \varphi^2}\right] \quad (69m)$$

$$r_{15}(0) = 0, \quad r_{16}(0) = R_{16}B^3 + O(B^2), \quad R_{16} = -\frac{\bar{a}(\varphi_3 - \varphi_4)\left(\bar{a}^2 - \bar{b}^2\right)}{4(\Omega^2 - 4\varphi^2)} \quad (69n)$$

$$r_{17}(0) = 0, \quad r_{18}(0) = R_{18}B^3 + O(B^2), \quad (69o)$$

$$R_{18} = -\bar{a}(\varphi_3 - \varphi_4)\left(\bar{a}^2 - \bar{b}^2\right)\left[\frac{3}{2\Omega^2} + \frac{2}{\Omega^2 - \varphi^2} + \frac{1}{2(\Omega^2 - 4\varphi^2)}\right] \quad (69p)$$

$$r_{19}(0) = 0, \quad r_{20}(0) = R_{20}B^3 + O(B^2), \quad (69q)$$

$$R_{20} = \frac{(\varphi_3 - \varphi_4)}{2}\bar{a}\left(\bar{a}^2 - \bar{b}^2\right)\left[\frac{3}{2\Omega^2} + \frac{2}{\Omega^2 - \varphi^2} + \frac{1}{2(\Omega^2 - 4\varphi^2)}\right] \quad (69r)$$

$$r_{21}(0) = 0, \quad r_{22}(0) = R_{20}(0) = R_{20}B^3 + O(B^2), \quad r_{23}(0) = 0 \quad (69s)$$

We also simplify the following

$$U_2^{(1)}U_2^{(2)} = r_{24} + r_{25}cos\varphi t + r_{26}sin\varphi t + r_{27}cos2\varphi t + r_{28}sin2\varphi t + r_{29}cos3\varphi t + r_{30}sin3\varphi t + r_{31}cos\Omega t + r_{32}sin\Omega t + r_{33}cos(\varphi - \Omega)t + r_{34}sin(\varphi - \Omega)t + r_{35}cos(\varphi + \Omega)t + r_{36}sin(\varphi + \Omega)t \quad (70a)$$

where,

$$r_{24} = \left[\frac{\delta_2^{(1)}r_6}{2(\Omega^2-\varphi^2)} + \frac{\beta_2^{(1)}r_7}{2(\Omega^2-\varphi^2)} + \frac{\bar{b}Br_5}{\Omega^2}\right] \quad (70b)$$

$$r_{25} = \left[\frac{\delta_2^{(1)}r_5}{\Omega^2} + \frac{\delta_2^{(1)}r_8}{2(\Omega^2-\varphi^2)} + \frac{\beta_2^{(1)}r_9}{\Omega^2-4\varphi^2} + \frac{\bar{b}Br_6}{\Omega^2-\varphi^2}\right] \quad (70c)$$

$$r_{26} = \left[\frac{r_9\delta_2^{(1)}}{2(\Omega^2-4\varphi^2)} + \frac{r_5\beta_2^{(1)}}{\Omega^2} - \frac{r_8\beta_2^{(1)}}{\Omega^2-4\varphi^2} + \frac{\bar{b}Br_7}{\Omega^2-\varphi^2}\right] \quad (70d)$$

$$r_{27} = \left[\frac{r_6\delta_2^{(1)}}{2(\Omega^2-\varphi^2)} - \frac{r_7\beta_2^{(1)}}{2(\Omega^2-\varphi^2)} + \frac{\bar{b}Br_8}{\Omega^2-4\varphi^2}\right] \quad (70e)$$

$$r_{28} = \left[\frac{r_7\delta_2^{(1)}}{2(\Omega^2-\varphi^2)} + \frac{r_6\beta_2^{(1)}}{2(\Omega^2-\varphi^2)} + \frac{\bar{b}Br_9}{\Omega^2-4\varphi^2}\right] \quad (70f)$$

$$r_{29} = \left[\frac{r_8\delta_2^{(1)}}{2(\Omega^2-\varphi^2)} - \frac{r_9\beta_2^{(1)}}{\Omega^2-4\varphi^2}\right], \quad r_{30} = \left[\frac{r_9\delta_2^{(1)}}{2(\Omega^2-\varphi^2)} + \frac{r_8\beta_2^{(1)}}{\Omega^2-4\varphi^2}\right] \quad (70g)$$

$$r_{31} = \bar{b}B\delta_2^{(2)}, \quad r_{32} = \bar{b}B\beta_2^{(2)}, \quad r_{33} = \frac{1}{2}\left(\delta_2^{(1)}\delta_2^{(2)} + \beta_2^{(1)}\beta_2^{(2)}\right) \quad (70h)$$

$$r_{34} = \frac{1}{2}\left(\delta_2^{(2)}\beta_2^{(1)} + \delta_2^{(2)}\beta_2^{(1)}\right), \quad r_{35} = \frac{1}{2}\left(\delta_2^{(1)}\delta_2^{(2)} - \beta_2^{(1)}\beta_2^{(2)}\right) \quad (70i)$$

$$r_{36} = \frac{1}{2}\left(\delta_2^{(1)}\beta_2^{(2)} + \delta_2^{(2)}\beta_2^{(1)}\right) \quad (70j)$$

$$r_{24}(0) = R_{24}B^3 + O(B^2), \quad R_{24} = \frac{\bar{a}\bar{b}^2(\varphi_3+\varphi_4)}{\Omega^2} - \frac{2\bar{a}\bar{b}^2(\varphi_3+\varphi_4)}{2(\Omega^2-\varphi^2)} \quad (71a)$$

$$r_{25}(0) = R_{25}B^3 + O(B^2), \quad R_{25} = \bar{a}\bar{b}^2(\varphi_3+\varphi_4)\left[\frac{1}{\Omega^2} + \frac{1}{2(\Omega^2-4\varphi^2)}\right] \quad (71b)$$

$$r_{26}(0) = 0, \quad r_{27}(0) = R_{27}B^3 + O(B^2), \quad R_{27} = \frac{-\bar{a}\bar{b}^2(\varphi_3+\varphi_4)}{2(\Omega^2-4\varphi^2)} \quad (71c)$$

$$r_{28}(0) = 0, \quad r_{29}(0) = R_{29}B^3 + O(B^2), \quad R_{29} = \frac{-\bar{a}\bar{b}^2(\varphi_3+\varphi_4)}{2(\Omega^2-\varphi^2)} \quad (71d)$$

$$r_{30}(0) = 0, \quad r_{31}(0) = R_{31}B^3 + O(B^2), \quad R_{31} = -\bar{a}\bar{b}^2(\varphi_3+\varphi_4)\left[\frac{1}{\Omega^2} + \frac{2}{\Omega^2-\varphi^2} + \frac{1}{\Omega^2-4\varphi^2}\right] \quad (71e)$$

$$r_{32}(0) = 0, \quad r_{33}(0) = R_{33}B^3 + O(B^2), \quad R_{33} = \frac{1}{2}\bar{a}\bar{b}^2(\varphi_3+\varphi_4)\left[\frac{1}{\Omega^2} + \frac{2}{\Omega^2-\varphi^2} + \frac{1}{\Omega^2-4\varphi^2}\right] \quad (71f)$$

$$r_{34}(0) = 0, \quad r_{35}(0) = R_{33}(0) = R_{33}B^3 + O(B^2), \quad r_{36}(0) = 0 \quad (71g)$$

