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Truncated Cauchy Power–Inverse Exponential Distribution: Theory and Applications

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Abstract:
In this article, a new distribution is introduced, which is generated from the truncated Cauchy power-G family of distribution named as truncated Cauchy power- inverse exponential distribution (TCP-IE). We have explored various statistical and mathematical properties, shapes and behavior of the proposed distribution through probability density function (PDF) plot, cumulative distribution function (CDF) plot, and hazard rate function. We illustrated the estimation of the parameters and their corresponding confidence interval using the maximum likelihood estimation (MLE) method for the (TCP-IE) distribution. Two real data sets are taken to assess the suitability and applicability of purposed distribution. It is observed that it can be used quite effectively to analyze lifetime data and performs betteras compared to theother three distributions namelyexponential power, Marshall-Olkin Extended Exponential (MOEE) distribution and generalized Rayleigh distributions.

Keywords: *Truncated Cauchy power-G family, Inverse Exponential distribution, Hazard function,Maximum likelihood estimation.*

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I. Introduction

Statistical models are very useful in analyzing and predicting real-world phenomena. Several classical probability distributions have been widely used over the past decades for modeling data in several areas. Recently, authors focused on introducing a new probability model adding an extra parameter(s) to the well-known classical distributions, which are found more flexible in modeling data. Many recent generators of distributions have been defined to develop new distributions in the statistical literature.

Some of the well-known general families of distributions are,the Marshall-Olkin-Generator introduced by (Marshall and Olkin, 1997), Eugene et al. (2002) has introduced lambda-Generator family, the transmuted-G family by (Shaw & Buckley, 2009),Kumaraswamy-Generator (KW-G) family introduced by (Cordeiro et al., 2010), the Weibull-Generator developed by (Alzaatreth et al., 2013),exponentiated extended-G (Elgarhy et al. (2017), power Lindley-G (Hassan and Nassr (2018) and the truncated inverted Kumaraswamy-Generator family proposed by (Bantan et al., 2019),

The Cauchy distribution is often used in statistics as the counter-example of a "pathological" distribution since both its expected value and its variance are undefined. The Cauchy distribution does not have finite moments of order greater than or equal to one and does not exist moment generating function, but only fractional absolute moments exist. To overcome these drawbacks Johnson &Kotz (1970) has defined the truncated Cauchy distribution having the cumulative distribution function (CDF),

$$G(x;\mu,\theta)=\frac{\arctan\left[(x-\mu)/\theta\right]-\arctan\left[(a-\mu)/\theta\right]}{\arctan\left[(b-\mu)/\theta\right]-\arctan\left[(a-\mu)/\theta\right]};\ x\in(a,b)$$

$where (a,b)\in\Re\bigcup\{-\infty,\infty\},\ \mu\in\Re\ and\ \theta>0.$

The truncated Cauchy distribution has finite moments, and it is more flexible for modeling real data sets which are generally defined over finite ranges of values. A truncated version of the Cauchy distribution was introduced by (Nadarajah&Kotz, 2006) and they had calculated thefinite moments of all ordersand proved that it isa better model for certain practical situations such asfinance, economics, medicine etc.Alzaatreh et al. (2016) has introduced the gamma half-Cauchy distribution, Ashani&Bakar (2016) has presented a skewed truncated Cauchy logistic distribution and its Moments,Cordeiro et al. (2017) has developed the generalized odd half-

Cauchy family of distributions, Tahir et al. (2017) has introduced the Weibull-power Cauchy distribution. Similarly,Alizadeh et al. (2018) has studied the odd power Cauchy family of distributions.

Recently, Aldahlan et al. (2020) has introduced the truncated Cauchy power family of distributions, whose cumulative distribution function (CDF) and probability density function (PDF) respectively defined as,

$$F\left(t;\alpha,\Omega\right)=F_{(0,1)}\left[G(t;\Omega)^{\alpha}\right]=\frac{4}{\pi}\arctan\left[G(t;\Omega)^{\alpha}\right];\ \ t\in\Re,\alpha>0 \quad (1.1)$$

and

$$f\left(t;\alpha,\Omega\right)=\frac{4\alpha}{\pi}\frac{g(t;\Omega)G(t;\Omega)^{\alpha-1}}{1+G(t;\Omega)^{2\alpha}};\ \ t\in\Re,\alpha>0 \quad (1.2)$$

where $G(t;\Omega)$ $g(t;\Omega)$ and are CDF and PDF of baseline distribution respectively and Ω is the parameter space of baseline distribution.

This article aims to provide a gentle introduction of the truncated Cauchy power inverse exponential distribution and discuss some of its recent developments. This new distribution has several advantages, and it will give the practitioner one more option for analyzing real lifetime data. We hope this article will help the practitioner to get the necessary background and the relevant references for this distribution.

The contentsof the proposed study are organized as follows. The truncated Cauchy powerinverse exponential (TCPIE) distribution is introduced and explored various distributional properties in Section 2. The maximum likelihood estimation procedure to estimatethe model parameters and associated confidence intervals using the observed informationmatrix is discussed in Section 3. In Section 4, two real data setshave been analyzed to explore the applications and appropriateness of the proposed distribution. In thissection, we performed some statistical tests for goodness of fit and compared the proposed distribution with some other distributions. Finally, Section 5 ends up with some general concluding remarks.