Now going back to (63) and retaining only the terms that are cubic in displacement components and making use of the simplifications earlier initiated, we have

$$U_{1,tt}^{(3)} + \varphi^2U_1^{(3)} = \varphi_9U_1^{(1)}U_1^{(2)} + \varphi_{10}U_2^{(1)}U_2^{(2)} + \varphi_{11}U_1^{(1)}{U_1^{(1)}}^2 - 2\left(U_{1,t}^{(1)} + U_{1,t\tau}^{(1)}\right) \quad (72a)$$

where,

$$\varphi_9 = \varphi_0\varphi_6 - \varphi_6\varphi_7 - \varphi_5 - \varphi_6\varphi_8 \quad (72b)$$

$$\varphi_{10} = \varphi_0\varphi_6 + \varphi_6\varphi_7 - \varphi_5, \quad \varphi_{11} = -\varphi_6\varphi_8 \quad (72c)$$

Substituting into (62) gives, after simplification

$$U_{1,tt}^{(3)} + \varphi^2U_1^{(3)} = r_{44} + (\varphi_9r_{12} + \varphi_{10}r_{25} + \varphi_{11}r_{38})cos\varphi t + (\varphi_9r_{13} + \varphi_{10}r_{26} + \varphi_{11}r_{39})sin\varphi t + r_{45}cos2\varphi t + r_{46}sin2\varphi t + r_{47}cos3\varphi t + r_{48}sin3\varphi t + r_{49}cos\Omega t + r_{50}sin\Omega t + r_{51}cos(\varphi - \Omega)t + r_{52}sin(\varphi - \Omega)t + r_{53}cos(\varphi + \Omega)t + r_{54}sin(\varphi + \Omega)t + 2\varphi\left({\delta_1^{(1)}}' + \delta_1^{(1)}\right)sin\varphi t - 2\varphi\left({\beta_1^{(1)}}' + \beta_1^{(1)}\right)cos\varphi t \quad (73a)$$

$$U_1^{(3)}(0,0) = 0, \quad U_{1,t}^{(3)}(0,0) + U_{1,\tau}^{(1)}(0,0) = 0 \quad (73b)$$

where, $\frac{d}{dx}(\ldots) = ()'$ and

$$r_{44} = \varphi_9r_{11} + \varphi_{10}r_{27} + \varphi_{11}r_{37}, \quad r_{45} = \varphi_9r_{14} + \varphi_{10}r_{27} + \varphi_{11}r_{40} \quad (74a)$$

$$r_{46} = \varphi_9r_{15} + \varphi_{10}r_{28} + \varphi_{11}r_{41}, \quad r_{47} = \varphi_9r_{16} + \varphi_{10}r_{29} + \varphi_{11}r_{42} \quad (74b)$$

$$r_{48} = \varphi_9r_{17} + \varphi_{10}r_{30} + \varphi_{11}r_{43}, \quad r_{49} = \varphi_9r_{15} + \varphi_{10}r_{31}, \quad r_{50} = \varphi_9r_{19} + \varphi_{10}r_{32} \quad (74c)$$

$$r_{51}=\varphi_9 r_{20}+\varphi_{10}r_{33},\quad r_{52}=\varphi_9 r_{22}+\varphi_{10}r_{35} \tag{74d}$$

$$r_{53}=\varphi_9 r_{20}+\varphi_{10}r_{33},\quad r_{54}=\varphi_9 r_{23}+\varphi_{10}r_{36} \tag{74e}$$

$$r_{44}(0)=R_{44}B^3+O(B^2),\quad R_{44}=\varphi_9\left\{2\bar{a}(\varphi_3-\varphi_4)\left(\frac{\bar{a}^2-\bar{b}^2}{2(\Omega^2-\varphi^2)}+\frac{(\varphi_3+\varphi_4)}{\Omega^2}\right)\right\}+\varphi_{10}(\varphi_3+\varphi_4)\bar{a}\bar{b}^2\left(\frac{1}{\Omega^2}+\frac{1}{\Omega^2-\varphi^2}\right)+3\varphi_{11}\left(3\bar{a}\bar{b}^2+2\bar{a}^3\right) \tag{74f}$$

$$r_{45}(0)=R_{45}B^3+O(B^2),\quad R_{45}=\varphi_9(\varphi_3-\varphi_4)\left(\bar{a}^2-\bar{b}^2\right)\bar{a}\left(\frac{1}{2(\Omega^2-4\varphi^2)}-\frac{2}{\Omega^2-\varphi^2}\right)-\frac{\varphi_{10}(\varphi_3+\varphi_4)\bar{a}\bar{b}^2}{2(\Omega^2-4\varphi^2)}+\frac{3\varphi_{11}\bar{a}\bar{b}^2}{2} \tag{74g}$$

$$r_{46}(0)=0,\quad r_{47}(0)=R_{47}B^3+O(B^2),\quad R_{47}=-\left[\varphi_9\bar{a}(\varphi_3-\varphi_4)\left(\bar{a}^2-\bar{b}^2\right)+\varphi_{10}(\varphi_3+\varphi_4)\bar{a}\bar{b}^2+\frac{\varphi_{11}\bar{a}\bar{b}^2}{4}\right] \tag{74h}$$

$$r_{48}(0)=0,\quad r_{49}(0)=R_{49}B^3+O(B^2),\quad R_{49}=\left[\varphi_9(\varphi_3-\varphi_4)\left(\bar{a}^2-\bar{b}^2\right)\left(\frac{3}{2\Omega^2}+\frac{2}{\Omega^2-\varphi^2}+\frac{1}{2(\Omega^2-4\varphi^2)}\right)-\varphi_{10}(\varphi_3+\varphi_4)\bar{a}\bar{b}^2\left(\frac{1}{\Omega^2}+\frac{2}{\Omega^2-\varphi^2}+\frac{1}{\Omega^2-4\varphi^2}\right)\right] \tag{74i}$$

$$r_{50}(0)=0,\quad r_{51}(0)=R_{51}B^3+O(B^2),\quad R_{51}=\left[-\varphi_9\bar{a}\left(\bar{a}^2-\bar{b}^2\right)\left(\frac{3}{2\Omega^2}+\frac{2}{\Omega^2-\varphi^2}+\frac{1}{2(\Omega^2-4\varphi^2)}\right)+\varphi_{10}(\varphi_3-\varphi_4)\bar{a}\bar{b}^2\left(\frac{1}{\Omega^2}+\frac{2}{\Omega^2-\varphi^2}+\frac{1}{\Omega^2-4\varphi^2}\right)\right] \tag{74j}$$

$$r_{52}(0)=0,\quad r_{53}(0)=R_{51}B^3+O(B^2),\quad R_{51}=\varphi_9 R_{20}+\varphi_{10}R_{33},\ r_{52}(0)=0 \tag{74l}$$

To ensure a uniformly valid solution in terms of t, we equate to zero in (73a) the coefficients of $cos\varphi t$ and $sin\varphi t$. This yields, separately

$$\beta_1^{(1)'}+\beta_1^{(1)}=\frac{1}{2\varphi}\left[\varphi_9 r_{12}+\varphi_{10}r_{25}+\varphi_{11}r_{38}\right] \tag{75a}$$

and

$$\delta_1^{(1)'}+\delta_1^{(1)}=-\frac{1}{2\varphi}\left[\varphi_9 r_{13}+\varphi_{10}r_{26}+\varphi_{11}r_{39}\right] \tag{75b}$$

Equations (75a, b) are coupled but their explicit solutions are not needed. We only need

$$\delta_1^{(1)'}(0)=-\delta_1^{(1)}=\bar{a}B,\quad \beta_1^{(1)'}(0)=\frac{1}{2\varphi}\left[\varphi_9 r_{12}(0)+\varphi_{10}r_{25}(0)+\varphi_{11}r_{38}(0)\right] \tag{76}$$

The solution of the remaining equation in (73a) is

$$U_1^{(3)}(t,\tau)=\delta_1^{(3)}(\tau)cos\varphi t+\beta_1^{(3)}(\tau)sin\varphi t+\frac{r_{44}}{\varphi^2}-\frac{1}{3\varphi^2}\left(r_{45}cos2\varphi t+r_{46}sin2\varphi t\right)-\frac{1}{8\varphi^2}\left(r_{47}cos3\varphi t+r_{48}sin3\varphi t\right)+\frac{1}{\varphi^2-\Omega^2}\left(r_{49}cos\Omega t+r_{50}sin\Omega t\right)+\frac{1}{\Omega(2\varphi-\Omega)}\left(r_{51}cos(\varphi-\Omega)t+r_{50}sin(\varphi-\Omega)t\right)-\frac{1}{\Omega(2\varphi+\Omega)}\left(r_{52}cos(\varphi-\Omega)t+r_{53}sin(\varphi-\Omega)t\right) \tag{77a}$$

where,

$$\delta_1^{(3)}(0) = -\left[\frac{r_{44}}{\varphi^2} - \frac{r_{45}}{3\varphi^2} - \frac{r_{47}}{8\varphi^2} + \frac{r_{49}}{\varphi^2 - \Omega^2} + \frac{r_{51}}{\Omega(2\varphi - \Omega)} - \frac{r_{53}}{\Omega(2\varphi + \Omega)}\right]|_{\tau=0}, \quad \beta_1^{(3)}(0) = \frac{-\bar{a}B}{\varphi} \quad (77b)$$

After simplifying (65) and retaining only the terms that are cubic in displacement, we get

$$U_{2,tt}^{(3)} + \varphi^2 U_2^{(3)} = \varphi_{12} U_1^{(1)} U_2^{(2)} - U_2^{(1)} U_1^{(2)} - \varphi_8 U_2^{(1)} {U_1^{(1)}}^2 + \varphi_8 {U_2^{(1)}}^3 U_1^{(2)} + \varphi_{13} U_1^{(1)} f_2^{(2)} - U_2^{(1)} U_1^{(2)} - 2U_{2,t}^{(1)} + 2U_{2,t\tau}^{(1)} \quad (78a)$$

$$U_2^{(3)}(0,0) = 0, \quad U_{2,t}^{(3)}(0,0) + U_{2,\tau}^{(1)}(0,0) = 0 \quad (78b)$$

where,

$$\varphi_{12} = \varphi_5 + \varphi_0 + \varphi_6, \quad \varphi_{13} = (\varphi_7 - \varphi_5 - \varphi_0\varphi_6) \quad (78c)$$

The simplification of terms on the right hand side of (78a) is as follows:

$$U_1^{(1)} U_2^{(2)} = r_{69} + r_{70} cos\varphi t + r_{71} sin\varphi t + r_{72} cos2\varphi t + r_{73} sin2\varphi t + r_{74} cos3\varphi t + r_{75} sin3\varphi t + r_{76} cos(\varphi + \Omega)t + r_{77} sin(\varphi + \Omega)t + r_{78} cos(\varphi - \Omega)t + r_{79} sin(\varphi - \Omega)t \quad (79a)$$

where,

$$r_{69} = \left[\frac{r_6\delta_1^{(1)}}{2(\Omega^2 - \varphi^2)} + \frac{r_7\beta_1^{(1)}}{2(\Omega^2 - \varphi^2)} + \frac{\bar{a}Br_5}{\Omega^2}\right] \quad (79b)$$

$$r_{70} = \left[\frac{r_5\delta_1^{(1)}}{\Omega^2} + \frac{1}{2(\Omega^2 - 4\varphi^2)}\left(r_8\delta_1^{(1)} + r_9\beta_1^{(1)}\right) + \frac{\bar{a}Br_6}{\Omega^2 - \varphi^2}\right] \quad (79c)$$

$$r_{71} = \left[-\frac{1}{2(\Omega^2 - 4\varphi^2)}\left(r_9\delta_1^{(1)} + r_8\beta_1^{(1)}\right) + \frac{r_5\beta_1^{(1)}}{\Omega^2} + \frac{\bar{a}Br_7}{\Omega^2 - \varphi^2}\right] \quad (79d)$$

$$r_{72} = \left[\frac{1}{2(\Omega^2 - \varphi^2)}\left(r_9\delta_1^{(1)} + r_8\beta_1^{(1)}\right) + \frac{\bar{a}Br_8}{\Omega^2 - \varphi^2}\right] \quad (79e)$$

$$r_{73} = \left[\frac{1}{2(\Omega^2 - 4\varphi^2)}\left(r_7\delta_1^{(1)} + r_6\beta_1^{(1)}\right) + \frac{\bar{a}Br_9}{\Omega^2 - 4\varphi^2}\right] \quad (79f)$$

$$r_{74} = \left[\frac{\left(r_8\delta_1^{(1)} - r_9\beta_1^{(1)}\right)}{2(\Omega^2 - 4\varphi^2)}\right], \quad r_{75} = \frac{\left(r_9\delta_1^{(1)} + r_8\beta_1^{(1)}\right)}{2(\Omega^2 - 4\varphi^2)} \quad (79g)$$

$$r_{76} = \frac{1}{2}\left(\delta_1^{(1)}\delta_2^{(2)} - \beta_1^{(1)}\beta_2^{(2)}\right), \quad r_{77} = \frac{1}{2}\left(\delta_1^{(1)}\beta_2^{(2)} + \beta_1^{(1)}\beta_2^{(2)}\right) \quad (79h)$$

$$r_{78} = \frac{1}{2}\left(\delta_1^{(1)}\delta_2^{(2)} + \beta_1^{(1)}\beta_2^{(2)}\right), \quad r_{79} = \frac{1}{2}\left(\beta_1^{(1)}\delta_2^{(2)} - \delta_1^{(1)}\beta_2^{(2)}\right) \quad (79i)$$

$$r_{69}(0) = R_{69}B^3 + O(B^2), \quad R_{69} = \frac{-(\varphi_3 + \varphi_4)\bar{a}^2\bar{b}}{\Omega^2 - \varphi^2} \quad (80a)$$

$$r_{70}(0) = R_{70}B^3 + O(B^2), \quad R_{70} = (\varphi_3 + \varphi_4)\bar{a}^2\bar{b}\left[\frac{1}{\Omega^2 - \varphi^2} - \frac{1}{\Omega^2} - \frac{1}{2(\Omega^2 - 4\varphi^2)}\right] \quad (80b)$$

$$r_{71}(0) = 0, \quad r_{72}(0) = R_{72}B^3 + O(B^2), \quad R_{72} = (\varphi_3 + \varphi_4)\bar{a}^2\bar{b}\left[\frac{1}{\Omega^2 - 4\varphi^2} - \frac{1}{\Omega^2 - \varphi^2}\right] \quad (80c)$$