II. Truncated Cauchy Power Inverse Exponential (TCPIE) Distribution

The Inverse Exponential distribution was introduced by (Keller &Kamath, 1982), and it has been studied and discussed as a lifetime model. If a random variable X has anexponential distribution, the variable $W=\frac{1}{X}$ will have an Inverse exponential distribution. A random variable X is said to have an Inverse Exponential distribution withparameter λ if its PDF and CDF are given respectively by;

$$G\left(x\right)=e^{-\lambda/x};\ \ \lambda>0,x>0$$

(2.1)

and

$$g\left(x\right)=\frac{\lambda}{x^{2}}e^{-\lambda/x};\ \ \lambda>0,x>0 \quad (2.2)$$

By using (1.1) and (1.2) the CDF and PDF of truncated Cauchy power family can be written as

$$F(x)=1-\frac{4}{\pi}\arctan\left\{[1-G(x)]^{\alpha}\right\}$$

(2.3)

and

$$f(x)=\frac{4}{\pi}\frac{\alpha g(x)[1-G(x)]^{\alpha-1}}{1+[1-G(x)]^{2\alpha}} \quad (2.4)$$

After substituting (2.1) and (2.2) in (2.3) and (2.4), we obtained the CDF and PDF of truncated Cauchy power inverse exponential (TCPIE) distribution with parameters α and λ respectively are

$$F(x)=1-\frac{4}{\pi}\arctan\left\{(1-e^{-\lambda/x})^{\alpha}\right\};\ \ \alpha>0,\lambda>0,x>0 \quad (2.5)$$

and

$$f(x)=\frac{4\alpha\lambda}{\pi x^{2}}\frac{e^{-\lambda/x}(1-e^{-\lambda/x})^{\alpha-1}}{1+(1-e^{-\lambda/x})^{2\alpha}};\ \ \ \alpha>0,\lambda>0,x>0 \quad (2.6)$$

Reliability/Survival function

The reliability/survival function of truncated Cauchy power inverse exponential (TCPIE) distribution is

$$R(x)=1-F(x)=\frac{4}{\pi}\arctan\left\{(1-e^{-\lambda/x})^{\alpha}\right\};\ \ \ \alpha>0,\lambda>0,x>0 \quad (2.7)$$

Hazard function

Suppose that T is a random variable with a continuous distribution on [0, ∞). Suppose we interpret T as the lifetime of a device. In that case, the right tail distribution function G is called the reliability function: G(t) is the probability that the device lasts at least t time units. Moreover, the function h defined below is called the failure rate function:

$$h(x)=\frac{f(x)}{R(x)}=\frac{\alpha\lambda e^{-\lambda/x}(1-e^{-\lambda/x})^{\alpha-1}}{x^2\arctan\left\{(1-e^{-\lambda/x})^{\alpha}\right\}[1+(1-e^{-\lambda/x})^{2\alpha}]};\quad \alpha>0,\ \lambda>0,\ x>0 \quad (2.8)$$

In Figure 1, we have displayed the plots of the PDF and hazard rate function of TCPIE distribution for different values of α and λ.

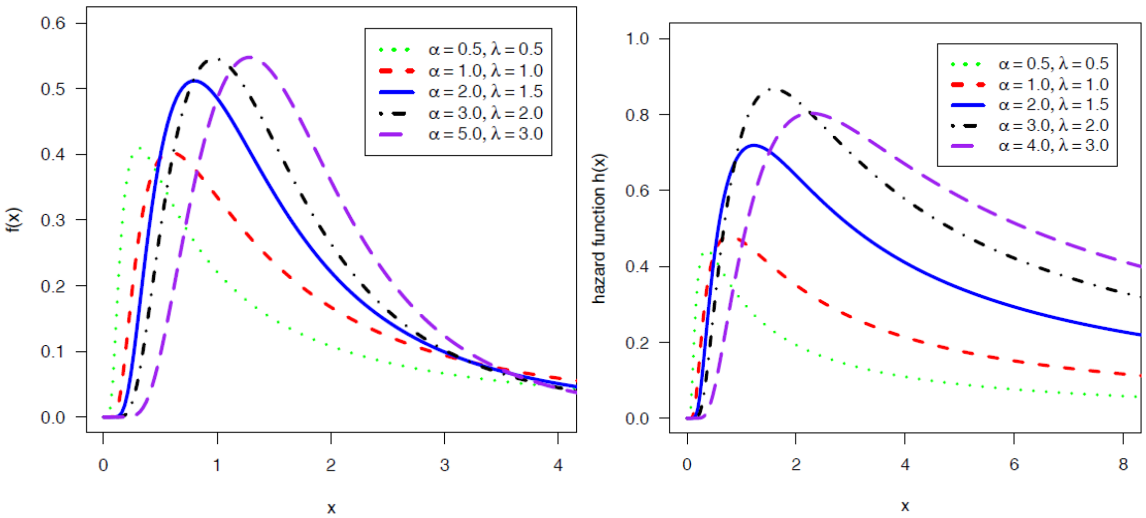

Figure 1. Graph of PDF (left panel) and hazard function (right panel) for different values of α and λ.