$$r_{73}(0) = 0, \quad r_{74}(0) = R_{74}B^3 + O(B^2), \quad R_{74} = \frac{-\bar{a}^2\bar{b}}{2(\Omega^2 - 4\varphi^2)} \quad (80d)$$

$$r_{75}(0) = 0, \quad r_{76}(0) = R_{76}B^3 + O(B^2), \quad R_{76} = \bar{a}^2\bar{b}\frac{(\varphi_3 + \varphi_4)}{2}\left[\frac{1}{\Omega^2} + \frac{1}{\Omega^2 - \varphi^2} + \frac{1}{\Omega^2 - 4\varphi^2}\right] \quad (80e)$$

$$r_{77}(0) = 0, \quad r_{78}(0) = r_{76}(0), \quad r_{79}(0) = 0 \quad (80f)$$

Similarly, we have

$$U_2^{(1)}U_1^{(1)^2} = r_{62} + r_{63}cos\varphi t + r_{64}sin\varphi t + r_{65}cos2\varphi t + r_{66}sin2\varphi t + r_{67}cos3\varphi t + r_{68}sin3\varphi t \quad (81a)$$

$$r_{62} = \delta_2^{(1)}\delta_1^{(1)}(\bar{a}B) - \beta_2^{(1)}\beta_1^{(1)}(\bar{a}B) + (\bar{a}B)\left(\frac{1}{2}\left(\delta_1^{(1)^2} + \beta_1^{(1)^2}\right) + (\bar{a}B)^2\right) \quad (81b)$$

$$r_{63} = \delta_2^{(1)}\left(\frac{1}{2}\left(\delta_1^{(1)^2} + \beta_1^{(1)^2}\right) + (\bar{a}B)^2\right) + \delta_2^{(1)}\left(\delta_1^{(1)^2} - \beta_1^{(1)^2}\right) + \frac{1}{2}\left(\beta_2^{(1)}\delta_1^{(1)}\beta_1^{(1)}\right) + 2(\bar{a}B)^2\delta_1^{(1)} \quad (81c)$$

$$r_{64} = \frac{1}{2}\left(\delta_2^{(1)}\delta_1^{(1)}\beta_1^{(1)}\right) + \beta_2^{(1)}\left(\frac{1}{2}\left(\delta_1^{(1)^2} + \beta_1^{(1)^2}\right) + (\bar{a}B)^2\right) - \frac{\beta_2^{(1)}}{4}\left(\delta_1^{(1)^2} - \beta_1^{(1)^2}\right) + 2\beta_1^{(1)}(\bar{a}B)^2 \quad (81d)$$

$$r_{65} = \delta_2^{(1)}\delta_1^{(1)}(\bar{a}B) - \beta_2^{(1)}\beta_1^{(1)}(\bar{a}B) + \frac{\bar{a}B}{2}\left(\delta_1^{(1)^2} - \beta_1^{(1)^2}\right) \quad (81e)$$

$$r_{66} = \delta_2^{(1)}\beta_1^{(1)}(\bar{a}B) + \delta_1^{(1)}\beta_2^{(1)}(\bar{a}B) + \delta_1^{(1)}\beta_1^{(1)}(\bar{a}B) \quad (81f)$$

$$r_{67} = \frac{\delta_2^{(1)}}{4}\left(\delta_1^{(1)^2} - \beta_1^{(1)^2}\right) - \frac{\beta_2^{(1)}\delta_1^{(1)}\beta_1^{(1)}}{2} \quad (81g)$$

$$r_{68} = \frac{\delta_2^{(1)}\delta_1^{(1)}\beta_1^{(1)}}{2} + \frac{\beta_2^{(1)}}{4}\left(\delta_1^{(1)^2} - \beta_1^{(1)^2}\right) \quad (81h)$$

$$r_{62}(0) = R_{62}B^3 + O(B^2), \quad R_{62} = \frac{1}{2}\left(3\bar{a}^2\bar{b} + 3\bar{a}^3\right) \quad (81i)$$

$$r_{63}(0) = R_{63}B^3 + O(B^2), \quad R_{63} = -\frac{1}{2}\left(5\bar{a}^2\bar{b} + 4\bar{a}^3\right) \quad (81j)$$

$$r_{64}(0) = 0, \; r_{65}(0) = R_{65}B^3 + O(B^2), \quad R_{65} = \frac{1}{2}\left(\bar{a}^2\bar{b} + \bar{a}^3\right) \quad (81k)$$

$$r_{66}(0) = 0, \; r_{67}(0) = R_{67}B^3 + O(B^2), \quad R_{67} = \frac{-\bar{a}^2\bar{b}}{4}, \qquad r_{68}(0) = 0 \quad (81l)$$

We similarly expand $U_2^{(1)^3}$

$$U_2^{(1)^3} = \left(\delta_2^{(1)}cos\varphi t + \beta_1^{(1)}sin\varphi t + \bar{b}B\right)^3 = r_{80} + r_{81}cos\varphi t + r_{82}sin\varphi t + r_{83}cos2\varphi t + r_{84}sin2\varphi t + r_{85}cos3\varphi t + r_{86}sin3\varphi t \quad (82a)$$

where,

$$r_{80} = \frac{3\delta_2^{(1)^2}\left(\bar{b}B\right)}{2} + 3\beta_2^{(1)^2}\left(\bar{b}B\right) + \left(\bar{b}B\right)^3 \quad (82b)$$

$$r_{81} = \frac{3\delta_2^{(1)^3}}{4} + \frac{3\delta_2^{(1)^2}\beta_2^{(1)}}{4} + 3\left(\bar{b}B\right)^2\delta_2^{(1)}, \quad r_{82} = \frac{1}{4}\left(3\delta_2^{(1)^2}\beta_2^{(1)} + \beta_2^{(1)^2}\right) + 3\left(\bar{b}B\right)^2\beta_2^{(1)} \quad (83c)$$

$$r_{83} = \frac{1}{2}\left(\frac{3\delta_2^{(1)^2}\left(\bar{b}B\right)}{2} - \frac{3\left(\bar{b}B\right)\beta_2^{(1)^2}}{2}\right), \quad r_{84} = 3\left(\bar{b}B\right)\delta_2^{(1)}\beta_2^{(1)} \quad (82d)$$

$$r_{85} = \frac{1}{4}\left(\delta_2^{(1)^3} - 3\delta_2^{(1)}\beta_2^{(1)}\right), \quad r_{86} = \frac{1}{4}\left(3\delta_2^{(1)^2}\beta_2^{(1)} - \beta_2^{(1)^3}\right) \quad (82e)$$

$$r_{80}(0) = \frac{5\left(\bar{b}B\right)^3}{2}, \quad r_{81}(0) = \frac{-15\left(\bar{b}B\right)^3}{4}, \quad r_{82}(0) = 0 \quad (82f)$$

$$r_{83}(0) = \frac{3\left(\bar{b}B\right)^3}{2}, \quad r_{84}(0) = 0, \quad r_{85}(0) = \frac{-\left(\bar{b}B\right)^3}{4}, \quad r_{86}(0) = 0 \quad (82g)$$

In the same way, we get

$$U_2^{(1)}U_1^{(2)} = r_{87} + r_{88}cos\varphi t + r_{89}sin\varphi t + r_{90}cos2\varphi t + r_{91}sin2\varphi t + r_{92}cos3\varphi t + r_{93}sin3\varphi t + r_{94}cos\Omega t + r_{95}sin\Omega t + r_{96}cos(\varphi - \Omega)t + r_{97}sin(\varphi - \Omega)t + r_{98}cos(\varphi + \Omega)t + r_{99}sin(\varphi + \Omega)t \quad (83a)$$

where,

$$r_{87} = (\varphi_3 - \varphi_4)\left[\frac{r_1\delta_2^{(1)} + r_2\beta_2^{(1)}}{2(\Omega^2 - \varphi^2)} + \frac{\bar{b}Br_0}{\Omega^2}\right] \quad (83b)$$

$$r_{88} = (\varphi_3 - \varphi_4)\left[\frac{r_0\delta_2^{(1)}}{2} + \frac{r_3\beta_2^{(1)}}{2(\Omega^2 - 4\varphi^2)} + \frac{\bar{b}Br_1}{\Omega^2 - \varphi^2}\right] \quad (83c)$$

$$r_{89} = (\varphi_3 - \varphi_4)\left[\frac{r_3\delta_2^{(1)}}{2(\Omega^2 - 4\varphi^2)} + \frac{r_0\beta_2^{(1)}}{\Omega^2} + \frac{\bar{b}Br_2}{\Omega^2 - \varphi^2} - \frac{r_4\beta_2^{(1)}}{2(\Omega^2 - 4\varphi^2)}\right] \quad (83d)$$

$$r_{90} = (\varphi_3 - \varphi_4)\left[\frac{r_1\delta_2^{(1)}}{2(\Omega^2 - \varphi^2)} - \frac{r_2\beta_2^{(1)}}{2(\Omega^2 - \varphi^2)} + \frac{\bar{b}Br_4}{\Omega^2 - 4\varphi^2}\right] \quad (83e)$$

$$r_{91} = (\varphi_3 - \varphi_4)\left[\frac{r_2\delta_2^{(1)}}{2(\Omega^2 - \varphi^2)} + \frac{r_1\beta_2^{(1)}}{2(\Omega^2 - \varphi^2)} + \frac{\bar{b}Br_3}{\Omega^2 - 4\varphi^2}\right] \quad (83f)$$

$$r_{92} = (\varphi_3 - \varphi_4)\left[\frac{r_4\delta_2^{(1)}}{2(\Omega^2 - 4\varphi^2)} - \frac{r_2\beta_2^{(1)}}{2(\Omega^2 - 4\varphi^2)}\right] \quad (83g)$$

$$r_{93} = (\varphi_3 - \varphi_4)\left[\frac{r_3\delta_2^{(1)} + r_4\beta_2^{(1)}}{2(\Omega^2 - 4\varphi^2)}\right], \quad r_{94} = \bar{b}B\delta_1^{(2)}, \quad r_{95} = \bar{b}B\beta_2^{(2)} \quad (83h)$$

$$r_{96} = \frac{1}{2}\left(\delta_2^{(1)}\delta_1^{(2)} + \beta_1^{(2)}\beta_2^{(1)}\right), \quad r_{97} = \frac{1}{2}\left(\beta_2^{(1)}\delta_1^{(2)} - \delta_1^{(2)}\beta_2^{(1)}\right) \quad (83i)$$

$$r_{98} = \frac{1}{2}\left(\delta_2^{(1)}\delta_1^{(2)} - \beta_1^{(2)}\beta_2^{(1)}\right), \quad r_{99} = \frac{1}{2}\left(\beta_2^{(1)}\delta_2^{(1)} + \delta_1^{(2)}\beta_2^{(1)}\right) \quad (83j)$$

$$r_{87}(0) = R_{87}B^3 + O(B^2), \quad R_{87} = (\varphi_3 - \varphi_4)\left(\bar{a}^2 - \bar{b}^2\right)\bar{b}\left(\frac{3}{\Omega^2} - \frac{1}{\Omega^2 - \varphi^2}\right) \quad (83k)$$

$$r_{88}(0) = R_{88}B^3 + O(B^2), \quad R_{88} = (\varphi_3 - \varphi_4)\left(\bar{a}^2 - \bar{b}^2\right)\bar{b}\left(\frac{2}{\Omega^2 - \varphi^2} - \frac{3}{\Omega^2}\right) \quad (83l)$$

$$r_{89}(0) = 0, \; r_{90}(0) = R_{90}B^3 + O(B^2), \quad R_{90} = (\varphi_3 - \varphi_4)\left(\bar{a}^2 - \bar{b}^2\right)\bar{b}\left(\frac{1}{2(\Omega^2 - 4\varphi^2)} - \frac{1}{\Omega^2 - \varphi^2}\right) \quad (83m)$$

$$r_{91}(0) = 0, \; r_{92}(0) = R_{92}B^3 + O(B^2), \quad R_{92} = \frac{(\varphi_3 - \varphi_4)\left(\bar{a}^2 - \bar{b}^2\right)\bar{b}}{4(\Omega^2 - \varphi^2)} \quad (83n)$$

$$r_{93}(0) = 0, \; r_{94}(0) = R_{94}B^3 + O(B^2), \quad R_{94} = (\varphi_3 - \varphi_4)\left(\bar{a}^2 - \bar{b}^2\right)\bar{b}\left(\frac{3}{2\Omega^2} + \frac{2}{\Omega^2 - 4\varphi^2}\right) \quad (83o)$$

$$r_{95}(0) = 0, \; r_{96}(0) = R_{96}B^3 + O(B^2), \quad R_{96} = \frac{\bar{b}}{2}(\varphi_3 - \varphi_4)\left(\bar{a}^2 - \bar{b}^2\right)\left(\frac{3}{2\Omega^2} + \frac{2}{\Omega^2 - 4\varphi^2} + \frac{1}{2(\Omega^2 - 4\varphi^2)}\right) \quad (83p)$$

$$r_{97}(0) = 0, \quad r_{98}(0) = r_{96}(0), \quad r_{99}(0) = 0 \quad (83g)$$

Substituting all these simplifications in (78a) and maintaining only the terms that are cubic in the displacement, we get

$$U_{2,tt}^{(3)} + \varphi^2 U_2^{(3)} = r_{100} + r_{101}cos\varphi t + r_{102}sin\varphi t + r_{103}cos2\varphi t + r_{104}sin2\varphi t + r_{105}cos3\varphi t + r_{106}sin3\varphi t + r_{107}cos(\varphi - \Omega)t + r_{108}sin(\varphi - \Omega)t + r_{109}cos(\varphi + \Omega)t + r_{110}sin(\varphi + \Omega)t - 2\varphi\left[\left(\beta_2^{(1)'} + \beta_2^{(1)}\right)cos\varphi t - \left(\delta_2^{(1)'} + \delta_2^{(1)}\right)sin\varphi t\right] \quad (84a)$$

$$U_2^{(3)}(0,0) = 0, \quad U_{2,t}^{(3)}(0,0) + U_{2,\tau}^{(1)}(0,0) = 0 \quad (84b)$$