The quantile function

The quantile function for a probability distribution has many uses in both the theory and application of probability. If *F* is a probability distribution function, the quantile function may be used to constructa random variable having *F* as its distributions function.It is also called the inverse cumulative distribution function.

$$Q(u)=F^{-1}(u)$$

$$Q(u)=-\lambda\left\{\ln\left\{1-\left[\tan\left(\frac{(1-u)\pi}{4}\right)\right]^{\frac{1}{\alpha}}\right\}\right\}^{-1};0<u<1 \quad (2.9)$$

Let V denote a uniform random variable in (0,1), then the simulated values of X can be generated by setting,

$$x=-\lambda\left\{\ln\left\{1-\left[\tan\left(\frac{(1-v)\pi}{4}\right)\right]^{\frac{1}{\alpha}}\right\}\right\}^{-1};\quad 0<v<1 \quad (2.10)$$

Skewness and Kurtosis:

Skewness is a measure of the asymmetry of a univariate distribution.Bowley'sskewnessis a way to figure out if we have a positively-skewed or negatively skewed distribution.These measures are used mostly in data analysis to study the shape of the distribution or data set. Skewness and Kurtosis based on quantile function are

$$Skewness=\frac{Q(0.75)+Q(0.25)-2Q(0.5)}{Q(3/4)-Q(1/4)}, \text{ and}$$

Coefficient of kurtosis based on octiles given by (Moors, 1988) is

$$K_M=\frac{Q(7/8)-Q(5/8)+Q(3/8)-Q(1/8)}{Q(3/4)-Q(1/4)}$$

The skewness and kurtosis can easily be calculated using (2.9),whereQ(.) represents the quantile function. When the distribution is symmetric, skewness= 0 and when the distribution is right (or left) skewed, skewness>0 (or S <0). As $K_M$increases, the tail of the distribution becomes heavier. These measures are less sensitive to outliers and they exist even for distributions without moments.

Table 1. Skewness and Kurtosis of the TCPIE distribution for different values of parameter α

lambda	alpha	Skewness	Kurtosis
1.00	0.25	0.9086	16.5970
1.00	0.50	0.6861	4.0860
1.00	0.75	0.5495	2.6096
1.00	1.00	0.4651	2.1108
1.00	1.25	0.4081	1.8700
1.00	1.50	0.3668	1.7306
1.00	2.00	0.3101	1.5777
1.00	3.00	0.2452	1.4464
1.00	5.00	0.1825	1.3556
1.00	10.00	0.1210	1.2955

We have presented the skewness and kurtosis of the TCPIE distribution for different values of parameter α keeping λ = 1 in Table 1. It is observed that as α increase, the value of both skewness and kurtosis are decreased, and there is no effect of λ on skewness and kurtosis.

III. Maximum Likelihood Estimates

In this section the maximum likelihood estimators for $TCPIE(\alpha,\lambda)$ are considered, where (α,λ) are unknown. If $x_1, x_2, ...x_n$ is a random sample from $TCPIE(\alpha,\lambda)$ then the likelihood function, $L(\alpha,\lambda)$ is given by,

$$L(\theta; x_1, x_2......x_n) = f(x_1, x_2, ...x_n / \theta) = \prod_{i=1}^{n} f(x_i/\theta)$$

$$L(\alpha,\lambda) = \frac{4\alpha\lambda}{\pi}\prod_{i=1}^{n}\frac{1}{x_i^2}\frac{e^{-\lambda/x_i}(1-e^{-\lambda/x_i})^{\alpha-1}}{1+(1-e^{-\lambda/x_i})^{2\alpha}}; \quad \alpha>0, \lambda>0, x>0$$

Now log-likelihood density is

$$\begin{aligned} l = \log L(\alpha,\lambda) &= n\ln 4 + n\ln\alpha + n\ln\lambda - n\ln\pi - 2\sum_{i=1}^{n}\ln x_i - \sum_{i=1}^{n}\frac{\lambda}{x_i} \\ &+ (\alpha-1)\sum_{i=1}^{n}\ln(1-e^{-\lambda/x_i}) - \sum_{i=1}^{n}\ln\left\{1+(1-e^{-\lambda/x_i})^{2\alpha}\right\} \end{aligned} \tag{3.1}$$

Differentiating (3.1) with respect to α and λ we get,

$$\frac{\partial l}{\partial\alpha} = \frac{n}{\alpha} + \sum_{i=1}^{n}\ln(1-e^{-\lambda/x_i}) - \sum_{i=1}^{n}\frac{2\ln(1-e^{-\lambda/x_i})}{1+(1-e^{-\lambda/x_i})^{-2\alpha}} \tag{3.2}$$

$$\frac{\partial l}{\partial\lambda} = -\frac{n}{\lambda} - \sum_{i=1}^{n}\frac{1}{x_i} + \sum_{i=1}^{n}\frac{(\alpha-1)e^{-\alpha/x_i}}{x_i(1-e^{-\alpha/x_i})} - \sum_{i=1}^{n}\frac{2\alpha e^{-\lambda/x_i}(1-e^{-\lambda/x_i})^{2\alpha-1}}{x_i\left\{1+(1-e^{-\lambda/x_i})^{2\alpha}\right\}} \tag{3.3}$$

Equating (3.2) and (3.3) to zero and solving them for α and λ, we get the maximum likelihood estimate $\hat{\alpha}$ *and* $\hat{\lambda}$ of the parameters α and λ. Maximization of (3.1) can be obtained by using computer software like R, Matlab etc. For the interval estimation of α and λ and testing of the hypothesis, we have to calculate the observed information matrix. The observed information matrix for α and λ can be obtained as

$$O=\begin{bmatrix} O_{11} & O_{12} \\ O_{21} & O_{22} \end{bmatrix}$$

where

$$O_{11}=\frac{\partial^2 l}{\partial\alpha^2}=-\frac{n}{\alpha^2}-\sum_{i=1}^{n}\frac{4\,[\ln(1-e^{-\lambda/x_i})]^2(1-e^{-\lambda/x_i})^{-2\alpha}}{[1+(1-e^{-\lambda/x_i})^{-2\alpha}]^2}$$