where,

$$r_{100} = \varphi_{12}r_{69} - \varphi_{13}r_{87} - \varphi_8 r_{62} + \varphi_8 r_{80} \quad (85a)$$
$$r_{101} = \varphi_{12}r_{70} - \varphi_{13}r_{88} - \varphi_8 r_{63} + \varphi_8 r_{81} \quad (85b)$$
$$r_{102} = \varphi_{12}r_{71} - \varphi_{13}r_{89} - \varphi_8 r_{64} + \varphi_8 r_{82} \quad (85c)$$
$$r_{103} = \varphi_{12}r_{72} - \varphi_{13}r_{92} - \varphi_8 r_{65} + \varphi_8 r_{83} \quad (85d)$$
$$r_{104} = \varphi_{12}r_{73} - \varphi_{13}r_{93} - \varphi_8 r_{66} + \varphi_8 r_{84} \quad (85e)$$
$$r_{105} = \varphi_{12}r_{74} - \varphi_{13}r_{94} - \varphi_8 r_{67} + \varphi_8 r_{85} \quad (85f)$$
$$r_{106} = \varphi_{12}r_{75} - \varphi_{13}r_{95} - \varphi_8 r_{68} + \varphi_8 r_{86} \quad (85g)$$
$$r_{107} = \varphi_{12}r_{78} - \varphi_{13}r_{96}, \quad r_{108} = \varphi_{12}r_{79} - \varphi_{13}r_{97}, \quad r_{109} = \varphi_{12}r_{76} - \varphi_{13}r_{98} \quad (85h)$$
$$r_{110} = \varphi_{12}r_{77} - \varphi_{13}r_{99}, \quad \varphi_{12} = (\varphi_5 + \varphi_0\varphi_6), \quad \varphi_{13} = (\varphi_5 + \varphi_0\varphi_6 - \varphi_7) \quad (85i)$$

To obtain a uniformly valid solution in terms of t, we equate to zero in (84a) the coefficients of $cos\varphi t$ and $sin\varphi t$.This yields

$$\beta_2^{(1)'} + \beta_2^{(1)} = \frac{r_{101}}{2\varphi}, \quad \delta_2^{(1)'} + \delta_2^{(1)} = \frac{-r_{102}}{2\varphi} \quad (86)$$

We don't need to expressly solve for $\beta_2^{(1)}$ and $\delta_2^{(1)}$ but however need

$$\beta_2^{(1)'}(0) = \frac{r_{101}(0)}{2\varphi} = \frac{1}{2\varphi}[r_{101} = \varphi_{12}r_{70} - \varphi_{13}r_{88} - \varphi_8 r_{63} + \varphi_8 r_{81}]|_{\tau=0} = B^3 R_{110} \qquad (87a)$$

$$R_{110} = \frac{1}{2\varphi}\left[r_{101} = \varphi_{12}R_{70} - \varphi_{13}R_{88} - \varphi_8 R_{63} - \frac{15}{4}\bar{b}\varphi_8\right], \quad \delta_2^{(1)'}(0) = -\delta_2^{(1)}(0) = \bar{b}B \qquad (87b)$$

The remaining equation in (84a) is solved to get

$$U_2^{(3)}(t,\tau) = \delta_2^{(3)}(\tau)cos\varphi t + \beta_2^{(3)}(\tau)sin\varphi t + \frac{r_{100}}{\varphi^2} - \frac{1}{3\varphi^2}(r_{103}cos2\varphi t + r_{104}sin2\varphi t)$$
$$- \frac{1}{8\varphi^2}(r_{105}cos3\varphi t + r_{106}sin3\varphi t) + \frac{1}{\Omega(2\varphi-\Omega)}(r_{107}cos(\varphi-\Omega)t + r_{108}sin(\varphi-\Omega)t)$$
$$- \frac{1}{\Omega(2\varphi+\Omega)}(r_{109}cos(\varphi+\Omega)t + r_{110}sin(\varphi+\Omega)t) \qquad (88a)$$

where,

$$\delta_2^{(3)}(0) = -\left[\frac{r_{100}}{\varphi^2} - \frac{r_{103}}{3\varphi^2} - \frac{r_{105}}{8\varphi^2} + \frac{r_{107}}{\Omega(2\varphi-\Omega)} - \frac{r_{109}}{\Omega(2\varphi+\Omega)}\right]|_{\tau=0} \qquad (89a)$$

$$\beta_2^{(3)}(0) = \frac{-\delta_2^{(1)'}(0)}{\varphi} = \frac{-\bar{b}B}{\varphi} \qquad (89b)$$

By substituting (60) and (61) into (54) and simplifying, we can determine $U_{1(m,2n)}^{(3)}$ and $U_{2(m,2n)}^{(3)}$. This will not be done here so as not to obscure the main focus of the analysis. Nevertheless, the general deformation so far is

$$\binom{U}{f} = \epsilon\left\{\binom{U_1^{(1)}}{f_1^{(1)}}cosny + \binom{U_2^{(1)}}{f_2^{(1)}}sinny\right\}sinmx + \epsilon^2\left\{\binom{U_1^{(2)}}{f_1^{(2)}}cos2ny + \binom{U_2^{(2)}}{f_2^{(2)}}sin2ny\right\}sinmx$$
$$+ \left[\epsilon^3\left\{\binom{U_1^{(3)}}{f_1^{(3)}}cosny + \binom{U_2^{(3)}}{f_2^{(3)}}sinny + \binom{U_{1(m,3n)}^{(3)}}{f_{1(m,3n)}^{(3)}}cos3ny + \binom{U_{2(m,3n)}^{(3)}}{f_{2(m,3n)}^{(3)}}sin3ny\right\}sinmx\right]$$
$$+ \cdots \qquad (90)$$

VII. Maximum Displacement U_a

In determining the dynamic buckling load λ_D, we shall admit only the buckling modes that are in the shape of imperfection so that we write

$$U = \epsilon[U_1^{(1)}cosny + U_2^{(1)}sinny]sinmx + \epsilon^3[(U_1^{(3)}cosny + U_2^{(3)}sinny)sinmx] + \cdots \qquad (91)$$

The conditions for maximum displacement, which is obtained in space and time, are

$$U_{,x} = U_{,y} = 0; \qquad U_{,t} + \epsilon^2 U_{,\tau} = 0 \qquad (92)$$

Let x_a, y_a, t_a and τ_a be the critical values of x, y, t and τ at maximum displacement and let

$$y_a = y_0 + \epsilon^2 y_2 + \epsilon^3 y_3 + \cdots, \quad t_a = t_0 + \epsilon^2 t_2 + \ldots \quad \therefore \quad \tau_a = \epsilon^2 t_a = \epsilon^2(t_0 + \epsilon^2 t_2 + \ldots) \quad (93)$$

From the first two terms in (92), we get

$$\epsilon[U_1^{(1)}cosny_0 + U_2^{(1)}sinny_0]cosmx_a + \epsilon^3[(U_1^{(3)}cosny_0 + U_2^{(3)}sinny_0)cosmx_a] = 0 \qquad (94)$$

and

$$\epsilon[-U_1^{(1)}sinny_0 + U_2^{(1)}cosny_0]sinmx_a + \epsilon^3[(-U_1^{(3)}sinny_0 + U_2^{(3)}cosny_0)sinmx_a] = 0 \qquad (95)$$