Let $U_i=1-e^{-\lambda/x_i}$

$$O_{22}=\frac{\partial^2 l}{\partial\lambda^2}=-\frac{n}{\lambda^2}-\sum_{i=1}^{n}\frac{(\alpha-1)e^{-\lambda/x_i}}{x_i^2(U_i)^2}$$
$$-\sum_{i=1}^{n}\frac{2\alpha e^{-\lambda/x_i}(U_i)^{2\alpha-1}\left\{[1+(U_i)^{2\alpha}][(2\alpha-1)(U_i)^{2\alpha-1}-1]-2\alpha e^{-\lambda/x_i}(U_i)^{2\alpha-1}\right\}}{\left\{1+(U_i)^{2\alpha}\right\}^2}$$

$$O_{22}=\frac{\partial^2 l}{\partial\lambda^2}=-\frac{n}{\lambda^2}-\sum_{i=1}^{n}\frac{(\alpha-1)e^{-\lambda/x_i}}{x_i^2(U_i)^2}$$
$$-\sum_{i=1}^{n}\frac{2\alpha e^{-\lambda/x_i}(U_i)^{2\alpha-1}\left\{[1+(U_i)^{2\alpha}][(2\alpha-1)(U_i)^{2\alpha-1}-1]-2\alpha e^{-\lambda/x_i}(U_i)^{2\alpha-1}\right\}}{\left\{1+(U_i)^{2\alpha}\right\}^2}$$

$$O_{12}=\frac{\partial^2 l}{\partial\alpha\partial\lambda}=\sum_{i=1}^{n}\frac{e^{-\lambda/x_i}}{x(U_i)}-\sum_{i=1}^{n}\frac{2e^{-\lambda/x_i}}{x_i}\left\{\frac{[1+U_i^{-2\alpha-1}+2\alpha U_i^{-2\alpha-1}\ln(U_i)}{[1+(1-(U_i)^{2\alpha}]^2}\right\}$$

Let $\eta=(\alpha,\lambda)$ denote the parameter space and the corresponding MLE of η as $\hat{\eta}=(\hat{\alpha},\hat{\lambda})$, then the asymptotic normality results in, $(\hat{\eta}-\eta)\rightarrow N_2\left[0,(O(\eta))^{-1}\right]$ where $O(\eta)$ is the Fisher's information matrix. By applying the Newton-Raphson algorithm to maximize the likelihood produces the observed information matrix and hence the variance-covariance matrix is obtained as,

$$\left[O(\eta)\right]^{-1}=\begin{pmatrix} \mathrm{var}(\hat{\alpha}) & \mathrm{cov}(\hat{\alpha},\hat{\lambda}) \\ \mathrm{cov}(\hat{\alpha},\hat{\lambda}) & \mathrm{var}(\hat{\lambda}) \end{pmatrix} \qquad (3.4)$$

Hence from the asymptotic normality of MLEs, approximate 100(1-α) % confidence intervals for α and λ can be constructed as,

$\hat{\alpha}\pm z_{\alpha/2}SE(\hat{\alpha})$ and (3.5)

$\hat{\lambda}\pm z_{\alpha/2}SE(\hat{\lambda})$

where $z_{\alpha/2}$ is the upper percentile of standard normal variate.

IV. Real Data Analysis

In this section, we illustrate the applicability of TCPIE distribution by considering two different data sets used by earlier researchers. We also fit exponential power, Marshall-Olkin Extended Exponential (MOEE) distribution and generalized Rayleigh distribution.

I. Exponential Power (EP) distribution:

The probability density function of EP introduced by (Smith and Bain, 1975) is

$$f_{EP}(x)=\alpha\,\lambda^{\alpha}\;x^{\alpha-1}\,e^{(\lambda\,x)^{\alpha}}\,\exp\left\{1-e^{(\lambda\,x)^{\alpha}}\right\}\quad;(\alpha,\lambda)>0,\quad x\geq 0\,.$$

where αand λare the shape and scale parameters, respectively.

II. Marshall-Olkin Extended Exponential (MOEE) distribution.

Marshall &Olkin (1997) has presented MOEE distribution whose probability density function is

$$f_{MOEE}(x) = \frac{\alpha\,\lambda\,e^{-\lambda x}}{\left\{1-(1-\alpha)\,e^{-\lambda x}\right\}^2}\;; \qquad (x>0, \lambda>0, \alpha>0),$$

III. Generalized Rayleigh (GR) distribution.

The generalized Rayleigh distribution was introduced by (Kundu&Raqab, 2005).The PDF of GR distribution is

$$f_{GR}(x) = 2\alpha\lambda^2 x \exp\left(-\lambda^2 x^2\right)\left[1-\exp\left(-\lambda^2 x^2\right)\right]^{\alpha-1}; \qquad \alpha>0, \lambda>0, x>0$$

The parameter of each of these distributions is estimated by using the MLE method. For the comparison purpose we use negative log-likelihood (-LL), Akaike information criterion (AIC), Bayesian information criterion (BIC), Corrected Akaike Information criterion (CAIC) and Hannan-Quinn information criterion (HQIC), which are used to select the best modelamong several models. The expressions to compute AIC, BIC, CAIC and HQIC are given below:

I. $AIC = -2l(\hat{\theta}) + 2k$

II. $BIC = -2l(\hat{\theta}) + k\log(n)$

III. $CAIC = AIC + \dfrac{2k(k+1)}{n-k-1}$

IV. $HQIC = -2l(\hat{\theta}) + 2k\log\left[\log(n)\right]$

wherekis the number of parameters, and n is the size of the sample in the model under consideration. The negative log-likelihood value and the value of AIC, BIC, CAIC and HQIC are displayed in Table 5 and Table 6. We conclude that the proposed model produces a better fit than other models.

Further, to compare the fits of the TCPIE distribution with other competing distributions, we consider the Kolmogorov-Simnorov (KS), the Anderson-Darling (AD) and the Cramer-Von Mises (CVM) statistics. These three statistics are widely used to compare non-nested models and to illustrate how closely a specific CDF fits the empirical distribution of a given data set. These statistics are computed as

$$KS = \max_{1\le i\le n}\left(z_i - \frac{i-1}{n}, \frac{i}{n} - z_i\right)$$

$$AD = -n - \frac{1}{n}\sum_{i=1}^{n}(2i-1)\left[\ln z_i + \ln\left(1-z_{n+1-i}\right)\right]$$

$$CVM = \frac{1}{12n} + \sum_{i=1}^{n}\left[\frac{(2i-1)}{2n} - z_i\right]^2$$

where $z_i = CDF(x_i)$; the x_i's being the ordered observations.

Data Set: 1

The data below are from an accelerated life test of 59 conductors, Lawless(2003). The failures can occur in microcircuits because of the movement of atoms in the conductors in the circuit; this is referred to as electro-migration. Thefailure times are in hours,and there are no censored observations.

6.545, 9.289, 7.543, 6.956, 6.492, 5.459, 8.120, 4.706, 8.687, 2.997, 8.591,6.129, 11.038, 5.381, 6.958, 4.288, 6.522, 4.137, 7.459, 7.495, 6.573, 6.538, 5.589, 6.087, 5.807, 6.725, 8.532, 9.663, 6.369, 7.024, 8.336, 9.218, 7.945, 6.869, 6.352, 4.700, 6.948, 9.254, 5.009, 7.489, 7.398, 6.033, 10.092, 7.496, 4.531, 7.974, 8.799, 7.683, 7.224, 7.365, 6.923, 5.640, 5.434, 7.937, 6.515, 6.476, 6.071, 10.491, 5.923.

Data Set 2

The data given here arose in tests on the endurance of deep groove ball bearings. The data are the number of million revolutions before failure for each of the 23 ball bearings in the life test (Lawless, 2003).

17.88, 28.92, 33.00, 41.52, 42.12, 45.60, 48.80, 51.84, 51.96, 54.12, 55.56, 67.80, 68.64, 68.64, 68.88, 84.12, 93.12, 98.64, 105.12, 105.84, 127.92, 128.04, 173.40

By maximizing the likelihood function in (3.1), we have computed the maximum likelihood estimates directly using *optim(.)* inR software(R Development Core Team, 2020) and Rizzo (2008). The MLE's with their standard errors (SE) and 95% asymptotic confidence interval (ACI) for α and λ are presented in Table 2, and Table 3 for the data sets 1 and 2 respectively.

Table 2

MLE, SE and 95% confidence interval (Data Set-1)

Parameter	MLE	SE	95% ACI	t-value	Pr(>t)
alpha	68.9168	3.1946	(62.6554,75.1782)	21.57	<2e-16
lambda	30.1073	0.9114	(28.3210,31.8936)	33.03	<2e-16

Table 3

MLE, SE and 95% confidence interval (Data Set-2)

Parameter	MLE	SE	95%ACI	t-value	Pr(>t)
alpha	5.0144	0.9686	(3.1159,6.9129)	5.177	2.26e-07
lambda	114.3167	4.3047	(105.8795,122.7539)	26.556	< 2e-16