From (94), we require that

$$cosmx_a = 0 \;\; i.e \quad x_a = \frac{\pi}{2m}, \quad m\; odd \qquad (96)$$

Similarly, by using (96) and expanding (95) and taking terms of order ϵ, we get

$$U_2^{(1)}cosny_0 - U_1^{(1)}sinny_0 = 0 \qquad (97)$$

which is evaluated at $t = t_0, \; \tau = 0$, to get

$$y_0 = \frac{1}{n}\tan^{-1}\left(\frac{\bar{b}}{\bar{a}}\right) \qquad (98)$$

where we have taken the least nontrivial values in each case. By expanding the last term of (92), using (96) and equating terms of ϵ, we get

$$U_{1,t}^{(1)}cosny_0 + U_{2,t}^{(1)}sinny_0 = 0 \qquad (99)$$

On expansion, it follows from

$$sin\varphi t_0 = 0 \quad i.e \quad t_0 = \frac{\pi}{\varphi} \qquad (100)$$

The maximum displacement U_a is next determined by evaluating (91) at maximum values of the variables. This yields

$$U_a = \epsilon\big[U_1^{(1)}cosny_a + U_2^{(1)}sinny_a\big]sinmx_a + \epsilon^3\big[\big(U_1^{(3)}cosny_a + U_2^{(3)}sinny_a\big)sinmx_a\big] + \cdots \qquad (101)$$

which is evaluated at t_a, τ_a.After simplification, using (98) and (100), we get, from (101),

$$U_a = \epsilon\big[U_1^{(1)}cosny_0 + U_2^{(1)}sinny_0\big] + \epsilon^3\big[\big(U_1^{(3)}cosny_0 + U_2^{(3)}sinny_0\big) + t_0\big(U_{1,\tau}^{(1)}cosny_0 + U_{2,\tau}^{(1)}sinny_0\big)\big] + \cdots \quad (102)$$

After simplifying (102), the only nontrivial terms are

$$U_a = \epsilon\big[U_1^{(1)}cosny_0 + U_2^{(1)}sinny_0\big] + \epsilon^3\big[\big(U_1^{(3)}cosny_0 + U_2^{(3)}sinny_0\big) + t_0\big(U_{1,\tau}^{(1)}cosny_0 + U_{2,\tau}^{(1)}sinny_0\big)\big] + \cdots \quad (103)$$

The terms in (102) are simplified as follows:

$$U_1^{(1)}cosny_0 + U_2^{(1)}sinny_0 = 2B\big(\bar{a}cosny_0 + \bar{b}sinny_0\big) \qquad (104a)$$

$$U_1^{(3)}cosny_0 = B^3\left[\frac{2R_{44}}{\varphi^2} - \frac{2R_{45}}{3\varphi^2} - \frac{R_{49}(1+cos\Omega t_0)}{\varphi^2-\Omega^2} + \frac{R_{51}(1-cos\Omega t_0)}{\Omega(2\varphi-\Omega)} - \frac{R_{53}(1-cos\Omega t_0)}{\Omega(2\varphi+\Omega)}\right]cosny_0 \qquad (104b)$$

$$U_2^{(3)}sinny_0 = B^3\left[\frac{2R_{100}}{\varphi^2} - \frac{2R_{103}}{3\varphi^2} + \frac{R_{107}(1-cos\Omega t_0)}{\Omega(2\varphi-\Omega)} - \frac{R_{109}(1-cos\Omega t_0)}{\Omega(2\varphi+\Omega)}\right]sinny_0 \qquad (105)$$

where,

$$r_{100} = \varphi_{12}r_{69} - \varphi_{13}r_{87} - \varphi_8 r_{62} + \varphi_8 r_{80} \qquad (106a)$$

$$r_{103} = \varphi_{12}r_{72} - \varphi_{13}r_{92} - \varphi_8 r_{65} + \varphi_8 r_{83} \qquad (106b)$$

$$r_{107} = \varphi_{12}r_{78} - \varphi_{13}r_{96}, \qquad r_{109} = \varphi_{12}r_{76} - \varphi_{13}r_{98} \qquad (106c$$

Finally, we have

$$U_a = 2B\epsilon\big(\bar{a}cosny_0 + \bar{b}sinny_0\big) + \frac{B^3\epsilon^3}{2\varphi^2}\left[\big(R_{111}cosny_0 + R_{112}sinny_0\big) + \frac{2\pi\varphi}{B^2}\big(\bar{a}cosny_0 + \bar{b}sinny_0\big)\right] + \cdots \qquad (107)$$

where,

$$R_{111} = \left[R_{44} - \frac{R_{45}}{3} - \frac{\varphi^2}{2}\left\{\frac{R_{51}(1-cos\Omega t_0)}{\Omega(2\varphi-\Omega)} - \frac{R_{49}(1+cos\Omega t_0)}{\varphi^2-\Omega^2} - \frac{R_{53}(1-cos\Omega t_0)}{\Omega(2\varphi+\Omega)}\right\}\right] \qquad (108a)$$

$$R_{112} = \left[R_{100} - \frac{R_{103}}{3} + \frac{\varphi^2}{2}\left\{\frac{R_{107}(1-cos\Omega t_0)}{\Omega(2\varphi-\Omega)} - \frac{R_{109}(1-cos\Omega t_0)}{\Omega(2\varphi+\Omega)}\right\}\right] \qquad (108b)$$

VIII. Dynamic Buckling Load, λ_D

For the purpose of determining the dynamic buckling load λ_D, we write U_a as

$$U_a = \epsilon C_1 + \epsilon^3 C_3 + \cdots \qquad (109a)$$

where,

$$C_1 = 2B\big(\bar{a}cosny_0 + \bar{b}sinny_0\big) \qquad (109b)$$

$$C_3 = \frac{B^3}{2\varphi^2}\left[\big(R_{111}cosny_0 + R_{112}sinny_0\big) + \frac{2\pi\varphi}{B^2}\big(\bar{a}cosny_0 + \bar{b}sinny_0\big)\right] \qquad (109c)$$

As is the usual process [5, 6], we first have to reverse the series (109a) such that

$$\epsilon = d_1 U_a + d_3 U_a^3 + \cdots \qquad (110a)$$

By substituting in (110a) for U_a from (109a) and equating the coefficients of powers of ϵ, we get

$$d_1 = \frac{1}{C_1}, \qquad d_3 = \frac{-C_3}{C_1^4} \qquad (110b)$$

The maximization (1) easily follows from (110a) to yield

$$\epsilon = \frac{2}{3}\sqrt{\frac{C_1}{3C_3}} \qquad (111)$$

A simplification of (111) yields

$$\left[(m^2+n^2\xi)^2 + \left\{\left(\frac{mA}{1+\xi}\right)^2 + n^2\xi r\right\}(1+\xi)^2\left(\frac{n^2\xi r - m^2}{(m^2+n^2\xi)^2}\right) - \lambda_D\left\{\frac{\alpha m^2}{2} + n^2\xi\left(1-\frac{\alpha}{2}\right)\right\}\right]^{3/2} = \frac{3\sqrt{6}}{4}\lambda_D\epsilon\left\{\frac{\alpha m^2}{2} + n^2\xi\left(1-\frac{\alpha}{2}\right)\right\}\sqrt{\frac{A_2}{A_1}} \qquad (112a)$$

where,

$$A_1 = \bar{a}cosny_0 + \bar{b}sinny_0 \qquad (112b)$$

$$A_2 = \frac{B^3}{2\varphi^2}\left[R_{111}cosny_0 + R_{112}sinny_0 + \frac{2\pi\varphi(\lambda_D)}{B^2(\lambda_D)}\left(\bar{a}cosny_0 + \bar{b}sinny_0\right)\right] \qquad (112c)$$

Equation (112a) is valid for small values of ϵ i.e $|\epsilon| < 1$ and determines the dynamic buckling load λ_D asymptotically. It is implicit in nature and valid for m odd.