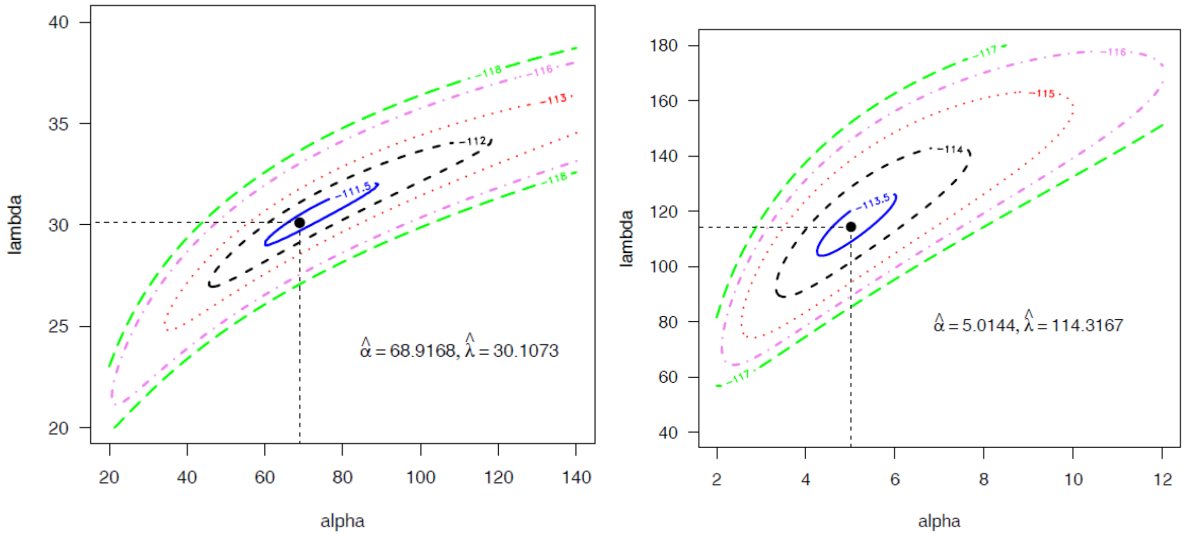

Figure 2. Contour plot for the parameters (α, λ) for the Data Set-1 and Data Set-2 respectively.

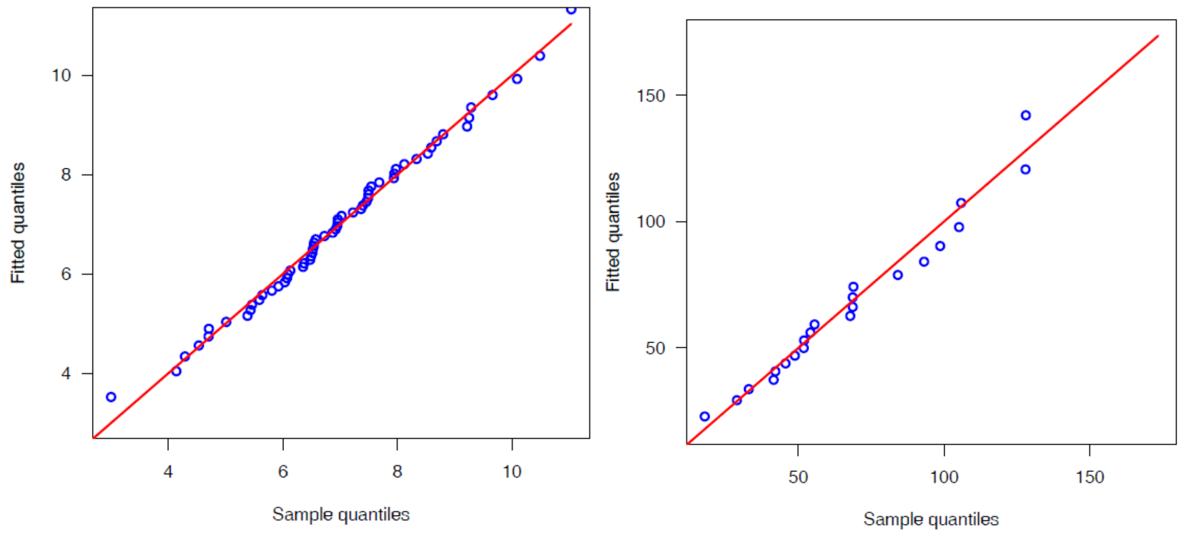

Figure 3. Quantile-Quantile (Q-Q) plot for the Data Set-1 and Data Set-2, respectively.

In Table 4 and Table 5, we have displayed the maximum likelihood estimators of all the models taken for comparison and their corresponding negative Log-likelihood value for the Data Set-1and Data Set-2, respectively.

Table 4
Maximum likelihood estimators and Log-likelihood (Data Set-1)

Model	MLEs		-LL
EP (α, λ)	3.1404	0.1138	116.5015
MOEE (α, λ)	305.0868	0.8422	114.4841
GR (α, λ)	6.406	0.2206	111.8717
TCPIE (α, λ)	68.9168	30.1073	111.3997

Table 5
Maximum likelihood estimators and Log-likelihood (Data Set-2)

Model	MLEs		-LL
EP (α, λ)	17.9214	0.04345	114.3503
MOEE (α, λ)	1.428	0.00888	115.1566
GR (α, λ)	1.199	0.0131	113.5442
TCPIE (α, λ)	5.0144	114.3167	113.3936

In Figure 3, we have plotted the graph of profile log-likelihood functions of α and λ. It is verified that the maximum likelihood estimators are unique.

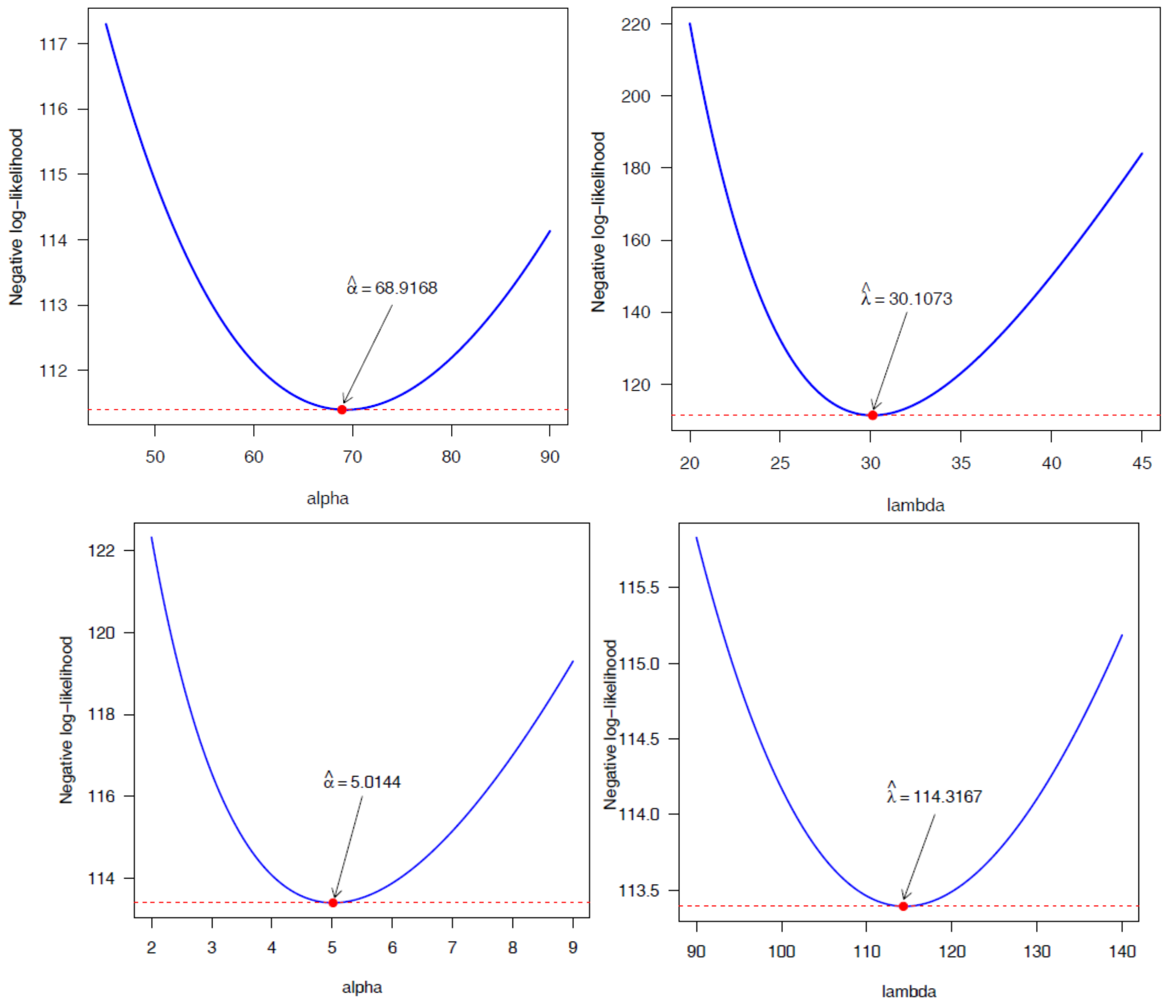

Figure 4. Profile log-likelihood functions of parameters α and λ (first row, Data Set-1) and (second row, Data Set-2).

Further, in Table 6 and Table 7, we have introduced (-LL) and the value of AIC, BIC, CAIC and HQIC for the Data Set-1 and Data Set-2. We conclude that the proposed modelTCPIE produces a better fit to the data taken than other competing models.

Table 6
Log-likelihood, AIC, BIC, CAIC and HQIC (Data Set-1)

Model	-LL	AIC	BIC	CAIC	HQIC
EP	116.5015	237.0030	241.1581	237.2099	238.6250
MOEE	114.4841	232.9682	237.1232	233.1750	234.5901
GR	111.8717	227.7434	231.8984	227.9576	229.3653
TCPIE	111.3997	226.7995	230.9545	227.0138	228.4214

Table 7
Log-likelihood, AIC, BIC, CAIC and HQIC (Data Set-2)

Model	-LL	AIC	BIC	CAIC	HQIC
EP	115.1566	234.3132	236.5842	234.8586	234.8843
MOEE	114.3503	232.7006	234.9716	233.2461	233.2718
GR	113.5442	231.0884	233.3594	231.6338	231.6595
TCPIE	113.3936	230.7873	233.0583	231.3327	231.3584

The histogram and the fitted density functions are displayed in Figure 5 (first row), which supports the results in Tables 8 and 9. Also, Figure 5 (second row), which compares the distribution functions for the different models with the empirical distribution function produces the same. Therefore, for the given data sets illustrates, the proposed distribution gets better fit and more reliable results from other alternatives.