Analysis of Result

As observed throughout the analyses, the dynamic buckling load λ_D, depends, among other things, on the Fourier coefficients $\bar{a}$ and $\bar{b}$. As in [6], the static buckling load λ_S can be derived from

$$\left[(m^2+n^2\xi)^2 + \left\{\left(\frac{mA}{1+\xi}\right)^2 + n^2\xi r\right\}(1+\xi)^2\left(\frac{n^2\xi r - m^2}{(m^2+n^2\xi)^2}\right) - \lambda_S\left\{\frac{\alpha m^2}{2} + n^2\xi\left(1-\frac{\alpha}{2}\right)\right\}\right]^{3/2}$$
$$= 3\sqrt{3}\lambda_S\epsilon\left\{\frac{\alpha m^2}{2} + n^2\xi\left(1-\frac{\alpha}{2}\right)\right\}\sqrt{Q_1} \qquad (113a)$$

where,

$$Q_1 = \frac{\varphi_9\bar{b}^3 sinny_0 - \varphi_{11}\bar{a}^3 cosny_0 + \bar{a}\bar{b}\left(\bar{a}\varphi_{10}sinny_0 - \bar{b}cosny_0\right)}{\bar{a}cosny_0 + \bar{b}sinny_0} \qquad (113b)$$

This is valid for the nomenclatures as per the cited publication. We have also been able to obtain the Airy stress function as in equation (89). Using (112) and (113a, b), we can relate the dynamic buckling load λ_D to its equivalent static buckling load λ_S to get

$$\left[\frac{(m^2+n^2\xi)^2 + \left\{\left(\frac{mA}{1+\xi}\right)^2 + n^2\xi r\left(1-\frac{\alpha}{2}\right)\right\}(1+\xi)^2\left(\frac{n^2\xi r - m^2}{(m^2+n^2\xi)^2}\right) - \lambda_D\left\{\frac{\alpha m^2}{2} + n^2\xi\left(1-\frac{\alpha}{2}\right)\right\}}{(m^2+n^2\xi)^2 + \left\{\left(\frac{mA}{1+\xi}\right)^2 + n^2\xi r\left(1-\frac{\alpha}{2}\right)\right\}(1+\xi)^2\left(\frac{n^2\xi r - m^2}{(m^2+n^2\xi)^2}\right) - \lambda_S\left\{\frac{\alpha m^2}{2} + n^2\xi\left(1-\frac{\alpha}{2}\right)\right\}}\right]^{3/2}$$
$$= \frac{1}{\sqrt{2}}\left(\frac{\lambda_D}{\lambda_S}\right)\sqrt{\frac{A_2}{A_1 Q_1}} \qquad (114)$$

Certainly, the relationship (114) is independent of the imperfection amplitude ϵ. Hence, if either of λ_D or λ_S is given, then the other buckling load can be conveniently evaluated.

With the aid of QBasic codes, we can obtain the numerical values for the relationship between the Dynamic Buckling Loads and the Imperfection parameters for some fixed values of r. Here, we take $A = 3.5, \xi = 0.3, H = 0.06, K(\xi) = 7, b = 1, m = n = 1, r = 3, 5, 7, 9$.The results are shown in Table 1and Figure 1.

The following are easily derived from Table 1 as well as from the graphical plot:

a) The dynamic buckling load decreases with increased imperfection

b) The higher the ratio of the radii of the toroidal shell r, the greater the dynamic buckling load.

Table 1: Relationship between the Dynamic Buckling Load λ_D and the Imperfection Parameter ϵ for some fixed values of r and for $\alpha = 1$.

IMPERFECTION PARAMETER ϵ	DYNAMIC BUCKLING LOAD λ_SFOR r = 5	DYNAMIC BUCKLING LOAD λ_SFOR r = 7	DYNAMIC BUCKLING LOAD λ_SFOR r = 9	DYNAMIC BUCKLING LOAD λ_SFOR r = 11
0.01	13.21456	52.33254	128.75078	256.54235
0.02	13.20762	51.28903	127.54681	256.23113
0.03	13.20489	51.20337	127.40127	256.10268
0.04	13.20339	50.64819	127.27463	256.02022
0.05	13.20246	49.94245	123.84598	253.20119
0.06	13.20182	49.89072	123.73169	253.14288
0.07	13.20133	49.85053	123.64689	253.09819
0.08	13.20097	49.81596	123.57958	253.06259
0.09	13.20068	49.78369	123.52379	253.03337
0.1	13.20044	49.75049	123.47615	253.00882

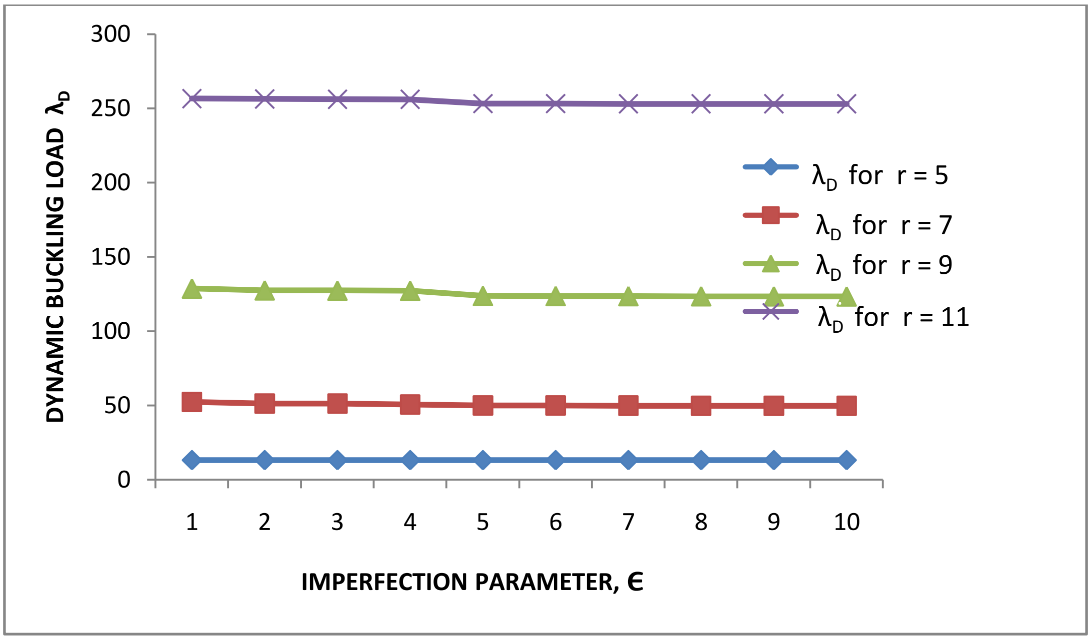

Figure 1: Graphical Plot showing the Relationship Between the Dynamic Buckling Load λ_D and the Imperfection Parameter ϵ for some fixed values of r and for $\alpha = 1$.

IX. Conclusion

This investigation has concerned itself with perturbation procedures in determining the deformation and dynamic buckling load of a deterministically imperfect toroidal shell segment trapped by a step load. The results are asymptotic in nature. The analysis is such that we are able to relate the dynamic buckling load to its static equivalent. Hence, if one of these buckling loads is known, the other one can be determined without necessarily carrying out the arduous perturbation process all over.

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