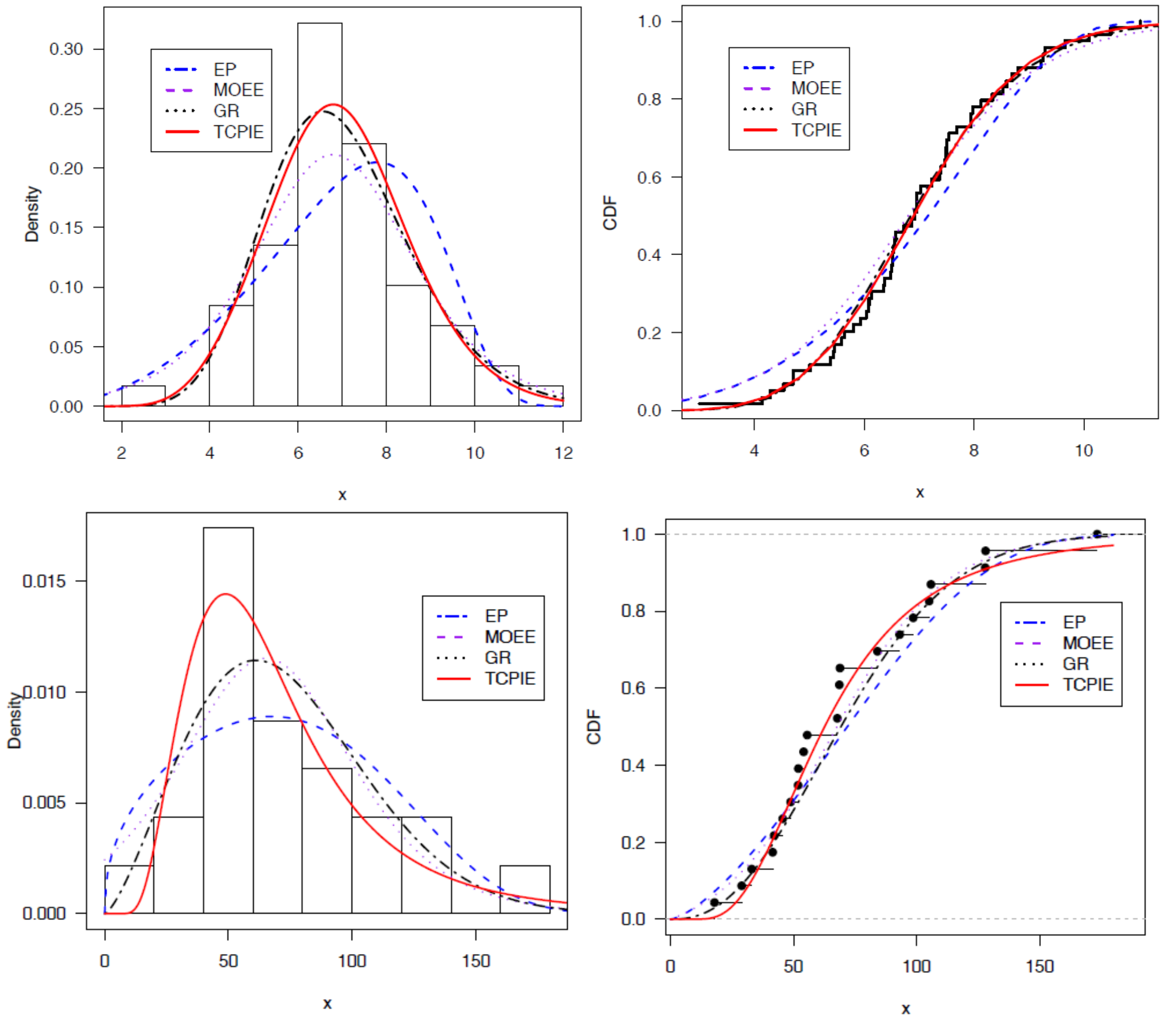

Figure 5. The Histogram and the PDF of fitted distributions and Empirical CDF with estimated CDF (first row, Data Set-1) and (second row, Data Set-2).

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In Table 8 and Table 9, we have displayed the test statistics and their corresponding p-value of competing models for both the data sets. The outcome shows that the proposed model has the minimum value of the test statistic and higher *p*-value; hence we conclude that the TCPIE is best in the analysis of goodness-of-fit.

Table 8

The goodness-of-fit statistics and their corresponding *p*-value (Data Set-1)

Model	*KS(p-value)*	*AD(p-value)*	*CVM(p-value)*
Data Set 1			
EP	0.1362(0.2042)	1.3699(0.2180)	0.2388(0.2035)
MOEE	0.1129(0.4091)	1.0667(0.3238)	0.1660(0.3449)
GR	0.0741(0.8785)	0.2555(0.9673)	0.0433(0.9169)
TCPIE	0.0585(0.9805)	0.1838(0.9943)	0.0307(0.9749)

Table 9
The goodness-of-fit statistics and their corresponding p-value (Data Set-2)

Model	*KS(p-value)*	*AD(p-value)*	*CVM(p-value)*
EP	0.1786(0.4551)	0.6172(0.6300)	0.1034(0.5723)
MOEE	0.1383(0.7714)	0.3795(0.8675)	0.0589(0.8255)
GR	0.1573(0.6199)	0.3428(0.9020)	0.0649(0.7882)
TCPIE	0.0840(0.9969)	0.2164(0.9853)	0.0312(0.9748)

V. Conclusion

In this study, we have introduced a new probability distribution named as two-parameter truncated Cauchy power inverse exponential (TCPIE) distribution. Some statistical and mathematical properties of the derived distribution are investigated. We have presented the PDF, the CDF, and the shape of the failure rate function and observed that the derived distribution could bear varieties of shapes. The parameters of the proposed distribution are estimated by using the maximum likelihood method. We have takentwo real data sets to demonstrate the methodology. We have computed the maximum likelihood estimates. The purposed distribution provides quite better for the dataset, as shown in the contour plots, profile log-likelihood plots, and Q-Q plots. We have also considered three other distributions for comparison. The comparison is done based on various information criteria such as AIC, BIC, CAIC, HQIC, and Kolmogorov-Simnorov (KS), the Anderson-Darling (AD) and the Cramer-Von Mises (CVM) statistics, and found that the proposed model is best as compared to three other distributions. We hope that this probability distribution may be an alternative in the field of probability distribution and applied statistics.

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